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# Maps Close to Identity and Universal Maps in the Newhouse Domain

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**Abstract:** Given an $n$-dimensional $C^r$-diffeomorphism $g$, its renormalized iteration is an iteration of $g$, restricted to a certain $n$-dimensional ball and taken in some $C^r$-coordinates in which the ball acquires radius 1. We show that for any $r \geq 1$ the renormalized iterations of $C^r$-close to identity maps of an $n$-dimensional unit ball $B^n$ ($n \geq 2$) form a residual set among all orientation-preserving $C^r$-diffeomorphisms $B^n \to R^n$. In other words, any generic $n$-dimensional dynamical phenomenon can be obtained by iterations of $C^r$-close to identity maps, with the same dimension of the phase space. As an application, we show that any $C^r$-generic two-dimensional map that belongs to the Newhouse domain (i.e., it has a so-called wild hyperbolic set, so it is not uniformly-hyperbolic, nor uniformly partially-hyperbolic) and that neither contracts, nor expands areas, is $C^r$-universal in the sense that its iterations, after an appropriate coordinate transformation, $C^r$-approximate every orientation-preserving two-dimensional diffeomorphism arbitrarily well. In particular, every such universal map has an infinite set of coexisting hyperbolic attractors and repellers.

## 1. Ruelle–Takens Problem and Universal Maps

*1.1. Ruelle–Takens construction.* A long-standing open problem in the theory of dynamical systems is to describe which kind of dynamical phenomena can be expected in close to identity maps. It started with a seminal paper [1] where it was shown that, given any $n \geq 2$, any $n$-dimensional dynamics can be implemented by a $C^n$-small perturbation of the identity map of an $n$-dimensional torus. The paper received a lot of attention by physicists, because it proposed a new view on the onset of hydrodynamical turbulence; at the same time it caused much criticism. One of the reasons for the critique was that the $C^n$-small perturbations constructed in [1] were not small in $C^{n+1}$, which is quite unphysical. The controversy was resolved in [2], where it was shown that for any $r \geq 1$ and $n \geq 2$, given any $C^r$-diffeomorphism $F$ of a closed $n$-dimensional ball, one can find a $C^r$-close to identity map $g$ of the $(n+1)$-dimensional closed unit ball $B^{n+1}$ such

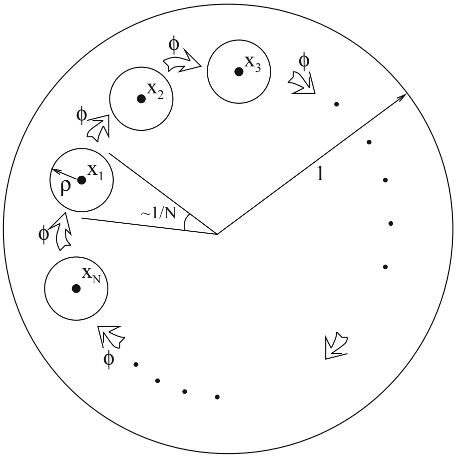


**Fig. 1.** An illustration to Ruelle–Takens construction

that the diffeomorphism $F$ coincides with some iteration of the map $g$ restricted to some $n$-dimensional invariant manifold. Thus, the restriction on smoothness of perturbations was removed by sacrificing one dimension of the phase space; anyway, other scenarios of the transitions to turbulence had already been known.

From the purely mathematical point of view, the question still remained unsolved: can an arbitrary $n$-dimensional dynamics be obtained by iterations of a $C^r$-close to identity map of $B^n$, i.e., in the same dimension of the phase space? The difficulty is that the straightforward construction proposed in [1] does not work for high $r$ in principle. Indeed, given an orientation-preserving diffeomorphism $F : B^n \to R^n$, one can imbed it into a continuous family $\mathcal{F}_t$ of the diffeomorphisms such that $\mathcal{F}_1 = F$ and $\mathcal{F}_0 = id$. Then, given any $N$, the map $F$ can be represented as a superposition of $N$ maps

$$F = F_N \circ \cdots \circ F_1, \quad \text{where} \quad F_s = \mathcal{F}_{s/N} \circ \mathcal{F}^{-1}_{(s-1)/N}, \tag{1.1}$$

that are $O(1/N)$-close to identity. One can then choose $N$ pairwise disjoint small balls $D_s \in B^n$ of radius $\rho \sim N^{-1/(n-1)}$ and define a map $\phi : B^n \to B^n$ such that $\phi(x)|_{x\in D_s} \equiv x_{s+1} + \rho F_s(\frac{x-x_s}{\rho})$ where $x_s$ is the center of $D_s$; the positions of the centers are chosen in such a way (Fig. 1) that the distances $\|x_{s+1} - x_s\|$ are uniformly close to zero for all $s$, hence the map $\phi$ is $C^0$-close to identity. By construction, $\phi^N|_{D_0}$ is linearly conjugate to $F$, i.e., the dynamics of $\phi^N|_{D_0}$ coincides with the dynamics of $F$. However, the derivatives of $\phi$ of order $k$ behave as $N^{-1}\rho^{1-k} \sim N^{\frac{k-n}{n-1}}$, i.e., at $k \geq n$ they do not, in general, tend to zero as $N \to +\infty$. Thus, an arbitrary $n$-dimensional dynamics can be implemented by iterations of $C^{n-1}$-close to identity maps of $B^n$, but the construction gives no clue of whether the same can be said about the $C^n$-close to identity maps.

One could try to position the regions $D_s$ differently, or make their radii vary, or change their shape. This, however, hardly can lead to an essential increase in the maximal order of the derivatives (of $\phi$) which tend to zero as $N \to +\infty$. The reason lies in a well-known fact from the averaging theory that the $O(\delta)$-close to identity map

$$\bar{x} = x + \delta f(x)$$

approximates a time shift of a certain autonomous flow with the accuracy $O(\delta^m)$ for an arbitrarily large $m$ (if $f \in C^\infty$). Hence, the number of iterations necessary in order to obtain a dynamics that is far from that of an autonomous flow, has to grow faster than $O(\delta^{-m})$, for every $m$. As we see, in order to obtain such kind of dynamics, one has to control a very large number of iterations of close to identity maps, hence decompositions much longer than that given by (1.1) have to be considered.

*1.2. Main result.* In this paper we propose such a decomposition (Theorem 7), using which we show that

*an arbitrary $C^r$-generic orientation-preserving n-dimensional dynamics can be obtained by iterations of $C^r$-close to identity maps of $B^n$, $n \geq 2$.*

To make the formulations precise, we borrow some definitions from [3]. Let $g$ be a $C^r$-diffeomorphism of a closed $n$-dimensional ball $D$. Take an integer $m > 0$ and any $C^r$-diffeomorphism $\psi$ of $R^n$ such that $\psi(B^n) \subset int(D)$. The map $g_{m,\psi} = \psi^{-1} \circ g^m \circ \psi|_{B^n}$ is a $C^r$-diffeomorphism that maps $B^n$ into $R^n$. We will call the maps $g_{m,\psi}$ obtained by this procedure *renormalized iterations of g*.

**Theorem 1.** *In the space of $C^r$-smooth orientation-preserving diffeomorphisms that map $B^n$ into $R^n (n \geq 2)$ there is a residual set $\mathcal{S}_r$ such that for every map $F \in \mathcal{S}_r$, for every $\delta > 0$, and for every n-dimensional ball D there exists a map $g : R^n \to R^n$, equal to identity outside D, such that $||g - id||_{C^r} < \delta$, and F is a renormalized iteration of g.*

In other words, a generic $C^r$-diffeomorphism $B^n \to R^n$ is, up to a smooth coordinate transformation, an iteration of an arbitrarily close to identity map. In fact, the theorem is an immediate consequence of the following

**Lemma 1.** *Given any ball $D \subset R^n$ and any orientation-preserving $C^r$-diffeomorphism $F : B^n \to R^n$, for any $\hat{\delta} > 0$ and $\hat{\varepsilon} > 0$ there exists a map g, equal to identity outside D, such that $||g - id||_{C^r} < \hat{\delta}$ and $||F - g_{m,\psi}||_{C^r} < \hat{\varepsilon}$ for some renormalized iteration $g_{m,\psi} = \psi^{-1} \circ g^m \circ \psi|_{B^n}$. Moreover, the map g and the coordinate transformation $\psi$ can be constructed in such a way that the sets $g^i(\psi(B^n))(i = 0, \ldots, m-2)$ do not intersect $g^{m-1}(\psi(B^n))$.*

It is easy to see that this lemma indeed implies Theorem 1. As the sets $g^i(\psi(B^n))$ $(i = 0, \ldots, m-2)$ do not intersect $g^{m-1}(\psi(B^n))$, it follows that by adding to $g$ small perturbations, localized in $g^{m-1}(\psi(B^n))$, we can make $g_{m,\psi}$ run an open subset in the space of $C^r$-diffeomorphisms $B^n \to R^n$. Thus, by Lemma 1, the set $\mathcal{S}_r(\delta)$ of all renormalized iterations of the maps $g$ such that $||g - id||_{C^r} < \delta$ (and $g$ equals identity outside $D$) contains a subset that is dense and open in the space of $C^r$-smooth orientation-preserving diffeomorphisms of $B^n$ into $R^n$, for every $\delta > 0$. Hence, the intersection $\mathcal{S}_r$ of these sets over all $\delta > 0$ is residual. The set $\mathcal{S}_r$ is independent of the choice of the ball $D$ (because there always exists a conjugacy that takes one such ball to another), which gives us the theorem.

We prove Lemma 1 in Sect. 4. The first step in our construction of the approximations of the given map $F$ by renormalized iterations is a representation of $F$ as a superposition of a pair of certain special maps and some volume-preserving diffeomorphisms (Lemma 3). Each of the "special maps" can be realized as a flow map through a kind of saddle-node bifurcation (see Fig. 2), reminiscent of the so-called Kotova–Stanzo–Ilyashenko lips (see [4–6]). For the volume-preserving diffeomorphisms we exploit the results obtained in [3] for symplectic diffeomorphisms and prove (Lemma 4) the existence of an arbitrarily good (in the $C^r$-norm on any compact) polynomial approximation by a superposition of volume-preserving Hénon-like maps. It is known that Hénon-like maps often appear as rescaled first-return maps near a homoclinic tangency.[1] In this paper we find a kind of homoclinic tangency that does incorporate all the Hénon-like maps that appear in our volume-preserving polynomial approximations.

Thus, we show that the map $F$ can be approximated arbitrarily well by a superposition of maps related to certain homoclinic bifurcations. The last step is to build a close to identity map which displays these bifurcations simultaneously. This is achieved by an arbitrarily small perturbation of the time-$\delta$ map of a certain $C^\infty$ flow (the time $\delta$ map of a flow is, obviously, $O(\delta)$-close to identity). A more detailed scheme of the proof is described in the beginning of Sect. 4.

Note that the approximation (that we construct in Lemma 4) of any volume-preserving diffeomorphism of a unit ball into $R^n$ by a polynomial volume-preserving diffeomorphism is not straightforward, because the Jacobian of the approximating diffeomorphism should be equal to 1 everywhere, and this constraint is quite strong for polynomial maps. Had the approximation result been true for all volume-preserving maps, i.e., not necessarily diffeomorphisms, it would produce a counterexample to the famous "Jacobian conjecture"; however, our approximation uses in an essential way the injectivity of the map that has to be approximated (we represent the map as a shift by the orbits of some smooth non-autonomous flow).

It should be mentioned that Theorem 1 does not hold true at $n = 1$. Indeed, if a map $F$ on the interval $B^1$ has two fixed points (with the multipliers different from 1), then every close map $\hat{F}$ has a pair of fixed points $P_{1,2}$ as well. If such $\hat{F}$ is a renormalized iteration of a diffeomorphism $g$, i.e., if $\hat{F} = \psi^{-1} \circ g^m \circ \psi$, then $g$ will also have a pair of fixed points, $\psi(P_1)$ and $\psi(P_2)$ (at $n > 1$ this is not true). The interval between $P_1$ and $P_2$ will therefore be invariant with respect to $\psi^{-1} \circ g \circ \psi$, hence $\psi^{-1} \circ g \circ \psi$ will be a root of degree $m > 1$ of the map $\hat{F}$ on this interval. Now note that the maps of the interval that have a root are not dense in $C^2$, according to [31]. Thus, we obtain that renormalized iterations are not dense either.

One can check through the proof of Theorem 1 that it holds true for finite-parameter families of orientation-preserving diffeomorphisms:

**Theorem 2.** *In the space of k-parameter families $F_\varepsilon$, $\varepsilon \in B^k$, of $C^r$-smooth orientation-preserving diffeomorphisms of $B^n$ into $R^n (n \geq 2)$ there is a residual set $\mathcal{S}_{kr}$ such that for every $F_\varepsilon \in \mathcal{S}_{kr}$ and for every $\delta > 0$ there exists $g_\varepsilon : D \times B^k \to D$ such that $||g_\varepsilon - id||_{C^r} < \delta$ and $F_\varepsilon = \psi_\varepsilon^{-1} \circ g_\varepsilon^m \circ \psi_\varepsilon|_{B^n}$ for some $m > 0$ and some family $\psi_\varepsilon$ of $C^r$-diffeomorphisms of $R^n$.*

The proof is obtained verbatim the proof of Theorem 1 in Sects. 3, 4 (just add the subscript $\varepsilon$ everywhere). Thus, we find that any dynamical phenomenon that occurs generically in finite-parameter families of dynamical systems can be encountered in maps arbitrarily close to identity (with the same dimension of the phase space).

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[1] This fact goes back to [7,8] and, since then, was exploited in many papers, e.g. in [9–22], and in the conservative case, which is most relevant for this paper, in [23–28] and many other works; see detailed reviews in [29,30].

*1.3. Guiding principle.* To put the result into a general perspective, we recall that one of the main sources of difficulties in the theory of dynamical systems is that structurally stable systems are not dense in the space of all systems [32–35]; moreover, most natural examples of chaotic dynamics are indeed structurally unstable (like e.g. the famous Lorenz attractor [37–40]). Understanding the dynamics of systems from the open regions of structural instability (i.e., the regions where arbitrarily close to every system there is a system that is not topologically conjugate to it) has been the subject of active research for the past four decades. It often happens, and helps a lot, that structurally unstable systems may possess certain robust properties, i.e., dynamical properties which are not destroyed by small perturbations. For example, systems with Lorenz attractor are pseudohyperbolic [30,41,42] (or volume-hyperbolic in the terminology of [43,44]), and this is, in fact, the very property (expansion of volumes transverse to a strong-stable invariant foliation) that allowed for a very detailed description of these systems [37–40,45–48]. Another robust property is uniform partial hyperbolicity, i.e., a robust existence of a strong-unstable invariant foliation [33,49–51]. These are partial cases of the so-called dominated splitting (see e.g. [52,53] for review). In fact, not so much of robust properties are known; it could even happen that beyond the mentioned partial hyperbolicity and volume-hyperbolicity no other robust dynamical properties exist.

There is significant progress in the theory of $C^1$-robust dynamical properties (see [53,54] for review). A similar general theory of $C^r$-robust properties with $r \geq 2$ does not exist. Moreover, there is no a priori reason why $C^r$-robust properties should be $C^1$-robust, i.e., why it so often happens that possible restrictions to the dynamics (like the above mentioned partial hyperbolicity and volume hyperbolicity) can be formulated in terms of the first derivatives only. Still, the following claim [74] can be demonstrated for various examples of *homoclinic bifurcations* (see [55–57]), and can be used as a guiding principle in the study of bifurcations of systems with a non-trivial dynamics:

*given an n-dimensional system with a compact invariant set that is neither partially hyperbolic nor volume-hyperbolic, every dynamics that is possible in $B^n$ should be expected to occur at the bifurcations of this particular system subject to $C^\infty$-small perturbations.*

This statement is not a theorem and it might not be true in some situations, still it gives a useful view on global bifurcations. In particular, it was explicitly applied in [58] to Galerkin approximations of damped nonlinear wave equations in order to obtain estimates from below on the dimension of the maximal attractor in the situation where classical methods [59] do not work.

Theorem 1 gives one more example to the above stated principle: the identity map has no kind of hyperbolic structure, neither it contracts nor expands volumes, so it should not be surprising that its bifurcations lead to an ultimately rich dynamics.

*1.4. Universal maps and coexistence of hyperbolic attractors.* The same idea can be expressed in somewhat different terms (cf. [3,28,56] and, for the $C^1$-perspective, see [54,60]). Let us call the set of all renormalized iterations of a map $g : D \to D$ its *dynamical conjugacy class*. The map will be called *$C^r$-universal* if its dynamical conjugacy class is $C^r$-dense among all orientation-preserving $C^r$-diffeomorphisms of the closed unit ball $B^n$ into $R^n$. By the definition, the dynamics of any single universal map is ultimately complicated and rich, and the detailed understanding of it is not simpler than understanding of all diffeomorphisms $B^n \to R^n$ altogether.

At first glance, the mere existence of $C^r$-universal maps of a closed ball is not obvious for sufficiently large $r$. However, Theorem 1 immediately implies the following

**Theorem 3.** *For every $r \geq 1$, $C^r$-universal diffeomorphisms of a given closed ball $D$ exist arbitrarily close, in the $C^r$-metric, to the identity map.*

*Proof.* Take an arbitrary sequence of pairwise disjoint closed balls $D_j \subset D$, a sequence of maps $F_j$ which is $C^r$-dense in space of orientation-preserving $C^r$-diffeomorphisms $B^n \to R^n$, and a sequence $\varepsilon_j \to +0$ as $j \to +\infty$. By Theorem 1, given any $\delta$, there exist maps $g_j$ such that $g_j$ is identity outside $D_j$, some renormalized iteration of $g_j$ is $\varepsilon_j$-close to $F_j$, and $\|g_j - id\|_{C^r} \leq \delta$. By construction, the map $g(x) \equiv g_j(x)$ at $x \in D_j$ $(j = 1, 2, \dots)$ and $g(x) \equiv x$ at $x \in D \setminus \cup_{j=1}^{\infty} D_j$ is $C^r$-universal and $\delta$-close to identity. □

As an immediate consequence of Theorem 3, we obtain the following result.

**Theorem 4.** *Arbitrarily close to the identity map in the space of $C^r$-diffeomorphisms $B^n \to B^n$ (any $r \geq 1$, any $n \geq 2$) there exist maps with infinitely many coexisting uniformly-hyperbolic attractors (and uniformly-hyperbolic repellers) of all topological types possible in $B^n$.*

*Proof.* This is true because a hyperbolic attractor is a structurally stable object: given a map with a uniformly-hyperbolic attractor, any $C^1$-close map has a hyperbolic attractor topologically conjugate to the original one. For every $n \geq 2$ there exists a $C^r$-diffeomorphism $B^n \to R^n$ with a hyperbolic attractor [61]. Hence, infinitely many of the maps $F_j$ defined in the proof of Theorem 3 have a hyperbolic attractor as well. It follows that each universal map constructed in Theorem 3 has infinitely many hyperbolic attractors; the same is true for hyperbolic repellers. □

We will discuss the coexistence of infinitely many attractors in more detail in the next Section. Here, the possibility of the birth of even a single hyperbolic attractor by a small perturbation of the identity map is enough for us. Thus, we can further pursue the approach of [1] and claim that a hyperbolic attractor can be born at the third Andronov–Hopf bifurcation. Recall that at the primary Andronov–Hopf bifurcation a limit cycle is born from an equilibrium state, and at the secondary Andronov–Hopf bifurcation a two-dimensional invariant torus is born from the limit cycle. The third Andronov–Hopf bifurcation occurs when a three-dimensional invariant torus is born from the two-dimensional one (filled by quasiperiodic orbits). The so-called Landau–Hopf scenario of the onset of turbulence envisioned a chain of further Andronov–Hopf bifurcations that would lead to a creation of an invariant torus of a sufficiently high dimension, i.e., to a quasiperiodic regime with a high number of rationally independent frequencies (see more discussion in [62]). However, as Ruelle and Takens pointed out in [1], the dynamics on the invariant torus is not necessarily quasiperiodic: at the moment the torus is born, the system can be perturbed in such a way that every orbit on the torus will become periodic, hence the Poincaré map will be the identity; hence, as it follows from our results above, a further small perturbation may lead to a chaotic dynamics for a flow on the torus of dimension 3 or higher.

## 2. Universal Maps in the Newhouse Domain

In general, it follows from Theorem 3 that every time we have a periodic orbit for which the corresponding Poincaré map is, locally, identity:

$$\bar{x} = x,$$

or coincides with identity up to flat (i.e. sufficiently high order) terms:

$$\bar{x} = x + o(\|x\|^r),$$

a $C^r$-small perturbation of the system can make the Poincaré map universal, i.e. bifurcations of this orbit can produce dynamics as complicated as it only possible for the given dimension of the phase space. Thus, arbitrarily complicated dynamical phenomena can be uncovered by the study of bifurcations of periodic orbits alone.

In order to show how powerful this observation can be, let us apply it to the analysis of the dynamics of two-dimensional diffeomorphisms from the Newhouse domain. In the space of $C^r$-smooth dynamical systems on any smooth manifold, we define the Newhouse domain as the interior of the closure of the set of systems that have a homoclinic tangency (a tangency between the stable and unstable manifolds of a saddle periodic orbit; for a saddle periodic orbit to exist, the dimension of the phase space should be at least 3 in the case of continuous time and at least 2 for discrete dynamical systems). A non-trivial fact [10,11,13,34,35] is that the Newhouse domain at $r \geq 2$ is always non-empty and adjoins to every system with a homoclinic tangency.

Importantly, most of known global bifurcations which lead to the emergence of chaotic dynamics or happen within the class of systems with complex (chaotic) behavior are accompanied by a creation of homoclinic tangencies. Therefore, Newhouse regions in the space of parameters can be detected for virtually every dynamical model with chaos (see more discussion in [3,15–18,28,56,63]). In my opinion, the ubiquitous presence of homoclinic tangencies in the dynamical models of a natural origin makes the study of maps from the Newhouse domain one of the most important problems of chaotic dynamics (at least for two-dimensional maps; in higher dimensions, the study of Diaz heterodimensional cycles [60,64,65] should be added to the list, as well as of other types of heteroclinic cycles). It should also be mentioned that a commonly shared believe (one of the Palis conjectures [66]) is that the space of two-dimensional $C^r$-diffeomorphisms with $r \geq 2$ is the closure of the union of just two open sets: axiom A systems (that includes Morse–Smale systems with simple dynamics), and the Newhouse domain. There is no hope yet of a proof of this conjecture (a known proof [67] of an analogous statement at $r = 1$ cannot be generalized to a higher smoothness). It might, of course, happen that this conjecture is not true at all. Still, it is an empirical fact that homoclinic tangencies appear easily in various models, i.e. if a two-dimensional map with a chaotic dynamics is not uniformly hyperbolic, it quite probably lies in the Newhouse domain.

Typically, a map from the Newhouse domain possesses a basic (i.e. compact, non-trivial, zero-dimensional, locally-maximal, and transitive) hyperbolic set which is wild—the term meaning that for the map itself, and for every $C^r$-close map ($r \geq 2$), there exists an orbit within this set such that its stable and unstable manifolds have a quadratic tangency [34,35,68]. For a given orbit, the corresponding tangency is a codimension-1 bifurcation phenomenon, so it can always be removed by a small smooth perturbation; the wildness, nevertheless, means that once the original tangency is removed a new tangency always appears, corresponding to some other orbit from the same basic hyperbolic set.

A wild hyperbolic set of a $C^r$-diffeomorphism of a two-dimensional smooth manifold will be called *ultimately wild* if it contains a pair of periodic orbits such that at one periodic orbit the so-called saddle value is less than 1 and at the other periodic orbit it is greater than 1. The open subset of the Newhouse domain which corresponds to maps with ultimately wild hyperbolic sets will be called the *absolute Newhouse domain*. If $Q$ is a period $q$ point of a map $f$ (i.e. $f^q Q = Q$), the saddle value is, by definition, the absolute value of the Jacobian of the map $f^q$ at $Q$. Thus, if it is greater than 1, then the map $f$

expands area near $Q$, and it is area-contracting near $Q$ if the saddle value is less than 1. So, no map from the absolute Newhouse domain is area-contracting, nor area-expanding. Moreover, the persistent tangencies between the stable and unstable manifolds of the wild hyperbolic set mean that none of these maps is uniformly hyperbolic, nor uniformly partially-hyperbolic. Thus, there is no obvious restrictions on the dynamics of two-dimensional diffeomorphisms from the absolute Newhouse domain, and the following theorem is, therefore, in agreement with the general "guiding principle" formulated in Sect. 1.3.

**Theorem 5.** *For every $r \geq 2$, a $C^r$-generic diffeomorphism from the absolute Newhouse domain in the space of two-dimensional diffeomorphisms is $C^r$-universal.*

*Proof.* Fix any $r \geq 2$ and let a $C^r$-diffeomorphism of a smooth two-dimensional manifold possess an ultimately wild basic hyperbolic set $\Lambda$. Let $P$ and $Q$ be two saddle periodic points in $\Lambda$ ($f^p P = P$ and $f^q Q = Q$) such that $f^p$ contracts areas near $P$ and $f^q$ expands areas near $Q$. As the periodic points $P$ and $Q$ are hyperbolic, they are preserved at all sufficiently small perturbations. Recall also that the unstable manifold of every point in the basic hyperbolic set has a transverse intersection with the stable manifold of every other orbit in this set. Therefore, the invariant unstable manifold $W^u(P)$ intersects the invariant stable manifold $W^s(Q)$ transversely at the points of some heteroclinic orbit $\Gamma_{PQ}$. By the transversality, this orbit is preserved for all maps sufficiently close to $f$. According to [28], given every $m \geq 1$, in any neighborhood of $f$ in the $C^r$-topology, there is a $C^\infty$-diffeomoprphism for which the unstable manifold $W^u(Q)$ has a tangency of order $m$ to the stable manifold $W^s(P)$ at the points of some heteroclinic orbit $\Gamma_{QP}$ (Fig. 3). This is a non-trivial statement: while the possibility to obtain a quadratic tangency (i.e. $m = 1$) by an arbitrarily small perturbation follows immediately from the wildness of $\Lambda$ and from the fact that the stable and unstable manifold of any given periodic orbit in the basic hyperbolic set are dense within the union of stable and, respectively, unstable manifolds of all orbits in this set [34,35], the tangencies of higher orders are due to the existence of moduli of local $\Omega$-conjugacy [16–18,69].

Let $\tilde{f} \in C^\infty$ be a $C^r$-close to $f$ diffeomorphism for which the above described heteroclinic cycle exists. This cycle is a closed set $C$ that consists of 4 orbits: two periodic orbits (the period $p$ orbit of $P$ and the period $q$ orbit of $Q$) such that $f^p$ contracts areas near $P$ and $f^q$ expands areas near $Q$, a transverse heteroclinic $\Gamma_{PQ}$ and the orbit $\Gamma_{QP}$ of heteroclinic tangency of a sufficiently high order $m$. We will further assume that $m \geq r + 1$. In Sect. 5 we prove the following

**Lemma 2.** *Given any neighborhood $U$ of the heteroclinic cycle $C$, arbitrarily $C^r$-close to $\tilde{f}$ there is a diffeomorphism $\hat{f}$ for which in $U$ there is an open region $V$ filled by periodic orbits of the same period.*

We will call the open region $V$ *a periodic spot*. For every orbit in the periodic spot the corresponding Poincaré map (the map $\hat{f}^k$ where $k$ is the period of the spot) is, locally, identity $\bar{x} = x$. Hence, as we mentioned in the beginning of this Section, it follows from Theorem 3 that in any $C^r$-neighborhood of $\hat{f}$ there exist $C^r$-universal maps.

We have just shown (modulo Lemma 2 a proof of which we postpone until Sect. 5) that the universal maps are $C^r$-dense in the absolute Newhouse domain. This immediately implies the genericity of the universal maps. Indeed, given an orientation-preserving $C^r$-diffeomorphism $g$ of the unit disk into $R^2$, denote as $\mathcal{S}(g, \delta)$ the set of all $C^r$-diffeomorphisms from the absolute Newhouse domain whose dynamical conjugacy class intersects the open $\delta$-neighborhood of $g$ (i.e. whose certain renormalized iteration is at a $C^r$-distance smaller than $\delta$ from $g$). This set, by definition, contains all universal maps—hence, it is dense in the absolute Newhouse domain. This set is also open by definition. Take a countable sequence of maps $g_i$ which is dense in the space of orientation-preserving $C^r$-diffeomorphisms of the unit disk into $R^2$, and a sequence $\delta_j$ of positive reals converging to zero. By construction, the countable intersection $\cap_{i,j}\mathcal{S}(g_i, \delta_j)$ is a residual subset of the absolute Newhouse domain, and every map that belongs to this set is universal. □

In essence, this theorem gives an exhaustive characterization of the richness of dynamical behavior in the absolute Newhouse domain: every two-dimensional dynamics can be approximated by iterations of any generic map from this domain. In a much simpler case of the Newhouse domain in the space of area-preserving maps, a similar statement is contained in [28]: iterations of a generic area-preserving map from the Newhouse domain approximate all symplectic dynamics in a two-dimensional disc. For *area-contracting* maps, it follows from [18] that the closure of the dynamical conjugacy class of a generic map from the Newhouse domain contains all *one-dimensional* maps (we cannot have truly two-dimensional dynamics there, as the areas are contracted). As we mentioned, any two-dimensional map without a uniformly-hyperbolic structure, falls, hypothetically, in one of the three types of the Newhouse domain (or in the boundary of these domains): the first is filled by area-contracting maps, the second by area-expanding maps (i.e. the maps inverse to the maps of the first type), and the third is the absolute Newhouse domain. If, indeed, this conjectural decomposition of the space of $C^r$-smooth two-dimensional diffeomorphisms holds true, then our Theorem 5 somehow completes the description of two-dimensional dynamics.

The three types of Newhouse domains of two-dimensional maps were introduced in [19]. It has been known since [34] that a generic area-contracting map from the Newhouse domain has an infinite set of stable periodic orbits, and the closure of this set contains a (wild) hyperbolic set. The latter fact is especially important: while chaotic dynamics is usually associated with hyperbolic sets, i.e., with saddle orbits, the Newhouse result shows that stable periodic motions can imitate chaos arbitrarily well, and they indeed do it generically [34, 36]. For various situations, one also establishes a generic coexistence of infinitely many *nontrivial* attractors (quasiperiodic [14, 15, 20], Henon-like [12], Lorenz-like [14, 21, 22]). In [19], for the Newhouse domain of the third type, it was shown that a generic map has both an infinite set of stable periodic orbits and an infinite set of repelling periodic orbits; moreover, the intersection of these sets is non-empty and contains an ultimately-wild hyperbolic set. Thus, not only the Newhouse phenomenon holds, we also have a new effect here: a generic inseparability of attractors from repellers (this effect was called *mixed dynamics* in [20]). Our Theorem 5 strengthens these observations: it implies the coexistence of infinitely many hyperbolic attractors and repellers for a generic map from the absolute Newhouse domain (see Theorem 4). As we obtain the hyperbolic attractors and repellers by a perturbation of periodic spots, and the periodic spots are found in an arbitrarily small neighborhood of any heteroclinic cycle of the type we consider in Theorem 5, it follows that the closures of the set of hyperbolic attractors and of the set of hyperbolic repellers that we construct here contain any transverse heteroclinic orbit connecting the points $P$ and $Q$. Such heteroclinic orbits are dense in the basic hyperbolic set $\Lambda$. Thus, we obtain the following

**Theorem 6.** *A $C^r$-generic two-dimensional map from the absolute Newhouse domain has an infinite set of uniformly-hyperbolic attractors and an infinite set of uniformly-*

*hyperbolic repellers, and the intersection of the closures of these sets contains a non-trivial hyperbolic set.*

In what follows we prove Theorem 1 (Sects. 3, 4) and finish the proof of Theorem 5 (Sect. 5).

## 3. An Approximation Theorem

Let $F$ be an orientation-preserving $C^r$-diffeomorphism ($r \geq 3$) which maps the ball $B^n : \{\sum_{i=1}^n x_i^2 \leq 1\}$ into $R^n$. Without loss of generality we may assume that $F$ is extended onto the whole $R^n$, i.e. it becomes a $C^r$-diffeomorphism $R^n \to R^n$, and it is identical (i.e. $F(x) = x$) at $\|x\|$ sufficiently large; such extension is always possible.[2] Let $K$ be a constant such that

$$\sup_{x\in R^n} \frac{\|\nabla J(x)\|}{J(x)} < K, \tag{3.1}$$

where $J(x)$ is the Jacobian of $F$. Denote $R^n_+ := \{x_n > 0\}$.

**Lemma 3.** *There exists a pair of volume-preserving, orientation-preserving $C^{r-2}$-diffeomorphisms $\Phi_1 : R^n_+ \to R^n_+$ and $\Phi_2 : R^n \to R^n$ such that*

$$F = \Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1, \tag{3.2}$$

*where*

$$\Psi_j := (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, \psi_j(x_n)) \quad (j = 1, 2), \tag{3.3}$$

*with*

$$\psi_1(x_n) = e^{Kx_n}, \quad \psi_2(x_n) = \ln x_n. \tag{3.4}$$

*Proof.* We will construct a volume-preserving diffeomorphism $\Phi_1 : (x_1, \ldots, x_n \geq 0) \mapsto (\bar{x}_1, \ldots, \bar{x}_n \geq 0)$ in such a way that

$$\det \frac{d}{dx} \Psi_2 \circ \Phi_1 \circ \Psi_1(x) \equiv J(x); \tag{3.5}$$

then the Jacobian of $\Phi_2 = F \circ (\Psi_2 \circ \Phi_1 \circ \Psi_1)^{-1}$ will be equal to 1 automatically. By (3.3), (3.4), condition (3.5) is equivalent to

$$\bar{x}_n = \phi(x_1, \ldots, x_n) \equiv \frac{Kx_n}{J(x_1, \ldots, x_{n-1}, \frac{1}{K} \ln x_n)}.$$

It follows from (3.1) that

$$\partial\phi/\partial x_n > 0 \tag{3.6}$$

[2] It can be done as follows. First, extend $F$ to a small convex neighbourhood $U$ of $B^n$ without loss of smoothness, so that $F$ will be now a $C^r$-diffeomorphism of $U$ into $R^n$. Take a smooth family $F_t$ of $C^r$-diffeomorphisms $U \to R^n$ such that $F_0 = id$ and $F_1 = F$, the existence of such family is a standard fact. The derivative $\frac{d}{dt} F_t$ defines a $C^r$-smooth vector field $Z(t, x)$ defined at $x \in F_t(U)$, so that $F(x_0)$ is the time-1 shift of the initial condition $x_0$ by the orbit of the non-autonomous differential equation $\dot{x} = Z(t, x)$. Now, if $\xi(t, x)$ is a $C^r$-function which equals to 1 everywhere on $F_t(B^n)$ and zero everywhere outside of $F_t(U)$, then the time-1 shift by the orbits of the system $\dot{x} = \xi(t, x)Z(t, x)$ is the sought $C^r$-diffeomorphism that equals to $F$ on $B^n$ and is identity outside of $U$.

everywhere. Moreover, as $F$ is the identity map outside a bounded region of $R^n$, we have that

$$\phi(x) = Kx_n \tag{3.7}$$

outside a compact subregion of $R^n_+$. Therefore, every trajectory of the vector field

$$\dot{x}_j = 0 \quad (j \leq n-2), \quad \dot{x}_{n-1} = \frac{\partial \phi}{\partial x_n}, \quad \dot{x}_n = -\frac{\partial \phi}{\partial x_{n-1}} \tag{3.8}$$

extends for all $x_{n-1} \in (-\infty, +\infty)$, and the time $\tau(x)$ that the trajectory of the point $x$ needs to get to $x_{n-1} = 0$ is a $C^{r-2}$-function of $x$, well defined everywhere in $R^n_+$. Moreover, as it follows from (3.7), (3.8)

$$\tau(x) = -\frac{1}{K}x_{n-1} + \tau_0(x), \tag{3.9}$$

where $\tau_0$ is a uniformly bounded function, vanishing identically at $x_n$ close to zero and at sufficiently large $x_n$. Thus, for every fixed values of $x_j$ $(j \leq n-2)$, given any $C \in (-\infty, +\infty)$ the level line $\tau(x) = C$ in the $(x_{n-1}, x_n)$-plane coincides with the straight line $x_{n-1} = -KC$ at $x_n$ close to zero and at sufficiently large $x_n$. Every such level line is a connected set (as it is the image of the line $\{x_{n-1} = 0, x_n \geq 0\}$ by the time-$(-C)$ map of the flow of (3.8). Thus, as $x_n$ runs from 0 to $+\infty$, the value of $\phi$ on this line runs all the values from 0 to $+\infty$ [see (3.7)]. It follows that the map $R^n_+ \to R^n_+$ defined by

$$\bar{x}_j = x_j \quad (j \leq n-2), \quad \bar{x}_{n-1} = -\tau(x), \quad \bar{x}_n = \phi(x) \tag{3.10}$$

is surjective.

By (3.8),

$$\frac{\partial \tau}{\partial x_{n-1}} \frac{\partial \phi}{\partial x_n} - \frac{\partial \tau}{\partial x_n} \frac{\partial \phi}{\partial x_{n-1}} = -1. \tag{3.11}$$

It follows that for every fixed values of $x_j$ $(j \leq n-2)$, the function $\phi$ changes monotonically along every level line of $\tau$, which implies the injectivity of map (3.10). Thus, map (3.10) is a $C^{r-2}$-diffeomorphism $R^n_+ \to R^n_+$. By (3.11), it is volume-preserving and orientation-preserving, i.e. it is the sought map $\Phi_1$. □

The maps $x \mapsto \bar{x}$ of the following form:

$$\bar{x}_1 = x_2, \ldots, \bar{x}_{n-1} = x_n, \quad \bar{x}_n = (-1)^{n+1}x_1 + h(x_2, \ldots, x_n) \tag{3.12}$$

(note no dependence on $x_1$ in $h$), will be called *Hénon-like volume-preserving maps*. Note that such maps are always one-to-one, and the inverse map is also Hénon-like.

**Theorem 7.** *Every orientation-preserving $C^r$-diffeomorphism $F : B^n \to R^n$ can be arbitrarily closely approximated, in the $C^r$-norm on $B^n$, by a map of the following form:*

$$H_{2q_2} \circ \cdots \circ H_{21} \circ \Psi_2 \circ H_{1q_1} \circ \cdots \circ H_{11} \circ \Psi_1, \tag{3.13}$$

*where the maps $\Psi_{1,2}$ are given by (3.3), and $H_{js}(j = 1, 2; s = 1, \ldots q_j)$ are certain polynomial Hénon-like volume-preserving maps.*

*Proof.* First, take a $C^{r+2}$-diffeomorphism $\hat{F}$ which approximates $F$ sufficiently closely in $C^r$. For the map $\hat{F}$ construct the decomposition $\hat{F} = \Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$ given by Lemma 3; all the maps in the decomposition are at least $C^r$. The map $\Phi_1$ can be extended onto $x_n \le 0$ by the rule $\bar{x}_n = Kx_n$, $\bar{x}_{n-1} = x_{n-1}/K$ [see (3.10), (3.7), (3.9)], so it becomes a volume-preserving, orientation-preserving $C^r$-diffeomorphism $R^n \to R^n$. Then the theorem follows immediately from Lemma 4 below. □

**Lemma 4.** *Every volume-preserving, orientation-preserving $C^r$-diffeomorphism $\Phi$ : $R^n \to R^n$ can be arbitrarily closely approximated, in the $C^r$-norm on any given compact, by a composition of polynomial Hénon-like volume-preserving maps.*

*Proof.* At $n = 2$ this result is immediately given by Theorem 1 in [3], so we proceed to the case $n \ge 3$. It is well known that $\Phi$ can be imbedded in a smooth in $t$ family $\mathcal{F}_t$ of volume-preserving $C^r$-diffeomorphisms $R^n \to R^n$ such that $\mathcal{F}_0 \equiv id$ and $\mathcal{F}_1 = \Phi$. The derivative $\frac{d}{dt}\mathcal{F}_t$ defines a divergence-free vector field $X(t, x)$, i.e. the diffeomorphism $\mathcal{F}_t$ is the time-$t$ shift by the flow generated by the field $X$. One can approximate $X$ arbitrarily closely on any given compact by a $C^\infty$-smooth divergence-free vector field which is defined and bounded for all $(x, t) \in R^n \times [0, 1]$. Therefore, it is enough to prove the lemma only for those $\Phi$ which can be obtained as the time-1 shift by the flow generated by such a vector field, i.e. we may assume that $X \in C_b^\infty$ with no loss of generality.

Let $T_{\tau,t} = \mathcal{F}_{t+\tau} \circ \mathcal{F}_t^{-1}$, i.e. it is the shift by the flow of $X$ from the time $t$ to $t+\tau$. This map is $O(\tau)$-close to identity, in the $C^r$-norm on any compact subset of $R^n \times [0, 1]$. By construction, given any arbitrarily large integer $N$,

$$\Phi = T_{\tau,(N-1)\tau} \circ \cdots \circ T_{\tau,k\tau} \circ \cdots \circ T_{\tau,0} \tag{3.14}$$

where $\tau = 1/N$, and $k = 0, \ldots, N-1$.

Note that the vector field $X$ admits the following representation:

$$X = \sum_{i=1}^{n-1} X^{(i)} \tag{3.15}$$

where $X^{(i)}$ is a $C^\infty$-smooth divergence-free vector field such that

$$\dot{x}_j \equiv 0 \quad \text{at} \quad j \ne i, i+1. \tag{3.16}$$

Indeed, if we write the field $X$ as

$$\dot{x}_i = \xi_i(x, t), \quad i = 1, \ldots, n,$$

where $\sum_{i=1}^n \frac{\partial \xi_i}{\partial x_i} \equiv 0$ (the zero divergence condition), then the fields $X^{(i)}$ are defined as

$$\dot{x}_i = \eta_i(x, t), \quad \dot{x}_{i+1} = \zeta_i(x, t)$$

with

$$\eta_1 \equiv \xi_1, \quad \eta_i \equiv \xi_i - \zeta_{i-1} \quad (i = 2, \ldots, n-1),$$

$$\zeta_i = -\int_0^{x_{i+1}} \frac{\partial}{\partial x_i} \eta_i(x_1, \ldots, x_i, s, x_{i+2}, \ldots, x_n, t) ds \quad (i = 1, \ldots, n-2), \quad \zeta_{n-1} \equiv \xi_n.$$

By construction, the fields $X^{(1)}, \ldots, X^{(n-2)}$ are divergence-free, and $X^{(n-1)} = X - X^{(1)} - \cdots - X^{(n-2)}$, so $X^{(n-1)}$ is also divergence-free, as $X$ is.

It follows from (3.15) that

$$T_{\tau,t} = T^{(n-1)}_{\tau,t} \circ \cdots \circ T^{(1)}_{\tau,t} + O(\tau^2), \tag{3.17}$$

where $T^{(i)}_{\tau,t}$ is the shift by the flow generated by the vector field $X^{(i)}$. Recall that the maps $T_{\tau,i\tau}$ in (3.14) are $O(1/N)$-close to identity. Therefore, it follows from (3.17), (3.14) that

$$\Phi = T^{(n-1)}_{\tau,(N-1)\tau} \circ \cdots \circ T^{(1)}_{\tau,(N-1)\tau} \circ \cdots \circ T^{(n-1)}_{\tau,k\tau} \circ \cdots \circ T^{(1)}_{\tau,k\tau} \circ \cdots \circ T^{(n-1)}_{\tau,0} \circ \cdots \circ T^{(1)}_{\tau,0} + O(\tau), \tag{3.18}$$

uniformly on compacta.

As $\tau$ can be taken arbitrarily small, it follows that in order to prove the lemma, it is enough to prove that every of the maps $T^{(i)}_{\tau,t}$ in the right-hand side of (3.18) can be approximated arbitrarily well by a composition of Hénon-like volume-preserving maps. The maps $T^{(i)}_{\tau,t}$ are volume-preserving and satisfy

$$\bar{x}_j = x_j \quad \text{at} \quad j \neq i, i+1 \tag{3.19}$$

[see (3.16)]. Therefore, if we denote

$$\bar{x}_i = p(x), \quad \bar{x}_{i+1} = q(x), \tag{3.20}$$

then

$$\det\left(\frac{\partial(p,q)}{\partial(x_i, x_{i+1})}\right) = 1. \tag{3.21}$$

Thus, we can view (3.20) as an $(n-2)$-parameter family of symplectic two-dimensional maps $(x_i, x_{i+1}) \mapsto (\bar{x}_i, \bar{x}_{i+1})$ parameterized by $(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n)$.

According to [3], every finite-parameter family of symplectic maps can be approximated (on any compact) by a composition of families of Hénon-like maps, i.e., in our case, maps of the form

$$\bar{x}_i = x_{i+1}, \quad \bar{x}_{i+1} = -x_i + h(x_{i+1}; x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n).$$

It follows that every map of the form (3.19), (3.20), (3.21) can be approximated arbitrarily closely by a composition of the maps of the form

$$\begin{aligned} \bar{x}_j &= x_j \quad \text{at} \quad j \neq i, i+1, \\ \bar{x}_i &= x_{i+1}, \\ \bar{x}_{i+1} &= -x_i + h(x_{i+1}; x_1, \ldots, x_{i-1}; x_{i+2}, \ldots, x_n). \end{aligned} \tag{3.22}$$

It just remains to note that every map of form (3.22) is a composition of volume-preserving Hénon-like maps; namely, it equals to

$$S^{n-i-1} \circ H \circ S \circ Q^{n-1} \circ S^{i+1},$$

where

$$S := (x_1, \dots, x_n) \mapsto (x_2, \dots, x_n, (-1)^{n+1} x_1),$$

$$Q := (x_1, \dots, x_n) \mapsto (x_2, \dots, x_n, \sum_{j=1}^{n} (-1)^{n+j} x_j),$$

$$H := \{\bar{x}_1 = x_2, \dots, \bar{x}_{n-1} = x_n,$$

$$\bar{x}_n = \sum_{j=1}^{n-1} (-1)^{n+j} x_j - x_n + h(x_n; x_{n-i+1}, \dots, x_{n-1}; (-1)^{n+1} x_2, \dots, (-1)^{n+1} x_{n-i})\}.$$

□

*Remark.* Consider the map

$$\Phi_0 := (x_1, \dots, x_{n-2}, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-2}, x_n, -x_{n-1}). \tag{3.23}$$

This is an orientation- and volume- preserving diffeomorphism of $R^n$, therefore we may rewrite (3.2) as follows:

$$F = \Phi_0 \circ \tilde{\Phi}_2 \circ \Psi_2 \circ \Phi_0 \circ \tilde{\Phi}_1 \circ \Psi_1,$$

where $\tilde{\Phi}_{1,2}$ are orientation-preserving, volume-preserving $C^{r-2}$-diffeomorphisms ($\tilde{\Phi}_j = \Phi_0^{-1} \circ \Phi_j$; we assume that $\Phi_1$ is extended onto the whole of $R^n$, like in Theorem 7). Now, by Lemma 4, we obtain the following, more convenient for us, analog of Theorem 7: *the map $F$ can be arbitrarily closely approximated by a map of the following form:*

$$\Phi_0 \circ \tilde{H}_{2q_2} \circ \dots \circ \tilde{H}_{21} \circ \Psi_2 \circ \Phi_0 \circ \tilde{H}_{1q_1} \circ \dots \circ \tilde{H}_{11} \circ \Psi_1, \tag{3.24}$$

*with polynomial Hénon-like volume-preserving maps $\tilde{H}_{js}$.*

## 4. Proof of Theorem 1

Now we can prove Theorem 1. As we explained in Sect. 1.2, it is enough to prove Lemma 1. Namely, given any orientation-preserving $C^r$-diffeomorphism $F : B^n \to R^n$ we will take a sufficiently large disc $D \subset R^n$ and construct a $C^r$-close to identity map $g$, equal to identity outside $D$, such that some renormalized iteration of $g$, i.e. a map $g_{m,\psi} = \psi^{-1} \circ g^m \circ \psi|_{B^n}$, is a close (as close as we want) approximation of $F$. Our construction allows for making $g$ as close to identity as we want, so once this is done for one particular ball $D$, the same can be done for every ball $D$, since there always exists a map that takes one such ball to another, we can take this map as a coordinate transformation on $D$ and this coordinate transformation changes the $C^r$-distance from $g$ to identity only to a finite factor. We will also check that $g$ and the rescaling map $\psi$ is constructed in such a way that $g^i(\psi(B^n)) \cap g^{m-1}(\psi(B^n)) = \emptyset$ $(i = 0, \dots, m-2)$, in full agreement with Lemma 1.

The idea of the construction is as follows. Given a diffeomorphism $F : B^n \to R^n$ we will take its sufficiently close approximation $\hat{F}$ in the form of (3.24). Then we construct the $C^r$-close to identity map $g$ whose some renormalized iteration is a close (as close as we want) approximation to $\hat{F}$. The map $g$ is a small perturbation (as small as we want, in the $C^r$-norm for any aforehand given $r$) of the time-$\delta$ map $Y_\delta$ of a certain $C^\infty$ flow

$Y$ in $R^n$; we allow the flow $Y$ to depend continuously (in $C^\infty$) on a certain finite set of parameters [$\mu$'s and $\gamma$'s; see (4.1.1), (4.2.1)] which can take values from [0, 1], the exact values of these parameters are defined at the end of the construction. The constant $\delta$ can be chosen as small as we need, so $Y_\delta$ is indeed $C^r$-close to the identity, uniformly with respect to the choice of the parameters $\mu, \gamma$. In our construction, the vector field of $Y$ vanishes identically outside a sufficiently large ball $D$ that depends on $F$ but it does not depend on the choice of the approximation $\hat{F}$, it does not depend on $\delta$ either. The small perturbations which we will apply to $Y_\delta$ will also be localized in $D$. So, our close to identity map $g$ equals to identity outside $D$.

We define the flow $Y$ by means of the following procedure: we give explicit formulas for the vector field inside certain blocks $U_{1\pm}$, $U_{2\pm}$, $V_{1,2}$ described below, while between the blocks we specify only the transition time from the boundary of one block to another and the corresponding Poincaré map. The existence of a $C^\infty$ flow with arbitrary (of class $C^\infty$) transition times and orientation-preserving Poincaré maps between blocks boundaries is a routine fact (at least for the given geometry of the blocks, see Fig. 2).

As we may always approximate our given diffeomorphism $F : B^n \to R^n$ by a $C^\infty$-diffeomorphism arbitrarily well, we assume that $F \in C^\infty$ from the beginning. Let $\Phi_{1,2}$ and $\Psi_{1,2}$ be the maps defined by decomposition (3.2) of $F$. We define the vector field inside the blocks $U_{1\pm}$, $U_{2\pm}$ in such a way that a kind of saddle-node bifurcation is created inside each of the blocks [see (4.1.1), (4.2.1); we build very degenerate saddle-nodes in order to make formulas for the time-$t$ map simpler—see (4.1.3), (4.2.4)]. The Poincaré maps from the boundary $\Sigma^{out}_{j+}$ of $U_{j+}$ to the boundary $\Sigma^{in}_{j-}$ of $U_{j-}$ ($j = 1, 2$) are chosen in such a way that the resulting flow map from entering $U_{j+}$ till exiting $U_{j-}$ is affine conjugate to the map $\Psi_j$ [see (4.1.5), (4.2.5), (3.3), (3.4)]. The aforementioned parameters $\mu$ and $\gamma$ of the flow $Y$ regulate the behaviour near the saddle-nodes inside the blocks $U_{1\pm}$, $U_{2\pm}$ in such a way that we can control the parameters of this affine conjugacy; they do not affect the flow outside these blocks. The regions from entering $U_{j+}$ till exiting $U_{j-}$ will be called "lips".

We make the flow inside $V_{1,2}$ volume-preserving and linear [see (4.1.6), (4.2.2), (4.2.3)]. Moreover, we put saddle equilibria into $V_{1,2}$. We define the Poincaré map between the boundaries $\Pi^{out}_{j+}$ and $\Pi^{in}_{j-}$ of $V_j$ (see Fig. 2) in such a way that a homoclinic loop to the saddle is created. For the time-$\delta$ map $Y_\delta$ of the flow $Y$ the saddle equilibrium is a saddle fixed point, and the homoclinic loop is a continuous family of homoclinic orbits. We perturb the map $Y_\delta$ in such a way that this family splits into a finite set of orbits of homoclinic tangency of sufficiently high orders and unfold these tangencies then. The exact form of the perturbation [see (4.1.17), (4.2.16)] may be chosen such that the iteration of the perturbed map $g$ for one round near the homoclinic loop is a close approximation (in appropriately scaled coordinates) to any given polynomial conservative Hénon-like map [see (4.1.22), (4.2.22)]. Hence, a multi-round iteration of $g$ can be made arbitrarily close to a superposition of a finite number of any given polynomial Hénon-like maps. By Lemma 4 such superpositions approximate any given volume-preserving maps, e.g. maps $\Phi_j$ from the decomposition (3.2). In this way the iteration of $g$ which after a large number of rounds near homoclinic orbits takes points entering $V_j$ to the points entering $U_{3-j}$, is made as close as we want to the map $\Phi_j$ ($j = 1, 2$), in some rescaled coordinates.

The small perturbation from $Y_\delta$ to $g$ is localized in a neighbourhood of a finite number of points; the number of these points depends on the choice of the maps $\Phi_j$ only. The positions of these points depend on $\delta$, and the size of the neighbourhood of these points where the perturbation is localized depends on one more parameter, an

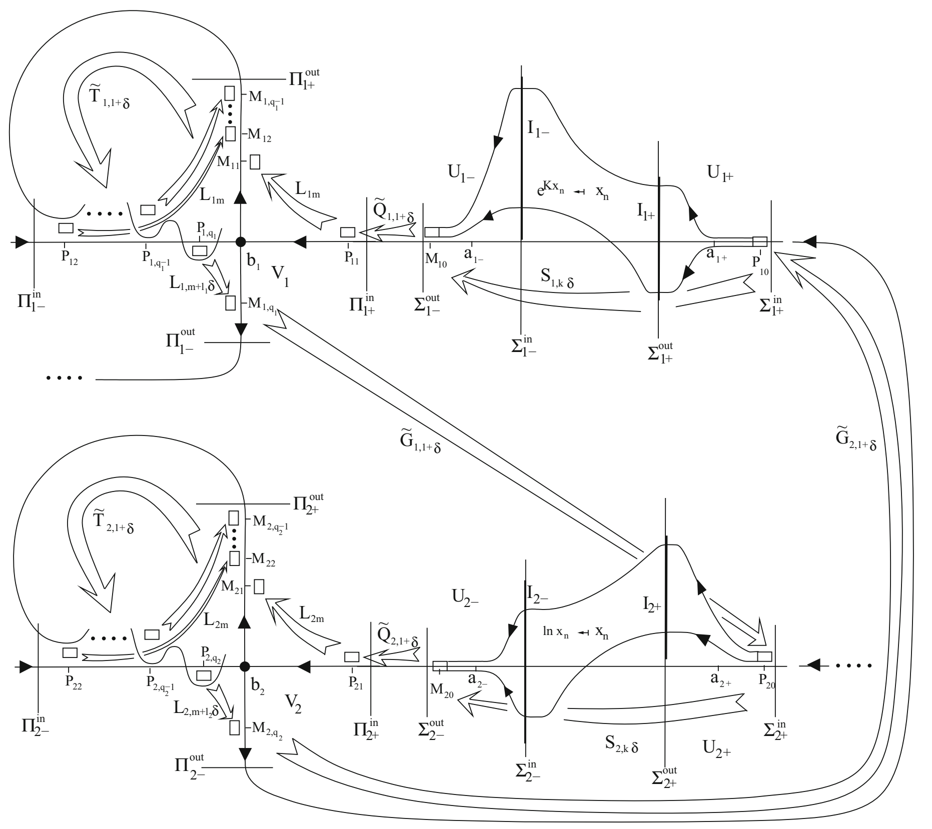


**Fig. 2.** An illustration to the proof of Theorem 1

integer $p$ (the size can shrink to zero as $p$ increases to infinity). More importantly, the increase of $p$ can make the $C^r$-norm of the perturbation as small as we need. We fix $\delta$ small enough, and then choose $p$ large enough to guarantee the sufficient closeness of $g$ to identity. The increase of the parameter $p$ also ensures the closeness (after an appropriate rescaling of coordinates) of the iteration of $g$, which takes points entering $V_j$ to the points entering $U_{3-j}$, to the map $\Phi_j$, and it also allows for making the scaling factor $\eta(p)$ arbitrarily small. We will show at the end of Sect. 4.1.5 that the smallness of the scaling factor guarantees that the images $g^i(\psi(B^n))$ $(i = 0, \ldots, m-2)$ do not intersect $g^{m-1}(\psi(B^n))$, as required by Lemma 1.

After the parameters $\delta$ and $p$ are chosen so that the map $g$ is sufficiently close to identity and the iterations of $g$ which take points entering $V_j$ to the points entering $U_{3-j}$ are sufficiently close to $\Phi_j$ in certain rescaled coordinates, we choose the parameters $\mu$ and $\gamma$ of the flow $Y$ within the lips in such a way that the coordinate scalings we make in the blocks $U_{1,2}$ and $V_{1,2}$ match each other [see (4.1.31), (4.1.32), (4.2.28), (4.2.29), (4.2.30)]. Thus, by construction, the (renormalized) iteration of $g$ that corresponds to passing from the entrance to $U_{1+}$ through $U_{1-}$ into $V_1$, then to many rounds near the homoclinic loop, then to exiting $V_1$ and entering $U_{2+}$, passing through $U_{2-}$ to $V_2$, a number of near-homoclinic rounds and return to $U_{1+}$, is a close approximation to $\Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$, i.e. to the original diffeomorphism $F$ indeed.

In the two-dimensional case, the key fact that every area-preserving diffeomorphism can be approximated by some multi-round iteration of a map with a homoclinic tangency was proven in [28]. To deal with dimensions higher than 2, we construct a very degenerate homoclinic tangency (in terms of [55], both critical and Lyapunov dimensions for this tangency are equal to $n$—this is necessary in order not to lose dimension at the rescaling of the first-return map). We do not undertake an analysis of the corresponding bifurcation; instead, we make explicit computations of the rescaled return maps for our particular example only.

*4.1. Two-dimensional case.* To make computations more transparent we start with the case $n = 2$.

*4.1.1. Flow Y in the lips.* Let $\Phi_{1,2}$ and $\Psi_{1,2}$ be the maps defined by (3.2). Let $I_{1\pm}$ and $I_{2\pm}$ be intervals of values of $x_2$ such that $x_2 \in I_{1+}$ at $(x_1, x_2) \in B^2$, $x_2 \in I_{1-}$ at $(x_1, x_2) \in \Psi_1(B^2)$, $x_2 \in I_{2+}$ at $(x_1, x_2) \in \Phi_1 \circ \Psi_1(B^2)$ and $x_2 \in I_{2-}$ at $(x_1, x_2) \in \Psi_2 \circ \Phi_1 \circ \Psi_1(B^2)$. Let $R$ be such that all the intervals $I_{j\pm}$ lie within $\{|x_2| \le R\}$. Choose numbers $a_{1+} = a_{1-} + 3 = b_1 + 6 = a_{2+} + 9 = a_{2-} + 12 = b_2 + 15$. Let the vector field of $Y$ in the regions $U_{j\sigma} : \{|x_1 - a_{j\sigma}\| \le 1, |x_2| \le R\}$, $j = 1, 2, \sigma = \pm 1$, be equal to

$$\dot{x}_1 = -\mu_j - (1 - \mu_j)(1 - \xi(x_1 - a_{j\sigma})),$$
$$\dot{x}_2 = \sigma\gamma_j\, x_2\, \xi(x_1 - a_{j\sigma}), \tag{4.1.1}$$

where $\mu_{1,2} > 0$ are small [see (4.1.31)], $\gamma_{1,2} \in [0, 1]$ [see (4.1.32)], and $\xi$ is a $C^\infty$ function such that

$$0 \le \xi \le 1, \quad \xi(0) = 1, \quad \xi(z) \equiv 0 \quad \text{at} \quad |z| \ge \frac{1}{2}. \tag{4.1.2}$$

As $\dot{x}_1 < 0$ in $U_{j\sigma}$, every orbit of $Y$ that starts in $U_{j\sigma}$ near $x_1 = a_{j\sigma} + 1$ must come in the vicinity of $x_1 = a_{j\sigma} - 1$ as time grows. For the corresponding time-$t$ map, we have

$$x_1(t) = x_1(0) - t + \frac{1}{2}\beta(\mu_j), \quad x_2(t) = e^{\sigma\gamma_j\alpha(\mu_j)}x_2(0), \tag{4.1.3}$$

where [see (4.1.1)]

$$\alpha(\mu) = \int_{x_{n-1}(t)-a_{j\sigma}}^{x_{n-1}(0)-a_{j\sigma}} \frac{\xi(z)dz}{\mu + (1-\mu)(1-\xi(z))} = \int_{-1/2}^{1/2} \frac{\xi(z)dz}{\mu + (1-\mu)(1-\xi(z))},$$
$$\beta(\mu) = 2\left(\int_{-1/2}^{1/2} \frac{dz}{\mu + (1-\mu)(1-\xi(z))} - 1\right). \tag{4.1.4}$$

Note that $\alpha(\mu)$ is positive, independent of $x_1(0)$ and $t$ [because we assume that the integration limits $(x_1(0) - a_{j\sigma})$ and $(x_1(t) - a_{j\sigma})$ are close to 1 in the absolute value, i.e. they fall in the region where $\xi(z) \equiv 0$; see (4.1.2)], and both $\alpha(\mu)$ and $\beta(\mu)$ tend to infinity as $\mu \to +0$ (the integrals diverge at $\mu = 0$ because $\xi(0) = 1$).

Denote $\Sigma^{in}_{j+} := \{x_1 = a_{j+} + 1, |x_2| \le 1\}$, $\Sigma^{out}_{j+} := \{x_1 = a_{j+} - 1, |x_2| \le R\}$, $\Sigma^{in}_{j-} := \{x_1 = a_{j-} + 1, |x_2| \le R\}$, $\Sigma^{out}_{j-} := \{x_1 = a_{j-} - 1, |x_2| \le 1\}$. Every orbit of $Y$ that intersects $\Sigma^{in}_{j+}$ at $x_2$ sufficiently small leaves $U_{j+}$ by crossing $\Sigma^{out}_{j+}$, and the orbits that intersect $\Sigma^{in}_{j-}$ leave $U_{j-}$ by crossing $\Sigma^{out}_{j-}$ [see (4.1.3)]. Define the vector field $Y$ in the region between $\Sigma^{out}_{j+}$ and $\Sigma^{in}_{j-}$ in such a way that the orbits starting in $\Sigma^{out}_{+}$ reach $\Sigma^{in}_{-}$ at time 1, and the corresponding Poincaré map $\Sigma^{out}_{j+} \to \Sigma^{in}_{j-}$ is

$$x_2 \mapsto \psi_j(x_2),$$

where $\psi_1(x_2) \equiv e^{Kx_2}$ at $x_2 \in I_{1+}$ and $\psi_2(x_2) \equiv \ln x_2$ at $x_2 \in I_{2+}$ [see (3.4)]. Then, the flow takes the points from the vicinity of $x_1 = a_{j+} + 1$ in $U_{j+}$ into the vicinity of $x_1 = a_{j-} - 1$ in $U_{j-}$. By (4.1.3), the corresponding time-$t$ map $S_{jt}$ is

$$x_1(t) = x_1(0) - t + \beta(\mu_j), \quad x_2(t) = e^{-\gamma_j \alpha(\mu_j)} \quad \psi_j\left(e^{\gamma_j \alpha(\mu_j)} x_2(0)\right). \tag{4.1.5}$$

*4.1.2. Flow Y near the homoclinic loops.* In the regions $V_j : \{|x_1 - b_j| \le 1, |x_2| \le 1\}$ $(j = 1, 2)$ we put the vector field of $Y$ to be equal to

$$\dot{x}_1 = -(x_1 - b_j), \quad \dot{x}_2 = x_2. \tag{4.1.6}$$

Thus, in $V_j$, the point $O_j : \{x_1 = b_j, x_2 = 0\}$ is a linear saddle. Its local stable manifold $W_j^s$ is $x_2 = 0$, and the local unstable manifold $W_j^u$ is $x_1 = b_j$. The time-$t$ map $L_{jt}$ within $V_j$ is given by

$$x_1(t) = b_j + e^{-t}(x_1(0) - b_j), \quad x_2(t) = e^t x_2(0). \tag{4.1.7}$$

Let us define $Y$ in the region between $\Sigma_{j-}^{out}$ and $\Pi_{j+}^{in} := \{x_1 = b_j + 1, |x_2| \le 1\}$ in such a way that all the orbits starting in a small neighborhood of $x_2 = 0$ in $\Sigma_{j-}^{out}$ intersect $\Pi_{j+}^{in}$ at time 1, and the corresponding Poincaré map is the identity:

$$x_2 \mapsto x_2.$$

Then the time-$t$ map $Q_{jt}$ from a small neighborhood of $(x_1 = a_{j-} - 1, x_2 = 0)$ in $U_{j-}$ into a small neighborhood of $(x_1 = b_j + 1, x_2 = 0)$ in $V_j$ is given by

$$x_1(t) - b_j = e^{-(t - x_1(0) - 2 + a_{j-})}, \quad x_2(t) = e^{t - x_1(0) - 2 + a_{j-}} x_2(0) \tag{4.1.8}$$

In order to see this, we recall that the vector field in $U_{j-}$ near $x_1 = a_{j-} - 1$ is given by

$$\dot{x}_1 = -1, \quad \dot{x}_2 = 0$$

[see (4.1.1)]. Therefore, the term $x_1(0) + 2 - a_{j-}$ in (4.1.8) is the time the orbit spends in order to get from $x(0)$ to $\Pi_{j+}^{in}$).

Every orbit that enters $V_j$ at $x_2 > 0$ leaves $V_j$ by crossing the cross-section $\Pi_{j+}^{out} := \{x_2 = 1, |x_1 - b_j| \le 1\}$, and every orbit that enters $V_j$ at $x_2 < 0$ leaves it by crossing the cross-section $\Pi_{j-}^{out} := \{x_2 = -1, |x_1 - b_j| \le 1\}$. We assume that the orbits that start at $\Pi_{j+}^{out}$ close to the point $W_j^u \cap \Pi_{j+}^{out} = (b_j, 1)$ return to $V_j$ at time 1 and cross $\Pi_{j-}^{in} := \{x_1 = b_j - 1, |x_2| \le 1\}$; we also assume that the corresponding Poincaré map $x_1 \mapsto \bar{x}_2$ is given by

$$\bar{x}_2 = -(x_1 - b_j)$$

(the minus sign stands to ensure the orientability of the flow map). It follows that the time-$t$ map $T_{jt}$ from a small neighborhood of $W_j^u \cap \Pi_{j+}^{out}$ in $V_j$ into a small neighborhood of $(x_1 = b_j - 1, x_2 = 0)$ in $V_j$ is given by

$$x_1(t) = b_j - e^{-(t-1)}/x_2(0), \quad x_2(t) = -e^{t-1} x_2(0)^2 (x_1(0) - b_j). \tag{4.1.9}$$

For the orbits that start at $\Pi_{j-}^{out}$ close to the point $W_j^u \cap \Pi_{j-}^{out} = (b_j, -1)$ we assume that they cross $\Sigma_{3-j,+}^{in}$ at time 1, and the corresponding Poincaré map is given by $\bar{x}_2 = -(x_1 - b_j)$. Thus [see (4.1.1), (4.1.2), (4.1.6)], the time-$t$ map $G_{jt}$ from a small neighborhood of $W_j^u \cap \Pi_{j-}^{out}$ in $V_j$ into a small neighborhood of $(x_1 = a_{3-j,+}+1, x_2 = 0)$ in $U_{3-j,+}$ is

$$x_1(t) = a_{3-j,+} + 2 - t - \ln|x_2(0)|, \quad x_2(t) = -(x_1(0) - b_j)|x_2(0)|. \tag{4.1.10}$$

Every $C^\infty$ flow $Y$, which satisfies (4.1.5), (4.1.8), (4.1.7), (4.1.9), (4.1.10), is good for our purposes. We may therefore assume that the vector field of $Y$ is identically zero outside some sufficiently large ball $D$. For small $\delta$, the time-$\delta$ map $Y_\delta$ of the flow is $O(\delta)$-close to identity in the $C^r$-norm, for any given $r$. It also equals to identity outside $D$. Let us fix a certain $r$, and take a sufficiently small $\delta$ (for convenience, we assume that $N := \delta^{-1}$ is an integer). Below we construct an arbitrarily small (in the $C^r$-norm), localized in $D$ perturbation of $Y_\delta$ as follows.

*4.1.3. Perturbation of $Y_\delta$ and rescaling near the homoclinic loops.* For the given diffeomorphism $F$, take its sufficiently close approximation in the form of (3.24); in the two-dimensional case the map $\Phi_0$ is given by

$$\Phi_0 := (x_1, x_2) \mapsto (x_2, -x_1). \tag{4.1.11}$$

As there is only a finite number $(q_1 + q_2)$ of the polynomial maps $\tilde{H}_{js}$ in (3.24), one can find some finite $d \geq 1$ (common for all $\tilde{H}_{js}$, $s = 1, \ldots, q_j$, $j = 1, 2$) such that the maps $\tilde{H}_{js}$ are written as follows:

$$\bar{x}_1 = x_2, \quad \bar{x}_2 = -x_1 + \sum_{0 \leq \nu \leq d} h_{js\nu} x_2^\nu. \tag{4.1.12}$$

Inside the segment $I_j^{out} := \{e^{-\delta} \leq x_2 < 1\}$ of $W_j^u$ $(j = 1, 2)$, we choose $q_j - 1$ different points $M_{j1}, \ldots, M_{j,q_j-1}$, and one point $M_{jq_j} \in W_j^u$ will be chosen in the segment $-e^{-\delta} \geq x_2 > -1$. Let $u_{js}$ denote the coordinate $x_2$ of $M_{js}$ $(s = 1, \ldots, q_j)$. As $N\delta = 1$ =flight time from $\Pi_{j+}^{out}$ to $\Pi_{j-}^{in}$, near the segment $I_j^{out}$ the $(N+1)$-th iteration of the time-$\delta$ map $Y_\delta$ is the map $T_{j,1+\delta}$ from (4.1.9), i.e. it is given by

$$\bar{x}_1 = b_j - e^{-\delta}/x_2, \quad \bar{x}_2 = -e^\delta x_2^2(x_1 - b_j). \tag{4.1.13}$$

This map takes the segment $I_j^{out}$ onto the segment $\{b_j - 1 \leq x_1 < b_j - e^{-\delta}, x_2 = 0\} \in W_j^s$. Let $P_{j,s+1} = T_{j,1+\delta} M_{js}$ $(s = 1, \ldots, q_j - 1)$, and let $P_{j1}$ be the point $(x_1, x_2 = (b_j + e^{-\delta/2}, 0)$, so $P_{j1} \in W_j^s \cap \{b_j + e^{-\delta} < x_1 < b_j + 1\}$. We further denote the coordinate $x_1$ of $P_{js}$ as $z_{js}$. By (4.1.13),

$$z_{j,s+1} = b_j - e^{-\delta}/u_{js}. \tag{4.1.14}$$

Recall also that

$$z_{j1} = b_j + e^{-\delta/2} \tag{4.1.15}$$

by the definition of $P_{j1}$. We stress that $u_{js}$ and $(b_j - z_{js})$ are bounded away from zero.

Take a sufficiently large integer $p$ and choose some points $P'_{js} = (z_{js}, z'_{js})$ and $M'_{js} = (u'_{js}, u_{js})$, sufficiently close to $P_{js}$ and $M_{js}$ respectively ($j = 1, 2$; $s = 1, \ldots, q_j$). We define the coordinates of $P'_{js}$ and $M'_{js}$ by the following rule:

$$\begin{aligned} z'_{js} &= e^{-p} u_{js}, \quad u'_{js} = b_j + e^{-p}(z_{js} - b_j) \quad \text{at } s \le q_j - 1 \\ z'_{jq_j} &= e^{-(p+l_j\delta)} u_{jq_j}, \quad u'_{jq_j} = b_j + e^{-(p+l_j\delta)}(z_{jq_j} - b_j), \end{aligned} \tag{4.1.16}$$

where $l_j$ is the maximal integer that is strictly less than $\frac{1}{\delta} \ln C_{jq_j}$, the coefficient $C_{jq_j}$ is defined inductively by formulas (4.1.23), (4.1.20). Note that $C_{jq_j}$ stay positive and uniformly bounded away from zero and infinity, so $l_j\delta$ remains uniformly bounded for arbitrarily small $\delta$.

As $p$ is assumed to be large and $l_j\delta$ is bounded, such defined points $P'_{js}$ and $M'_{js}$ are close to respective points $P_{js}$ and $M_{js}$ indeed. It follows from (4.1.16) that $M'_{js} = L_{jp} P'_{js}$ at $s \le q_j - 1$, where $L_{jt}$ is the map (4.1.7). At $s = q_j$ we have $M'_{jq_j} = L_{j,p+l_j\delta} P'_{jq_j}$.

Let us add to the map $Y_\delta$ a small perturbation, which is localized in a small neighborhood of the points $Y_\delta^{-1}(P_{j,s+1})$ (so outside these small neighborhoods $Y_\delta$ remains unchanged). We require that these localized perturbations are such that in a sufficiently small neighborhood of $M_{js} = Y_\delta^{-N}(Y_\delta^{-1} P_{j,s+1})$ the map $g^{N+1}$ (where $g$ denotes the perturbed map) is given by the following perturbation of (4.1.13):

$$\begin{aligned} \bar{x}_1 &= b_j - e^{-\delta}/x_2, \\ \bar{x}_2 &= z'_{j,s+1} - e^{\delta} x_2^2 (x_1 - u'_{js}) + \sum_{0 \le \nu \le d} \varepsilon_{js\nu} (x_2 - u_{js})^\nu, \end{aligned} \tag{4.1.17}$$

where $\varepsilon_{js\nu}$ are small coefficients to be determined later [see (4.1.21)]. By (4.1.16), $z'_{j,s+1}$ and $(u'_{js} - b_j)$ are small as well, so (4.1.17) is a small perturbation of (4.1.13) indeed. It is easy to see from (4.1.17), (4.1.14) that $P'_{j,s+1} = g^{N+1} M'_{js}$ at $\varepsilon = 0$.

At $\varepsilon = 0$ (hence at all small $\varepsilon$) the map $\tilde{T}_{j,1+\delta} \circ L_{jp} \equiv g^{pN+N+1}$ (where $\tilde{T}$ stands for the perturbed map $T$) takes a small neighborhood of $P'_{js} = (z_{js}, z'_{js})$ into a small neighborhood of $P'_{j,s+1} = (z_{j,s+1}, z'_{j,s+1})$. By (4.1.17), (4.1.7), (4.1.16), this map is written as

$$\begin{aligned} \bar{x}_1 &= b_j - e^{-(p+\delta)}/x_2, \\ \bar{x}_2 &= z'_{j,s+1} - e^{(p+\delta)} x_2^2 (x_1 - z_{js}) + \sum_{0 \le \nu \le d} \varepsilon_{js\nu} e^{\nu p} (x_2 - z'_{js})^\nu, \end{aligned} \tag{4.1.18}$$

We choose some $\eta(p)$ that tends to zero as $p \to +\infty$ and introduce rescaled coordinates $(v_1, v_2)$ near $P'_{js}$ by the rule

$$x_1 = z_{js} + C_{js}\, \eta\, v_1, \quad x_2 = z'_{js} + \frac{1}{C_{js}}\, \eta\, e^{-p} v_2, \tag{4.1.19}$$

where the independent of $p$ positive coefficients $C_{js}$ are determined later [see (4.1.20); note that $C_{js}$ are bounded away from zero and infinity]. Since $\eta$ tends to zero as $p \to +\infty$, any bounded region of values of $v$ corresponds to a small neighborhood of $P'_{js}$.

After the rescaling, map (4.1.18) takes the following form [see (4.1.16), (4.1.14)]:

$$C_{j,s+1}\bar{v}_1 = \frac{1}{e^{\delta}u_{js}^2 C_{js}} v_2 + O(\eta),$$
$$\frac{1}{C_{j,s+1}}\bar{v}_2 = -e^{\delta}u_{js}^2 C_{js} v_1 + \sum_{0\le\nu\le d} \varepsilon_{js\nu} e^{p} \eta^{\nu-1} C_{js}^{-\nu} v_2^{\nu} + O(\eta),$$

where $O(\eta)$ stands for $\eta$ times a bounded function of all variables. As we see, by putting

$$C_{j,s+1} = \frac{1}{e^{\delta}u_{js}^2 C_{js}}, \tag{4.1.20}$$

and

$$\varepsilon_{js\nu} = h_{js\nu} e^{-p} \eta^{1-\nu} C_{j,s+1}^{-1} C_{js}^{\nu}, \tag{4.1.21}$$

the map $\tilde{T}_{j,1+\delta} \circ L_{jp}$ near $P'_{js}$ takes the form

$$\bar{v}_1 = v_2 + O(\eta),$$
$$\bar{v}_2 = -v_1 + \sum_{0\le\nu\le d} h_{js\nu} v_2^{\nu} + O(\eta), \tag{4.1.22}$$

i.e. it can be made as close as we want to the map $\tilde{H}_{js}$, provided $p$ is taken large enough (recall that $\eta \to 0$ as $p \to +\infty$). We take $\eta$ tending to zero sufficiently slowly, so all $\varepsilon_{js\nu} \to 0$ [see (4.1.21); recall that $\nu \le d$ where $d$ is independent of $p$ and $\delta$]. Thus, our perturbation to $Y_\delta$ is arbitrarily small indeed.

It follows that in the rescaled coordinates the map $\left(\tilde{T}_{j,1+\delta} \circ L_{jp}\right)^{q_j} \equiv g^{q_j(pN+N+1)}$ from a small neighborhood of $P'_{j1}$ into a small neighborhood of $P'_{jq_j}$ can be made as close as we want to the map

$$\tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1},$$

provided $p$ is large enough. The rescaled coordinates near $P'_{j1}$ and $P'_{jq_j}$ are given by formulas (4.1.19), where the coefficients $C_{j1} > 0$ may be taken arbitrary, and the coefficients $C_{jq_j}$ are then recovered from the recursive formula (4.1.20). Further it is convenient to put

$$C_{j1} = e^{-\delta/2}. \tag{4.1.23}$$

Note that $p$ does not enter (4.1.20), (4.1.23), hence $C_{jq_j}$ stay bounded away from zero and infinity as $p \to +\infty$.

Now we will also fix the position of the point $M_{jq_j}$ by requiring that

$$\ln|u_{jq_j}| + \ln C_{jq_j} = l_j\delta \tag{4.1.24}$$

(recall that $u_{jq_j}$ is the coordinate $x_2$ of the point $M_{jq_j}$). Since $l_j$, by its definition, is the maximal integer that is strictly less than $\frac{1}{\delta}\ln C_{jq_j}$, it follows that

$$-e^{-\delta} \ge u_{jq_j} > -1. \tag{4.1.25}$$

From (4.1.7), (4.1.19), (4.1.16), (4.1.24) we obtain that the map $L_{j,p+l_j\delta}$ from a small neighborhood of $P'_{jq_j}$ into a small neighborhood of $M'_{jq_j}$ is identity (i.e. $\bar{v}_1 = v_1, \quad \bar{v}_2 = v_2$), provided the rescaled coordinates $(v_1, v_2)$ near $M'_{jq_j}$ are introduced as follows:

$$x_1 = u'_{jq_j} + |u_{jq_j}|^{-1}\,\eta\, e^{-p} v_1,$$
$$x_2 = u_{jq_j} - u_{jq_j}\,\eta\, v_2. \tag{4.1.26}$$

Therefore, for the given choice of the coordinates, by taking $p \to +\infty$, the map $L_{j,p+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jp}\right)^{q_j} \equiv g^{l_j+pN+q_j(mN+N+1)}$ from a small neighborhood of $P'_{j1}$ into a small neighborhood of $M'_{jq_j}$ can be made as close as we want to $\tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1}$ (we have already proved the same for the map $\left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j}$, and the map $L_{j,p+l_j\delta}$ is identity in the chosen coordinates).

*4.1.4. Perturbation and rescaling in the region between the loops and the lips.* Recall that, by construction, the point $g^{N+1}M'_{jq_j}$ lies in $U_{3-j,+}$ in the region $a_{3-j,+} + 1 - \delta < x_1 < a_{3-j,+} + 1$. We add to the map $Y_\delta$ a perturbation, localized near the point $Y_\delta^N M'_{jq_j}$, such that the corresponding map $\tilde{G}_{j,1+\delta} \equiv g^{N+1}$ will have the following form near $M'_{jq_j}$:

$$\bar{x}_1 = a_{3-j,+} + 1 - \delta - \ln|x_2|, \quad \bar{x}_2 = -|x_2|(x_1 - u'_{jq_j}); \tag{4.1.27}$$

since $u'_{jq_j} \to b_j$ as $p \to +\infty$ [see (4.1.16)], these fomulas define a small perturbation of the map $G_{j,1+\delta}$ given by (4.1.10) indeed.

Denote $P'_{3-j,0} = \tilde{G}_{j,1+\delta} M'_{jq_j}$. By (4.1.27), the coordinates of $P'_{3-j,0}$ are given by $\{x_1 = a_{3-j,+} + 1 - \delta - \ln|u_{jq_j}|,\ x_2 = 0\}$. Introduce rescaled coordinates near the points $P'_{3-j,0}$ $(j = 1, 2)$ by the rule

$$x_1 = a_{3-j,+} + 1 - \delta - \ln|u_{jq_j}| + \eta\, v_1, \quad x_2 = \eta\, e^{-p} v_2. \tag{4.1.28}$$

When coordinates are rescaled by rule (4.1.26) near $M'_{jq_j}$ and by rule (4.1.28) near $P'_{3-j,0}$, map (4.1.27) takes the form $(v_1, v_2) \mapsto (v_2, -v_1) + O(\eta)$, i.e. it becomes arbitrarily close to the map $\Phi_0$ [see (4.1.11)] as $p \to +\infty$. Thus, in the coordinates rescaled by rule (4.1.19) near $P'_{j1}$ and by rule (4.1.28) near $P'_{3-j,0}$, the map $\tilde{G}_{j,1+\delta} \circ L_{j,p+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jp}\right)^{q_j} \equiv g^{l_j+(1+q_j)(pN+N+1)}$ from a small neighborhood of $P'_{j1}$ into a small neighborhood of $P'_{3-j,0}$ , can be made as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1}$ as $p$ grows.

Analogously, we take the point $M'_{j0} : \{x_1 = a_{j-} - 1 + \delta/2,\ x_2 = 0\} \in U_{j-}$ and perturb the map $Y_\delta$ near $Y_\delta^N M'_{j0}$ in such a way that the map $\tilde{Q}_{j,1+\delta} \equiv g^{N+1}$ near $M'_{j0}$ will be given by

$$\bar{x}_1 - b_j = e^{-(\delta - x_1 - 1 + a_{j-})}, \quad \bar{x}_2 = e^{\delta - x_1 - 1 + a_{j-}} x_2 + e^{-p} u_{j1}. \tag{4.1.29}$$

It is a small perturbation of the map $Q_{j,1+\delta}$ given by (4.1.8), and it takes $M'_{j0}$ to $P'_{j1}$ [see (4.1.15), (4.1.16)]. When we introduce rescaled variables near $M'_{j0}$ by the rule

$$x_1 = a_{j-} - 1 + \delta/2 + \eta\, v_1, \quad x_2 = \eta\, e^{-p} v_2, \tag{4.1.30}$$

and near $P'_{j1}$ by rule (4.1.19), (4.1.23), map (4.1.29) will take the form $\bar{v} = v + O(\eta)$, i.e. it is close to the identity map. Thus, in the rescaled coordinates given by (4.1.30), (4.1.28), the map $\tilde{G}_{j,1+\delta} \circ L_{j,p+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jp}\right)^{q_j} \circ \tilde{Q}_{j,1+\delta} \equiv g^{l_j+(1+q_j)(pN+N+1)+N+1}$ from a small neighborhood of $M'_{j0}$ into a small neighborhood of $P'_{3-j,0}$, is as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1}$ at $p$ large enough, i.e. it is a close approximation of the map $\Phi_j$.

*4.1.5. Parameter fitting in the lips.* Let us now determine the form of the map $S_{jt}$ : $v \mapsto \bar{v}$ from a small neighborhood of $P'_{j0}$ into a small neighborhood of $M'_{j0}$ in the coordinates rescaled by the same rules (4.1.28) and (4.1.30) [note that when rescaling near $P'_{j0}$ one should change $j$ to $3-j$ in the right-hand side of (4.1.28)]. We will choose the coefficients $\mu_j$ in (4.1.1) such that

$$\beta(\mu_j) = (k + 3/2)\delta - 5 + \ln |u_{3-j,q_{3-j}}| \tag{4.1.31}$$

where $k$ is some sufficiently large integer. Since $\beta \to +\infty$ as $\mu \to +0$ [see (4.1.4)], and $\ln |u_{3-j,q_{3-j}}|$ is bounded [see (4.1.25)], Eq. (4.1.31) has a solution $\mu_j(k)$ for every sufficiently large $k$, and $\mu_j(k) \to +0$ as $k \to +\infty$. It follows that $\alpha(\mu_j(k)) \to +\infty$. Thus, for any sufficiently large $p$ we can find $\gamma_j \in (0, 1]$ and large $k$ such that

$$\eta(p)\, e^{-p} = e^{-\gamma_j \alpha(\mu_j(k))}. \tag{4.1.32}$$

From (4.1.5) we immediately obtain that with this choice of $\mu_j$, $\gamma_j$ and $k$ the map $S_{j,k\delta} \equiv Y_\delta^k$ from a small neighborhood the point $P'_{j0}$ into a small neighborhood of $M'_{j0}$ takes the following form in the coordinates rescaled by rules, respectively, (4.1.28) (with $(3 - j)$ changed to $j$) and (4.1.30):

$$\bar{v}_1 = v_1, \quad \bar{v}_2 = \psi_j(v_2).$$

As wee see, the map $S_{j,k\delta}$ in the rescaled coordinates coincides with the map $\Psi_j$ for $v$ from some open neighborhood of the unit disc $B^2$ (if $j = 1$) or of $\Phi_1 \circ \Psi_1(B^2)$ (if $j = 2$).

Summarizing, we obtain that the map

$$\begin{aligned} &\tilde{G}_{2,1+\delta} \circ L_{2,p+l_2\delta} \circ \left(\tilde{T}_{2,1+\delta} \circ L_{2p}\right)^{q_2} \circ \tilde{Q}_{2,1+\delta} \circ S_{2,k\delta} \circ \\ &\quad \circ \tilde{G}_{1,1+\delta} \circ L_{1,p+l_1\delta} \circ \left(\tilde{T}_{1,1+\delta} \circ L_{1p}\right)^{q_1} \circ \tilde{Q}_{1,1+\delta} \circ S_{1,k\delta} \\ &\equiv g^m, \end{aligned}$$

where $m = 2k + l_1 + l_2 + (2 + q_1 + q_2)(pN + N + 1) + 2(N + 1)$ is, in certain rescaled coordinates, a close approximation to the map $\Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$ (i.e. to the original map $F$), provided $\tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1}$ are sufficiently close approximations to $\tilde{\Phi}_j$ $(j = 1, 2)$ and $p$ is large enough.

This is in full agreement with the statement of Lemma 1. In order to complete the proof, let us verify that the sets $g^i(\psi(B^2))$ do not intersect $g^{m-1}(\psi(B^2))$ at $i = 0, \ldots, m - 2$ (where $\psi$ is the scaling transformation). Indeed, by construction $\psi(B^2)$ lies in a neighbourhood of the point $P_{10}$, at the "entrance" to the first lip (see Fig. 2), so the set $g^{m-1}(\psi(B^2))$ lies near the point $g^{-1}P_{10}$, just outside the lip. It follows from the rescaling formulas (4.1.19), (4.1.26), (4.1.28), (4.1.30) that those of the images

$g^i(\psi(B^2))(i = 0, \ldots, m-1)$ which do not lie inside the lips are each contained in an $O(\eta)$-size disc with the center at $g^i(Z)$, where $Z$ is an arbitrarily chosen point from $\psi(B^2)$. Since the set $g^{m-1}(\psi(B^2))$ lies outside the lips, it follows that $g^{m-1}(\psi(B^2))$ can intersect any of $g^i(\psi(B^2))(i = 0, \ldots, m-2)$ only if the latter lies at a distance of order no more than $\eta$ from $g^{m-1}(\psi(B^2))$, but we can choose $\eta$ as small as we want for a fixed $\delta$, so this contradicts to the fact that the map $g$ near the point $g^{-1}P_{10}$ shifts points to a distance at least of order $\delta$, so the first $(m-1)$ iterations, forward or backward, of any point that starts near $g^{-1}P_{10}$ stay at a distance at least of order $\delta \gg \eta$ from the initial point. This finishes the proof of Theorem 1 in the two-dimensional case.

*4.2. Higher-dimensional case.* In the case of arbitrary $n > 2$, the construction follows the same line as in the two-dimensional case.

*4.2.1. Flow in the lips.* As above, $\Phi_{1,2}$ and $\Psi_{1,2}$ are the maps defined by (3.2), and $I_{1\pm}$ and $I_{2\pm}$ are intervals of values of $x_n$ such that $x_n \in I_{1+}$ at $x \in B^n$, $x_n \in I_{1-}$ at $x \in \Psi_1(B^n)$, $x_n \in I_{2+}$ at $x \in \Phi_1 \circ \Psi_1(B^n)$ and $x_n \in I_{2-}$ at $x \in \Psi_2 \circ \Phi_1 \circ \Psi_1(B^n)$. A value $R$ is chosen such that all the intervals $I_{j\pm}$ lie within $\{|x_n| \le R\}$. Choose numbers $a_{1+} = a_{1-} + 3 = b_1 + 6 = a_{2+} + 9 = a_{2-} + 12 = b_2 + 15$. Define the regions $U_{j\sigma} : \{|x_{n-1} - a_{j\sigma}\| \le 1, |x_n| \le R, |x_i| \le 1\ (i \le n-2)\}$, and $V_j : \{|x_{n-1} - b_j| \le 1, |x_i| \le 1\ \ (i \ne n-1)\}$, $j = 1, 2, \sigma = \pm 1$. Let the vector field of a $C^\infty$ flow $Y$ in $U_{j\sigma}$ be equal to

$$\begin{aligned} \dot{x}_{n-1} &= -\mu_j - (1-\mu_j)(1 - \xi(x_{n-1} - a_{j\sigma})), \\ \dot{x}_i &= \sigma \gamma_{i\sigma}\, x_i\, \xi(x_{n-1} - a_{j\sigma}) \quad (i \ne n-1), \end{aligned} \tag{4.2.1}$$

where $\mu_{1,2} > 0$ are small [see (4.2.28), (4.2.29)], $\gamma_{i\pm} \in [0, 1]$ [see (4.2.29), (4.2.30)], and the $C^\infty$ function $\xi$ satisfies (4.1.2). In the regions $V_j$ we make $Y$ equal to

$$\dot{x}_i = -\lambda_i x_i \quad (i = 1, \ldots, n-2), \quad \dot{x}_{n-1} = -\lambda_{n-1}(x_{n-1} - b_j), \quad \dot{x}_n = x_n; \tag{4.2.2}$$

here $\lambda_i > 0$ are such that

$$\lambda_2 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_1 = 1 - (n-2)\lambda, \tag{4.2.3}$$

where the positive number $\lambda$ is specified below [see (4.2.11)].

As $\dot{x}_{n-1} < 0$ in $U_{j\sigma}$, every orbit of $Y$ that starts in $U_{j\sigma}$ near $x_{n-1} = a_{j\sigma} + 1$ must come in the vicinity of $x_{n-1} = a_{j\sigma} - 1$ as time grows. For the corresponding time-$t$ map, we have

$$x_i(t) = e^{\sigma \gamma_{i\sigma} \alpha(\mu_j)} x_i(0) \quad (i \le n-2), \quad x_n(t) = x_n(0) - t + \frac{1}{2}\beta(\mu_j), \tag{4.2.4}$$

where the tending to infinity, as $\mu \to +0$, functions $\alpha(\mu)$ and $\beta(\mu)$ are defined by (4.1.4).

Denote $\Sigma^{in}_{j+} := \{x_{n-1} = a_{j+} + 1, |x_n| \le 1\}$, $\Sigma^{out}_{j+} := \{x_{n-1} = a_{j+} - 1, |x_n| \le R\}$, $\Sigma^{in}_{j-} := \{x_{n-1} = a_{j-} + 1, |x_n| \le R\}$, $\Sigma^{out}_{j-} := \{x_{n-1} = a_{j-} - 1, |x_n| \le 1\}$ (we also assume that $|x_i| \le 1$ for $i \le n-2$ on $\Sigma^{in,out}_{j\pm}$). Every orbit of $Y$ that intersects $\Sigma^{in}_{j+}$ at $x_n, x_1, \ldots, x_{n-2}$ sufficiently small leaves $U_{j+}$ by crossing $\Sigma^{out}_{j+}$, and the orbits that intersect $\Sigma^{in}_{j-}$ leave $U_{j-}$ by crossing $\Sigma^{out}_{j-}$ [see (4.2.4)]. We define $Y$ in the region between $\Sigma^{out}_{j+}$ and $\Sigma^{in}_{j-}$ in such a way that the orbits starting in $\Sigma^{out}_{+}$ reach $\Sigma^{in}_{-}$ at time 1, and the corresponding Poincaré map $\Sigma^{out}_{j+} \to \Sigma^{in}_{j-}$ is $(x_1, \dots, x_{n-2}, x_n) \mapsto (x_1, \dots, x_{n-2}, \psi_j(x_n))$, where the functions $\psi_j$ are defined by (3.4). Then, the flow takes the points from the vicinity of $x_{n-1} = a_{j+} + 1$ in $U_{j+}$ into the vicinity of $x_{n-1} = a_{j-} - 1$ in $U_{j-}$. By (4.2.4), the corresponding time-$t$ map $S_{jt}$ is

$$\begin{aligned} x_i(t) &= e^{(\gamma_{i+} - \gamma_{i-})\alpha(\mu_j)} x_i(0) \quad (i \le n-2), \\ x_{n-1}(t) &= x_{n-1}(0) - t + \beta(\mu_j), \quad x_n(t) = e^{-\gamma_{n-}\alpha(\mu_j)} \psi_j(e^{\gamma_{n+}\alpha(\mu_j)} x_n(0)). \end{aligned} \tag{4.2.5}$$

*4.2.2. Flow near the homoclinic loops.* In the region between $\Sigma^{out}_{j-}$ and $\Pi^{in}_{j+} := \{x_{n-1} = b_j + 1, |x_i| \le 1\}$, we define $Y$ in such a way that all the orbits starting in a small neighborhood of $x_n = x_1 = \dots = x_{n-2} = 0$ in $\Sigma^{out}_{j-}$ intersect $\Pi^{in}_{j+}$ at time 1, and the corresponding Poincaré map is the identity: $(x_1, \dots, x_{n-2}, x_n) \mapsto (x_1, \dots, x_{n-2}, x_n)$. Then the time-$t$ map $Q_{jt}$ from a small neighborhood of $x_{n-1} = a_{j-} - 1, x_n = x_1 = \dots = x_{n-2} = 0$ in $U_{j-}$ into a small neighborhood of $x_{n-1} = b_j + 1$ in $V_j$ is given by

$$\begin{aligned} x_i(t) &= e^{-\lambda_i(t - x_{n-1}(0) - 2 + a_{j-})} x_i(0) \quad (i \le n-2), \\ x_{n-1}(t) - b_j &= e^{-\lambda(t - x_{n-1}(0) - 2 + a_{j-})}, \quad x_n(t) = e^{t - x_{n-1}(0) - 2 + a_{j-}} x_n(0) \end{aligned} \tag{4.2.6}$$

[see (4.2.2)].

In $V_j (j = 1, 2)$, the local stable manifold $W^s_j$ of the linear saddle equilibrium state $O_j : \{x_{n-1} = b_j, x_i = 0 \ (i \ne n-1)\}$ is $x_n = 0$, and the local unstable manifold $W^u_j$ is $x_{n-1} = b_j, x_1 = \dots = x_{n-2} = 0$. The time-$t$ map $L_{jt}$ within $V_j$ is given by

$$\begin{aligned} x_i(t) &= e^{-\lambda_i t} x_i(0) \quad (i \le n-2), \\ x_{n-1}(t) - b_j &= e^{-\lambda t}(x_{n-1}(0) - b_j), \quad x_n(t) = e^t x_n(0). \end{aligned} \tag{4.2.7}$$

Every orbit that enters $V_j$ at $x_n > 0$ leaves $V_j$ by crossing the cross-section $\Pi^{out}_{j+} := \{x_n = 1, |x_{n-1} - b_j| \le 1, |x_i| \le 1 \ (i \le n-2)\}$, and every orbit that enters $V_j$ at $x_n < 0$ leaves it by crossing the cross-section $\Pi^{out}_{j-} := \{x_n = -1, |x_{n-1} - b_j| \le 1, |x_i| \le 1 \ (i \le n-2)\}$. We assume that the orbits that start at $\Pi^{out}_{j+}$ close to the point $W^u_j \cap \Pi^{out}_{j+} = (x_1 = \dots = x_{n-2} = 0, x_{n-1} = b_j)$ return to $W_j$ at time 1 and cross $\Pi^{in}_{j-} := \{x_{n-1} = b_j - 1, |x_i| \le 1 (i \ne n-1)\}$; we also assume that the corresponding Poincaré map $(x_1, \dots, x_{n-1}) \mapsto (\bar{x}_1, \dots, \bar{x}_{n-2}, \bar{x}_n)$ is given by

$$\bar{x}_i = x_{i+1} \quad (i \le n-3), \quad \bar{x}_{n-2} = x_{n-1} - b_j, \quad \bar{x}_n = (-1)^{n+1} x_1$$

[the factor $(-1)^{n+1}$ stands to ensure the orientability]. It follows that the time-$t$ map $T_{jt}$ from a small neighborhood of $W^u_j \cap \Pi^{out}_{j+}$ in $V_j$ into a small neighborhood of $x_{n-1} = b_j - 1$ in $V_j$ is given by

$$\begin{aligned} x_i(t) &= e^{-\lambda_i(t-1)} x_n(0)^{\lambda_{i+1} - \lambda_i} x_{i+1}(0) \quad (i \le n-3), \\ x_{n-2}(t) &= e^{-\lambda_{n-2}(t-1)} x_n(0)^{\lambda_{n-1} - \lambda_{n-2}} (x_{n-1}(0) - b_j), \\ x_{n-1}(t) &= b_j - e^{-\lambda_{n-1}(t-1)} x_n(0)^{-\lambda_{n-1}}, \\ x_n(t) &= (-1)^{n+1} e^{t-1} x_n(0)^{1+\lambda_1} x_1(0). \end{aligned} \tag{4.2.8}$$

For the orbits that leave $V_j$ by crossing $\Pi^{out}_{j-}$, we assume that the orbits that start at $\Pi^{out}_{j-}$ close to the point $W^u_j \cap \Pi^{out}_{j-} = (x_1 = \dots = x_{n-2} = 0, x_{n-1} = b_j)$ cross $\Sigma^{in}_{3-j,+}$

at time 1, and the corresponding Poincaré map $(x_1,\dots,x_{n-1})\mapsto(\bar x_1,\dots,\bar x_{n-2},\bar x_n)$ is given by $\bar x_i=x_i$ at $i=1,\dots,n-2$ and $\bar x_n=-(x_{n-1}-b_j)$. Thus [see (4.2.1), (4.1.2), (4.2.2)], the time-$t$ map $G_{jt}$ from a small neighborhood of $W^u_j\cap\Pi^{out}_{j-}$ in $V_j$ into a small neighborhood of $x_{n-1}=a_{3-j,+}+1$ in $U_{3-j,+}$ is

$$\begin{aligned} x_i(t)&=|x_n(0)|^{\lambda_i}x_i(0)\quad(i\le n-2),\\ x_{n-1}(t)&=a_{3-j,+}+2-t-\ln|x_n(0)|,\\ x_n(t)&=-|x_n(0)|^{\lambda_{n-1}}(x_{n-1}(0)-b_j).\end{aligned}\tag{4.2.9}$$

These conditions define the flow $Y$ (we also assume that the vector field of $Y$ is identically zero outside some sufficiently large ball $D$). Let us take a sufficiently small $\delta$ such that $N:=\delta^{-1}$ is an integer, and proceed to the construction of a small (arbitrarily small in the $C^r$-norm, with any chosen in advance $r$) perturbation of $Y_\delta$, localized in $D$.

*4.2.3. Perturbation of $Y_\delta$ and rescaling near the homoclinic loops.* Take a sufficiently close approximation of the given diffeomorphism $F$ by the product (3.24). There exists some finite $d\ge 1$, common for all $\tilde H_{js}$ $(s=1,\dots,q_j,\ j=1,2)$ such that the polynomial maps $\tilde H_{js}$ in (3.24) are written as follows:

$$\bar x_i=x_{i+1}\quad(i\le n-1),\quad \bar x_n=(-1)^{n+1}x_1+\sum_{\substack{\nu_2\ge0,\dots,\nu_n\ge0\\ \nu_2+\dots+\nu_n\le d}}h_{js\nu}\prod_{2\le i\le n}x_i^{\nu_i}.\tag{4.2.10}$$

We will now fix the choice of the positive $\lambda$ in (4.2.3) such that

$$\lambda<\frac{1}{(n-1)d+r}.\tag{4.2.11}$$

In the segment $I^{out}_j:=\{e^{-\delta}\le x_n<1\}$ of $W^u_j$ $(j=1,2)$, we choose $q_j-1$ different points $M_{j1},\dots,M_{j,q_j-1}$, and one point $M_{jq_j}\in W^u_j$ will be chosen in the segment $-e^{-\delta}\ge x_n>-1$. Let $u_{js}$ denote the coordinate $x_n$ of $M_{js}$ $(s=1,\dots,q_j)$. As $N\delta=1$, near the segment $I^{out}_j$ the $(N+1)$th iteration of the time-$\delta$ map $Y_\delta$ is the map $T_{j,1+\delta}$ from (4.2.8). Thus, the map $Y_\delta^{N+1}$ near $I^{out}_j$ will be given by

$$\begin{aligned}&\bar x_1=e^{((n-2)\lambda-1)\delta}x_n^{(n-1)\lambda-1}\hat x_2,\quad \bar x_i=e^{-\lambda\delta}\hat x_{i+1}\ \ (2\le i\le n-2),\\ &\bar x_{n-1}=b_j-e^{-\lambda\delta}x_n^{-\lambda},\quad \bar x_n=(-1)^{n+1}e^{\delta}x_n^{2-(n-2)\lambda}x_1\end{aligned}\tag{4.2.12}$$

(where we denote $\hat x_i=x_i$ at $i\ne n-1$ and $\hat x_{n-1}=x_{n-1}-b_j$). This map takes the segment $I^{out}_j$ onto the segment $\{b_j-1<x_{n-1}<b_j-e^{-\lambda\delta},x_1=\dots=x_{n-2}=x_n=0\}\in W^s_j$. Let $P_{j,s+1}=T_{j,1+\delta}M_{js}(s=1,\dots,q_j-1)$, and let $P_{j1}$ be a point from $\{b_j+e^{-\lambda\delta}<x_{n-1}<b_j+1,x_1=\dots=x_{n-2}=x_n=0\}\in W^s_j$. By (4.2.12), the coordinate $x_{n-1}$ of $P_{j,s+1}$ equals to $b_j-e^{-\lambda\delta}u_{js}^{-\lambda}$.

We take sufficiently large integer $p$ and choose some points $P'_{js}$ and $M'_{js}$, sufficiently close to $P_{js}$ and $M_{js}$ respectively $(j=1,2;s=1,\dots,q_j)$, such that at $s\le q_j-1$ we have $M'_{js}=L_{jp}P'_{js}$ [where $L_{jt}$ is the map (4.2.7)]. At $s=q_j$ we assume $M'_{jq_j}=L_{j,p+l_j\delta}P'_{jq_j}$ where $l_j$ is an integer to be defined later [see (4.2.31); note that $l_j\delta$ is uniformly bounded]. Denote the coordinates of $P'_{js}$ and $M'_{js}$ as $(z'_{js1}, \ldots, z'_{js,n-2}, b_j + z'_{js,n-1}, z'_{jsn})$ and $(u'_{js1}, \ldots, u'_{js,n-2}, b_j + u'_{js,n-1}, u'_{jsn})$ respectively. By (4.2.7),

$$u'_{jsi} = e^{-\lambda_i p} z'_{jsi} \quad (i = 1, \ldots, n-1), \quad u'_{jsn} = e^{p} z'_{jsn} \tag{4.2.13}$$

at $s \le q_j - 1$. At $s = q_j$ we have

$$u'_{jq_ji} = e^{-\lambda_i (p+l_j\delta)} z'_{jq_ji} \quad (i = 1, \ldots, n-1), \quad u'_{jq_jn} = e^{p+l_j\delta} z'_{jq_jn}. \tag{4.2.14}$$

Note that $u'_{jsi}$ are small at $i \le n-1$, as $p$ is assumed to be large, and $l_j\delta$ is bounded. The values of $z'_{jsi}$ with $i \ne n-1$ will be taken sufficiently small as well, and we will keep

$$u'_{jsn} = u_{js} \quad \text{and} \quad z'_{j,s+1,n-1} = -e^{-\lambda\delta} u_{js}^{-\lambda}, \tag{4.2.15}$$

in order to ensure the closeness of $P'_{js}$ to $P_{js}$ and $M'_{js}$ to $M_{js}$.

The first of the small perturbations which we add to the map $Y_\delta$ is localized in a small neighborhood of the points $Y_\delta^{-1}(P_{j,s+1})$ (so outside these small neighborhoods $Y_\delta$ remains unchanged). We take these localized perturbations such that in a sufficiently small neighborhood of $M_{js} = Y_\delta^{-N}(Y_\delta^{-1} P_{j,s+1})$ the map $g^{N+1}$ (where $g$ denotes the perturbed map) is given by (4.2.12) with the following correction term

$$\begin{aligned} &z'_{j,s+1,n} - (-1)^{n+1} e^{\delta} x_n^{2-(n-2)\lambda} \left| \frac{x_{n-1} - b_j}{z'_{js,n-1}} \right|^{\frac{1}{\lambda}-(n-1)} e^{-\lambda p} z'_{js1} \\ &+ \sum_{\substack{\nu_2 \ge 0, \ldots, \nu_n \ge 0 \\ \nu_2 + \ldots + \nu_n \le d}} \varepsilon_{js\nu} \prod_{2 \le pi \le n} (\hat{x}_i - u'_{jsi})^{\nu_i} \end{aligned} \tag{4.2.16}$$

added into the equation for $\bar{x}_n$, where $\varepsilon_{js\nu}$ are small coefficients to be determined later [see (4.2.21)]. The first term in (4.2.16) is small as well [see (4.2.13), (4.2.14)]; in the second term the values of $x_n$ and $z'_{j,s,n-1}$ are bounded away from zero, and the exponent $\left(\frac{1}{\lambda} - (n-1)\right)$ is larger than $r$ [see (4.2.11)], hence the second term is also small with the derivatives up to the order $r$ at least. The first two terms in (4.2.16) ensure, in particular, that at $\varepsilon = 0$ the coordinate $x_n$ of $g^{N+1} M'_{js}$ coincides with that of $P'_{j,s+1}$ [see (4.2.13), (4.2.15), (4.2.3)]. We want $P'_{j,s+1} = g^{N+1} M'_{js}$ at $\varepsilon = 0$, so we put

$$\begin{aligned} z'_{j,s+1,1} &= e^{-(1-(n-2)\lambda)\delta - \lambda p} u_{js}^{(n-1)\lambda - 1} z'_{js2}, \\ z'_{j,s+1,i} &= e^{-\lambda(p+\delta)} z'_{js,i+1} \quad (2 \le i \le n-2), \end{aligned} \tag{4.2.17}$$

[see (4.2.12), (4.2.13), (4.2.15)]. At $s = 1$ we assume

$$z'_{j1i} = 0 \quad \text{at } i \le n-2, \quad z'_{j1,n-1} = e^{-\lambda\delta/2}. \tag{4.2.18}$$

Now, the values of $z'_{jsi}, u'_{jsi}$ are defined by (4.2.13), (4.2.14), (4.2.15), (4.2.17) for all $j, s, i$. As one can see, $z'_{jsi}$ at $i \ne n-1$ and $u'_{jsi}$ at $i \ne n$ tend to zero as $p \to +\infty$, i.e. $P'_{js} \to P_{js}$ and $M'_{js} \to M_{js}$ indeed.

At all small $\varepsilon$ the map $\tilde{T}_{j,1+\delta} \circ L_{jp} \equiv \tilde{Y}_\delta^{pN+N+1}$ takes a small neighborhood of $P'_{js}$ into a small neighborhood of $P'_{j,s+1}$. We choose some $\eta(p)$ that tends to zero as $p \to +\infty$

and some, independent of $p$, coefficients $C_{jsi} > 0$, and introduce rescaled coordinates $v_1, \dots, v_n$ near $P'_{js}$ by the rule

$$x_1|b_j - x_{n-1}|^{n-1-1/\lambda} = z'_{js1}|z'_{js,n-1}|^{n-1-1/\lambda} + C_{js1}\,\eta\, e^{-\lambda p(n-2)} v_1$$
$$\hat{x}_i = z'_{jsi} + C_{jsi}\,\eta\, e^{-\lambda p(n-i-1)} v_i \quad (2 \le i \le n-1), \quad x_n = z'_{jsn} + C_{jsn}\,\eta\, e^{-p} v_n \tag{4.2.19}$$

[recall that $|b_j - x_{n-1}|$ is close to 1 near $P_{js}$, hence (4.2.19) is a smooth coordinate transformation]. Since $\eta$ tends to zero as $p \to +\infty$, any bounded region of values of $v$ corresponds to a small neighborhood of $P'_{js}$.

After the rescaling, the map $\tilde{T}_{j,1+\delta} \circ L_{jp} \equiv g^{pN+N+1}$ from a small neighborhood of $P'_{js}$ into a small neighborhood of $P'_{j,s+1}$ takes the following form [see (4.2.12), (4.2.7), (4.2.3), (4.2.13), (4.2.15), (4.2.17), (4.2.19)]:

$$C_{j,s+1,i}\bar{v}_i = e^{-\lambda\delta} C_{j,s,i+1} v_{i+1} \quad (i \le n-2),$$
$$C_{j,s+1,n-1}\bar{v}_{n-1} = e^{-\lambda\delta}\,(u_{js}^{-\lambda} - (u_{js} + C_{jsn}\,\eta\, v_n)^{-\lambda})/\eta,$$
$$C_{j,s+1,n}\bar{v}_n = (-1)^{n+1}\phi_{js} C_{js1} v_1 + \sum_{\substack{\nu_2 \ge 0,\dots,\nu_n \ge 0 \\ \nu_2+\dots+\nu_n \le d}} \varepsilon_{js\nu} E_{js\nu} \prod_{2 \le i \le n} v_i^{\nu_i}$$

where we denote

$$\phi_{js} = e^{\delta}(u_{js} + \eta C_{jsn} v_n)^{2-(n-2)\lambda} |z'_{js,n-1} + \eta C_{j,s,n-1} v_{n-1}|^{\frac{1}{\lambda}-(n-1)},$$
$$E_{js\nu} = e^{p(1-\lambda\sum_{2\le i\le n-1}(n-i)\nu_i)}\, \eta^{(-1+\sum_{2\le i\le n}\nu_i)} \prod_{2\le i\le n} C_{jsi}^{\nu_i}.$$

Note that $\sum_{2\le i\le n-1}(n-i)\nu_i \le (n-2)d$, hence $1 - \lambda\sum_{2\le i\le n-1}(n-i)\nu_i > 0$ [see (4.2.11)]. Therefore, all the coefficients $E_{js\nu}$ tend to infinity as $p \to +\infty$ (provided $\eta(p)$ tends to zero sufficiently slowly).

As we see, by putting

$$C_{j,s+1,i} = e^{-\lambda\delta} C_{j,s,i+1} \quad (i \le n-2), \quad C_{j,s+1,n-1} = \lambda e^{-\lambda\delta} u_{js}^{-\lambda-1} C_{jsn},$$
$$C_{j,s+1,n} = e^{\delta} u_{js}^{2-(n-2)\lambda} |z'_{js,n-1}|^{\frac{1}{\lambda}-(n-1)} C_{js1}, \tag{4.2.20}$$

and

$$\varepsilon_{js\nu} = h_{jsi}\frac{C_{j,s+1,n}}{E_{js\nu}}, \tag{4.2.21}$$

the map $\tilde{T}_{j,1+\delta} \circ L_{jp}$ near $P'_{js}$ takes the form

$$\bar{v}_i = v_{i+1} \quad (i \le n-2), \quad \bar{v}_{n-1} = v_n + O(\eta),$$
$$\bar{v}_n = (-1)^{n+1} v_1 + O(\eta) + \sum_{\substack{\nu_2 \ge 0,\dots,\nu_n \ge 0 \\ \nu_2+\dots+\nu_n \le d}} h_{js\nu} \prod_{2\le i\le n} v_i^{\nu_i}, \tag{4.2.22}$$

i.e. it can be made as close as we want to the map $\tilde{H}_{js}$, provided $p$ is taken large enough (recall that $\eta \to 0$ as $p \to +\infty$). We take $\eta(p)$ tending to zero sufficiently slowly, so, as we mentioned, $E_{ij\nu} \to \infty$ as $p \to +\infty$, which implies that all $\varepsilon_{js\nu} \to 0$ (see (4.2.21)), i.e. our perturbation to $Y_\delta$ is arbitrarily small indeed.

It follows that in the rescaled coordinates the map $\left(\tilde{T}_{j,1+\delta} \circ L_{jp}\right)^{q_j} \equiv g^{q_j(pN+N+1)}$ from a small neighborhood of $P'_{j1}$ into a small neighborhood of $P'_{jq_j}$ can be made as close as we want to the map $\tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1}$, provided $p$ is large enough (the rescaled coordinates near $P'_{j1}$ and $P'_{jq_j}$ are given by formulas (4.2.19), where the coefficients $C_{j1i} > 0$ are taken arbitrary, and the coefficients $C_{jq_ji}$ are recovered from the recursive formula (4.2.20); since $p$ does not enter (4.2.20), it follows that $C_{jq_ji}$ stay bounded away from zero and infinity as $p \to +\infty$).

Now, from (4.2.7) we obtain that the same holds true for the map $L_{j,p+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jp}\right)^{q_j} \equiv g^{l_j+pN+q_j(pN+N+1)}$ from a small neighborhood of $P'_{j1}$ into a small neighborhood of $M'_{jq_j}$, where the rescaled coordinates $(v_1, \dots, v_n)$ are introduced as follows:

$$\begin{gathered} x_1|x_{n-1} - b_j|^{n-1-1/\lambda} = u'_{jq_j1}|u'_{jq_j,n-1}|^{n-1-1/\lambda} + C_{jq_j1}\ \eta\ e^{-\lambda(n-1)p-\lambda l_j\delta} v_1, \\ \hat{x}_i = u'_{jq_ji} + C_{jq_ji}\ \eta\ e^{-\lambda p(n-i)-\lambda l_j\delta} v_i \quad (2 \le i \le n-1), \quad x_n = u_{jq_j} + C_{jq_jn}\ \eta e^{l_j\delta} v_n, \end{gathered} \tag{4.2.23}$$

with the same constants $C_{jq_ji}$ as above.

*4.2.4. Perturbation and rescaling between loops and lips.* Recall that, by construction, the point $Y_\delta^{N+1} M'_{jq_j}$ lies in $U_{3-j,+}$ in the region $a_{3-j,+} + 1 - \delta < x_{n-1} < a_{3-j,+} + 1$. We add to the map $Y_\delta$ an additional perturbation, localized near the point $Y_\delta^N M'_{jq_j}$, such that the corresponding map $\tilde{G}_{j,1+\delta} \equiv g^{N+1}$ will have the following form near $M'_{jq_j}$:

$$\begin{aligned} \bar{x}_1 &= |x_n|^{1-(n-2)\lambda}(x_1 - e^{-\lambda(p+l_j\delta)} z'_{jq_j1} \left|(x_{n-1} - b_j)/z'_{jq_j,n-1}\right|^{\frac{1}{\lambda}-(n-1)}), \\ \bar{x}_i &= |x_n|^\lambda (x_i - u'_{jq_ji}) \quad (2 \le i \le n-2), \\ \bar{x}_{n-1} &= a_{3-j,+} + 1 - \delta - \ln|x_n|, \quad \bar{x}_n = -|x_n|^\lambda (x_{n-1} - b_j - u'_{jq_j,n-1}). \end{aligned} \tag{4.2.24}$$

Note that $u'_{jq_ji}$ at $i \le n-1$ tend to zero as $p \to +\infty$ [see (4.2.14)], while the values of $x_n$ near $M'_{jq_j}$ and $z'_{j,q_j,n-1}$ stay bounded away from zero; the exponent $\frac{1}{\lambda} - (n-1)$ in the first line is larger than $r$ [see (4.2.11)]. Thus, for sufficiently large $p$, map (4.2.24) is indeed a $C^r$-small perturbation of the map $G_{j,1+\delta}$ given by (4.2.9).

Denote $P'_{3-j,0} = \tilde{G}_{j,1+\delta} M'_{jq_j}$. By (4.2.24), this is the point with the coordinates $x_i = 0$ at $i \ne n-1$, and $x_{n-1} = a_{3-j,+} + 1 + (\kappa_j - 1)\delta$ [we assume that the coordinate $x_n$ of $M'_{jq_j}$ is $u'_{jq_jn} = u_{jq_j} = -e^{-\kappa_j\delta}$ where $\kappa_j \in (0, 1]$ is defined by (4.2.31)]. Introduce rescaled coordinates near $P'_{3-j,0}$ by the rule

$$\begin{aligned} x_i &= e^{-(l_j+\kappa_j)\delta\lambda_i} C_{jq_ji}\ \eta\ e^{-\lambda p(n-i)} v_i \quad (i \le n-2), \\ x_{n-1} &= a_{3-j,+} + 1 + (1-\kappa_j)\delta + e^{(l_j+\kappa_j)\delta} C_{jq_jn}\ \eta\ v_{n-1}, \\ x_n &= e^{-(l_j+\kappa_j)\delta\lambda} C_{j,q_j,n-1}\ \eta\ e^{-\lambda p} v_n \end{aligned} \tag{4.2.25}$$

(with the same constants $C_{jq_ji}$ as above). In coordinates (4.2.23), (4.2.25), map (4.2.24) takes the form $(v_1, \ldots, v_{n-1}, v_n) \mapsto (v_1, \ldots, v_n, -v_{n-1}) + O(\eta)$, i.e. it becomes arbitrarily close to the map $\Phi_0$ [see (3.23)] as $m \to +\infty$. Thus, in the rescaled coordinates, the map $\tilde{G}_{j,1+\delta} \circ L_{j,p+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jp}\right)^{q_j} \equiv g^{l_j+(1+q_j)(pN+N+1)}$ from a small neighborhood of $P'_{j1}$ into a small neighborhood of $P'_{3-j,0}$ , can be made as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1}$ as $p$ grows.

Analogously, we take the point $M'_{j0} : \{x_{n-1} = a_{j-} - 1 + \delta/2, x_i = 0 \ (i \neq n-1)\} \in U_{j-}$, and perturb the map $Y_\delta$ near $Y_\delta^N P'_{j0}$ in such a way that the map $\tilde{Q}_{j,1+\delta} \equiv g^{N+1}$ near $M'_{j0}$ will be given by

$$
\begin{aligned}
&\bar{x}_i = e^{-\lambda_i(\delta - x_{n-1} - 1 + a_{j-})} x_i \quad (i \leq n-2),\\
&\bar{x}_{n-1} - b_j = e^{-\lambda(\delta - x_{n-1} - 1 + a_{j-})}, \quad \bar{x}_n = e^{\delta - x_{n-1} - 1 + a_{j-}} x_n + e^{-p} u_{j1}.
\end{aligned}
\tag{4.2.26}
$$

It is a small perturbation of the map $Q_{j,1+\delta}$ from (4.2.6), and it takes $M'_{j0}$ to $P'_{j1}$ [see (4.2.13), (4.2.15)]. When we introduce rescaled variables near $M'_{j0}$ by the rule

$$
\begin{aligned}
x_i &= e^{\delta\lambda_i/2} C_{j1i}\ \eta\ e^{-\lambda p(n-i-1)} v_i \quad (i \leq n-2),\\
x_{n-1} &= a_{j-} - 1 + \delta/2 + \frac{1}{\lambda} e^{\lambda\delta/2} C_{j,1,n-1}\ \eta\ v_{n-1},\\
x_n &= e^{-\delta/2} C_{j1n}\ \eta\ e^{-p} v_n,
\end{aligned}
\tag{4.2.27}
$$

map (4.2.26) will take the form $\bar{v} = v + O(\eta)$, i.e. it is close to the identity map. Thus, in the rescaled coordinates given by (4.2.27), (4.2.25), the map $\tilde{G}_{j,1+\delta} \circ L_{j,p+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jp}\right)^{q_j} \circ \tilde{Q}_{j,1+\delta} \equiv g^{l_j+(1+q_j)(pN+N+1)+N+1}$ from a small neighborhood of $M'_{j0}$ into a small neighborhood of $P'_{3-j,0}$, is as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1}$ at $p$ large enough, i.e. it is a close approximation of the map $\Phi_j$.

*4.2.5. Parameter fitting in the lips.* Let us now determine the form of the map $S_{jt} : v \mapsto \bar{v}$ from a small neighborhood of $P'_{j0}$ into a small neighborhood of $M'_{j0}$ in the rescaled coordinates (4.2.27), (4.2.25). By (4.2.5), for an integer $k > 0$, the map $S_{j,k\delta} \equiv Y_\delta^k$ takes the point $P'_{j0}$ into $M'_{j0}$ if

$$
\beta(\mu_j) = (k + \kappa_{3-j} - 1/2)\delta - 5 \tag{4.2.28}
$$

[see (4.2.27), (4.2.25)]. Since $\beta \to +\infty$ as $\mu \to +0$ [see (4.1.4)], for every sufficiently large $k$ equation (4.2.28) has a solution $\mu_j(k)$, and $\mu_j(k) \to +0$ as $k \to +\infty$. It follows that $\alpha(\mu_j(k)) \to +\infty$. Thus, for any sufficiently large $p$ we can find $\gamma_{n\pm} \in (0, 1]$ and $k$ such that

$$
\begin{aligned}
e^{-\gamma_{n+}\alpha(\mu_j(k))} &= e^{-(l_{3-j}+\kappa_{3-j})\delta\lambda} C_{3-j,q_{3-j},n-1}\ \eta\ e^{-\lambda p},\\
e^{-\gamma_{n-}\alpha(\mu_j(k))} &= e^{-\delta/2} C_{j1n}\ \eta\ e^{-p}.
\end{aligned}
\tag{4.2.29}
$$

This guarantees that $\bar{v}_n = \psi_j(v_n)$ [see (4.2.27), (4.2.25), (4.2.5)].

We also obtain $\bar{v}_i = v_i$ at $i \leq n-2$ by choosing $\gamma_{i\pm} \in (0, 1]$ such that

$$
\begin{aligned}
e^{-\gamma_{i+}\alpha(\mu_j(k))} &= e^{\delta\lambda_i/2} C_{j1i}\ \eta\ e^{-\lambda p},\\
e^{-\gamma_{i-}\alpha(\mu_j(k))} &= e^{-(l_j+\kappa_j)\delta\lambda_i} C_{3-j,q_{3-j},i}\,\eta.
\end{aligned}
\tag{4.2.30}
$$

Finally, we fix the choice of the integer $l_j$ and $\kappa_j \in (0, 1]$ as follows:

$$e^{(l_j+\kappa_j)\delta} = \frac{1}{\lambda} e^{\lambda\delta/2} C_{3-j,1,n-1}/C_{jq_jn}. \tag{4.2.31}$$

This [along with (4.2.28)] gives us $\bar{v}_{n-1} = v_{n-1}$ for the map $S_{j,k\delta}$ in the coordinates (4.2.27), (4.2.25). As wee see, the map $S_{j,k\delta}$ in the rescaled coordinates coincides with the map $\Psi_j$ for $v$ from some open neighborhood of the unit ball $B^n$ (if $j = 1$) or of $\Phi_1 \circ \Psi_1(B^n)$ (if $j = 2$).

Thus, we see that the map

$$\begin{aligned} &\tilde{G}_{2,1+\delta} \circ L_{2,p+l_2\delta} \circ \left(\tilde{T}_{2,1+\delta} \circ L_{2p}\right)^{q_2} \circ \tilde{Q}_{2,1+\delta} \circ S_{2,k\delta} \\ &\qquad \circ \tilde{G}_{1,1+\delta} \circ L_{1,p+l_1\delta} \circ \left(\tilde{T}_{1,1+\delta} \circ L_{1p}\right)^{q_1} \circ \tilde{Q}_{1,1+\delta} \circ S_{1,k\delta} \\ &\equiv \; g^{2k+l_1+l_2+(2+q_1+q_2)(pN+N+1)+2(N+1)} \end{aligned}$$

is a close approximation to the map $F = \Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$, provided $\tilde{H}_{jq_j} \circ \cdots \circ \tilde{H}_{j1}$ are sufficiently close approximations to $\tilde{\Phi}_j$ $(j = 1, 2)$ and $p$ is large enough. This completes the proof of Theorem 1 (the absence of intersections of $g^{m-1}(\psi(B^n))$ with $g^i(\psi(B^n))$ at $i = 0, \ldots, m-2$ is established in the same way as in the case $n = 2$; see the end of Sect. 4.1.5).

## 5. Birth of Periodic Spots from a Heteroclinic Cycle

In this Section we prove Lemma 2, thus finishing the proof of Theorem 5. Namely, we consider a $C^\rho$-diffeomorphism $\tilde{f}$ of a smooth two-dimensional manifold and assume that $\tilde{f}$ has a pair of saddle periodic points $P$ and $Q$ of periods $p$ and, respectively, $q$. Denote as $T_{01}$ the map $\tilde{f}^p$ restricted onto a small neighborhood of $P$, and denote as $T_{02}$ the map $\tilde{f}^q$ restricted onto a small neighborhood of $Q$. By definition, $T_{01}P = P$ and $T_{02}Q = Q$. One can introduce coordinates $(x_1, y_1)$ in the neighborhood of $P$ and $(x_2, y_2)$ in the neighborhood of $Q$ such that the maps $T_{0j}$ will have the form

$$\bar{x}_j = \lambda_j x_j + \cdots, \quad \bar{y}_j = \gamma_j y_j + \cdots,$$

where $|\lambda_j| < 1$, $|\gamma_j| > 1$; the dots stand for nonlinearities. The numbers $\lambda_1, \gamma_1$ and $\lambda_2, \gamma_2$ are the multipliers of the periodic points $P$ and $Q$, respectively. We assume

$$J_1 := |\lambda_1\gamma_1| < 1 \quad \text{and} \quad J_2 := |\lambda_2\gamma_2| > 1. \tag{5.1}$$

Every saddle periodic point lies in the intersection of two $C^\rho$-smooth invariant manifolds: the points in the stable invariant manifold $W^s$ tend to the periodic orbit at forward iterations of the map, and the points in the unstable invariant manifold $W^u$ tend to the periodic orbit at backward iterations. With our choice of the coordinates, the unstable manifold is tangent at the periodic point to the $y$-axis, and the stable manifold is tangent to the $x$-axis. One can locally straighten $W^u$ and $W^s$, i.e. the coordinates $(x_j, y_j)$ can be chosen in such a way that $W^s_{loc}(P) = \{y_1 = 0\}$, $W^u_{loc}(P) = \{x_1 = 0\}$, $W^s_{loc}(Q) = \{y_2 = 0\}$, $W^u_{loc}(Q) = \{x_2 = 0\}$.

Assume that $W^u(P)$ has an orbit of a transverse intersection with $W^s(Q)$. This means that $M_2^+ := \tilde{f}^{k_{12}} M_1^- \in W^s_{loc}(Q)$ for some positive integer $k_{12}$ and some point

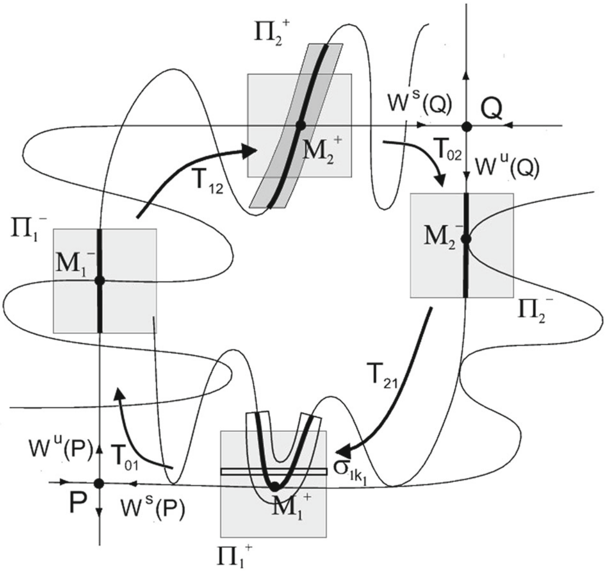


**Fig. 3.** A non-transverse heteroclinic cycle

$M_1^-(0, y_1^-) \in W^u_{loc}(P)$, and the curve $\tilde{f}^{k_{12}}(W^u_{loc}(P))$ intersects $W^s_{loc}(Q)$ at the point $M_2^+(x_2^+, 0)$ transversely. We denote the map $\tilde{f}^{k_{12}}$ restricted onto a small neighborhood of $M_1^-$ as $T_{12}$. It can be written as

$$\bar{x}_2 - x_2^+ = a_1 x_1 + b_1(y_1 - y_1^-) + \cdots, \quad \bar{y}_2 = c_1 x_1 + d_1(y_1 - y_1^-) + \cdots, \tag{5.2}$$

where $d_1 \neq 0$ because of the transversality of $T_{12}(W^u_{loc}(P))$ to $W^s_{loc}(Q)$; the dots stand for quadratic and higher order terms. As an orbit of intersection of $W^u(P)$ and $W^s(Q)$, the orbit $\Gamma_{PQ}$ of the point $M_1^-$ is heteroclinic: it tends to the orbit of $P$ at backward iterations of $\tilde{f}$ and to the orbit of $Q$ at forward iterations. Note that since $d_1 \neq 0$, we may rewrite (5.2) in the so-called cross-form:

$$d_1(\bar{x}_2 - x_2^+) = Dx_1 + b_1\bar{y}_2 + \cdots, \quad d_1(y_1 - y_1^-) = \bar{y}_2 - c_1 x_1 + \cdots, \tag{5.3}$$

where $D := a_1 d_1 - b_1 c_1$.

Another assumption is that $\tilde{f}$ has a heteroclinic orbit $\Gamma_{QP}$ at the points of which $W^u(Q)$ has a tangency of order $m$ with $W^s(P)$. This means that there exists a pair of points, $M_2^-(0, y_2^-) \in W^u_{loc}(Q)$ and $M_1^+(x_1^+, 0) \in W^s_{loc}(P)$, such that $M_1^+ = \tilde{f}^{k_{21}} M_2^-$ for some positive integer $k_{21}$, and the curve $\tilde{f}^{k_{21}}(W^u_{loc}(Q))$ has a tangency of order $m$ with $W^s_{loc}(P)$ at $M_1^+$ (see Fig. 3). We denote the map $\tilde{f}^{k_{21}}$ restricted onto a small neighborhood of $M_2^-$ as $T_{21}$. It can be written as

$$\bar{x}_1 - x_1^+ = a_2 x_2 + b_2(y_2 - y_2^-) + \cdots, \quad \bar{y}_1 = c_2 x_2 + d_2(y_2 - y_2^-)^{m+1} + \cdots, \tag{5.4}$$

where $d_2 \neq 0$; the dots stand for higher order terms. Obviously, to speak about the tangency of order $m$, the smoothness of $\tilde{f}$ has to be sufficiently high, i.e. $\rho \geq m + 1$.

The two periodic orbits (the orbit of $P$ and the orbit of $Q$) and two heteroclinic orbits, $\Gamma_{PQ}$ and $\Gamma_{QP}$ comprise a heteroclinic cycle $C$. Let $U$ be a small neighborhood of the heteroclinic cycle. We prove the following statement (Lemma 2):

*Given any neighborhood $U$ of the heteroclinic cycle $C = P \cup Q \cup \Gamma_{PQ} \cup \Gamma_{QP}$, for any $r \leq m - 1$, arbitrarily $C^r$-close to $\tilde{f}$ there is a diffeomorphism $\hat{f}$ which has a periodic spot in $U$.*

*Proof.* Let us fix the orientation in the neighborhood of $Q$ by requiring that the determinant $D$ of the derivative matrix $\partial T_{12}/\partial(x, y)$ is positive at the point $M_1^-$. Then, let us require that the determinant of $\partial T_{21}/\partial(x, y)$ at the point $M_2^-$ is also positive, i.e. we have

$$D > 0 \quad \text{and} \quad b_2 c_2 < 0. \tag{5.5}$$

This is always the case if the manifold on which $f$ is defined is orientable and $f$ is orientation-preserving.

We will prove the lemma under condition (5.5); moreover, instead of $r \leq m - 1$ we will do it for all $r \leq m$ (i.e. the periodic spot can be born in $U$ at a perturbation of $\tilde{f}$ which is arbitrarily small in $C^m$). If condition (5.5) is not satisfied, i.e. $Db_2c_2 > 0$ so the cycle $C$ is "non-orientable", then it is easy to show (see Lemma 8 in [28]) that an arbitrarily small perturbation can be added to $\tilde{f}$ such that a new (double-round) orbit of heteroclinic tangency of order $(m - 1)$ between $W^u(Q)$ and $W^s(P)$ is born, and the corresponding cycle is now orientable. Thus, by adding to the map with this orientable heteroclinic cycle a perturbation which is arbitrarily small in $C^{m-1}$, we obtain periodic spots in the non-orientable case as well.

In what follows we assume (5.5) and $r = m$. We will also assume $\ln|\gamma_1|\ln|\gamma_2| \neq \ln|\lambda_1|\ln|\lambda_2|$ (one can always achieve this by an arbitrarily small perturbation of $\tilde{f}$ without destroying the order $m$ tangency between $W^u(Q)$ and $W^s(P)$). In fact, we can always assume

$$\ln|\gamma_1|\ln|\gamma_2| < \ln|\lambda_1|\ln|\lambda_2|, \tag{5.6}$$

as the case $\ln|\gamma_1|\ln|\gamma_2| > \ln|\lambda_1|\ln|\lambda_2|$ reduces to the given one by considering the map $\tilde{f}^{-1}$ instead of $\tilde{f}$ (and interchanging $P$ with $Q$).

Let us imbed $\tilde{f}$ in any $(m+1)$-parameter family $f_\nu$ of maps, of class $C^\rho$ with respect to coordinate and parameters. The corresponding map $T_{21}$ will also depend smoothly on parameters. Since it has form (5.4) at $\nu = 0$, i.e. all the derivatives of $\bar{y}_1$ with respect to $(y_2 - y_2^-)$ vanish up to the order $m$, it follows that at non-zero $\nu$ the map $T_{21}$ can be written in the form

$$\bar{x}_1 - x_1^+ = a_2 x_2 + b_2(y_2 - y_2^-) + \cdots, \quad \bar{y}_1 = c_2 x_2 + \sum_{s=0}^{m-1} \mu_s (y_2 - y_2^-)^s + d_2 (y_2 - y_2^-)^{m+1} + \cdots, \tag{5.7}$$

where $\mu_s$ are smooth functions of $\nu$ (note that $x_1^+$ and $y_2^-$ also depend on $\nu$ now: the value of $y_2^-$ is fixed by the condition $\partial^m \bar{y}_1/\partial y_2^m = 0$ at $y_2 = y_2^-$; thus $y_2^-$ is a $C^{\rho-m}$-function of $\nu$, and $\mu_s$ are also $C^{\rho-m}$; the high-order terms that are denoted by dots in (5.7) and (5.3) depend now also on $\nu$, $C^{\rho-m}$-smoothly).

Denote $\theta = |\ln J_2/\ln J_1|$. The value of $\theta$ is a $C^{\rho-1}$-smooth function of $\nu$. We may always put a family $f_\nu$ in general position, i.e. we may further assume

$$\frac{\partial(\mu_0, \ldots, \mu_{m-1}, \theta)}{\partial(\nu_1, \ldots, \nu_{m+1})} \neq 0. \tag{5.8}$$

This, in particular, means that by changing $\nu$ we may change the values of any of the parameters $\theta$ and $\mu_s$ while keeping the other parameters constant. For example, one may change the value of $\theta$ and keep $\mu = 0$, i.e. keep the order $m$ tangency between $W^u(Q)$ and $W^s(P)$.

Varying the parameters $\nu$ can lead to the destruction of the heteroclinic cycle and to bifurcations of other orbits in $U$. We will further focus on one instance of such bifurcations. We call a periodic orbit of the map $f_\nu$ *single-round* if it stays entirely in $U$ and visits a small neighborhood of each of the points $M^{\pm}_{1,2}$ only once, i.e. the point of intersection of the single-round orbit with a small neighborhood of $M_1^+$ is a fixed point of the map $T_{21}T_{02}^{k_2}T_{12}T_{01}^{k_1}$ for some positive integers $k_1$ and $k_2$; the image of this point by the map $T_{01}^{k_1}$ lies in a small neighborhood of $M_1^+$, the image by $T_{12}T_{01}^{k_1}$ lies in a small neighborhood of $M_2^-$, and the image by $T_{02}^{k_2}T_{12}T_{01}^{k_1}$ lies in a small neighborhood of $M_2^+$. Below we will prove the following

**Lemma 5.** *Let assumptions (5.1), (5.6), (5.5), (5.8) be satisfied. Then, there exist a sequence $\nu_l \to 0$ and sequences of integers $k_{1l} \to +\infty$ and $k_{2l} \to +\infty$ such that at $\nu = \nu_l$ the map $f_\nu$ has a single-round periodic orbit which corresponds to a fixed point of the map $\mathcal{T}_l := T_{21}T_{02}^{k_{2l}}T_{12}T_{01}^{k_{1l}}$, and the map $\mathcal{T}_l$ near this point is, in some $C^\rho$-coordinates $(u, v)$, given by*

$$(\bar{u}, \bar{v}) = \Phi(u, v) + o(|u|^m + |v|^m), \tag{5.9}$$

*where $\Phi$ denotes the time-1 map by the flow*

$$\dot{u} = v, \quad \dot{v} = -\Psi(u)(1 + v) \tag{5.10}$$

*near $(u, v) = (0, 0)$; here $\Psi$ is a polynomial such that $\Psi(0) = 0$ and $\Psi'(0) \geq 0$.*

Before we start proving Lemma 5, we remark that system (5.10) has an integral $H(u, v) = \int \Psi(u)du + v - \ln(1 + v) = \frac{v^2}{2} + \Psi'(0)\frac{u^2}{2} + \cdots$. Thus, when $\Psi'(0) > 0$, the equilibrium at zero is a center: every orbit of (5.10) is in this case a closed curve surrounding the origin. The corresponding time-1 map $\Phi$ is, therefore, conservative (it preserves the area form $\frac{1}{1+v}\, du \wedge dv$) and has an elliptic point at the origin. Note that map (5.9) can, by an arbitrarily $C^m$-small perturbation, be made equal to $(\bar{u}, \bar{v}) = \Phi(u, v)$ identically in a sufficiently small neighborhood of zero. Hence, Lemma 5 implies, that by an arbitrarily $C^m$-small perturbation of the map $\tilde{f}$ a periodic point can be born in a small neighborhood of the heteroclinic cycle $C$ such that the first-return map near this point will be *area-preserving* and the point will be elliptic (i.e. its multipliers lie on the unit circle). The area-preservation property of the first-return map in a neighbourhood of this point is crucial because, by [70,71], an arbitrarily $C^m$-small perturbation of an area-preserving map in a neighborhood of an elliptic point creates an orbit of homoclinic tangency; the perturbation does not destroy the area-preserving property. By Theorem 5 of [28], an arbitrarily $C^m$-small perturbation of any area-preserving map with a homoclinic tangency leads to the birth of a periodic spot (again, the area-preservation was crucially important for the quoted result of [28]). As we see, once Lemma 5 provides us with an elliptic periodic orbit whose normal form is area-preserving up to a sufficiently high order, the birth of periodic spots from the heteroclinic cycles under consideration follows from the results of [28] on perturbations of area-preserving maps. Thus, in order to finish the proof of Lemma 2 and Theorem 5, it remains to prove Lemma 5.

*Proof of Lemma 5.* At $m = 1$ Lemma 5 claims only that a periodic orbit can be born with the eigenvalues on the unit circle. Such statement can be found in [19,20], so we further focus on the much more involved case $m \geq 2$. We prove the lemma in two steps. First, we show that for an appropriate choice of $k_1$ and $k_2$ the first-return map $\mathcal{T}$ can be brought, by a rescaling of coordinates, to a standard form (5.22) (a similar form for the rescaled first-return map was obtained in [20] for the case $m = 1$; map (5.22) was also obtained in [28] as the rescaled first-return map near a homoclinic tangency of order $m$). Next, we choose the values of the parameters of (5.22) to be close to a bifurcation of a parabolic point; we bring the map to a normal form near this point and show that this normal form coincides with (5.9) for a proper choice of the parameters.

Since the periodic points $P$ and $Q$ are saddle, it follows that given any small $x_{j0}$ and $y_{jk}$ (where $j = 1, 2$) and any $k \geq 0$ there exist uniquely defined small $x_{jk}$ and $y_{j0}$ such that $(x_{jk}, y_{jk}) = T_{0j}^k(x_{j0}, y_{j0})$ and all the points in the orbit $\{(x_{j0}), y_{j0}), T_{0j}(x_{j0}, y_{j0}), \ldots, T_{0j}^k(x_{j0}, y_{j0})\}$ lie in a small neighborhood of $P$ (at $j = 1$) or $Q$ (at $j = 2$), see [19] and, for a more detailed exposition, Chapter 3 of [72]. Denote

$$x_{jk} = \lambda_j^k x_{j0} + \xi_{jk}(x_{j0}, y_{jk}, \nu), \quad y_{j0} = \gamma_j^{-k} y_{jk} + \eta_{jk}(x_{j0}, y_{jk}, \nu). \tag{5.11}$$

By [15], one can introduce $C^\rho$-coordinates $(x_j, y_j)$ near the saddle periodic points in such a way that

$$\xi_{jk} = o(\lambda_j^k), \quad \eta_{jk} = o(\gamma_j^{-k}), \tag{5.12}$$

i.e. the map $T_{0j}^k$ written in the "cross-form" (5.11) is linear in the main order.[3] By [15], the same $o(\lambda_j^k)$ and, resp., $o(\gamma_j^{-k})$ estimates hold for the derivatives of the functions $\xi_{jk}$ and $\eta_{jk}$ up to the order $(\rho - 1)$ with respect to $(x_{j0}, y_{jk})$ and up to the order $(\rho - 2)$ with respect to parameters; while for the higher order derivatives up to the order $\rho$ for which the number of differentiations with respect to the parameters does not exceed $(\rho - 2)$ we have that they uniformly tend to zero as $k \to \infty$.

Let $\delta > 0$ be sufficiently small. Consider rectangular neighborhoods $\Pi_j^+ : \{|x_j - x_j^+| < \delta, |y_j| < \delta\}$ and $\Pi_j^- : \{|y_j - y_j^-| < \delta, |x_j| < \delta\}$ of $M_j^+$ and $M_j^-$ $(j = 1, 2)$, and take any sufficiently large integers $k_1$ and $k_2$. By (5.11), (5.12), the set $\sigma_{jk_j} = \Pi_j^+ \cap T_{0j}^{-k_j} \Pi_j^-$ is non-empty: it is a strip of the form $\{|\gamma_j^{k_j} y_j - y_j^-| < \delta + o(1)_{k\to+\infty}\}$. We introduce a new coordinate $y_j$ on $\sigma_{jk_j}$ such that

$$y_{j,old} = \gamma_j^{-k_j} y_{j,new} + \eta_{jk_j}(x_j, y_{j,new}) \tag{5.13}$$

[i.e. $y_{j,new}$ equals to $y_{jk_j}$ from (5.11)].

By construction, the map $T_{12}T_{01}^{k_1}$ is defined on $\sigma_{1k_1}$; in the new coordinates, when this map takes a point $(x_1, y_1) \in \sigma_{1k_1}$ into a point $(\bar{x}_2, \bar{y}_2) \in \sigma_{2k_2}$ it can be written in the following form [see (5.3), (5.13)]:

$$\begin{aligned} d_1(\bar{x}_2 - x_2^+) &= D\lambda_1^{k_1} x_1 + \phi_1(x_1) + b_1\gamma_2^{-k_2}\bar{y}_2 + o(\gamma_2^{-k_2}), \\ d_1(y_1 - y_1^-) &= -c_1\lambda_1^{k_1} x_1 - \phi_2(x_1) + \gamma_2^{-k_2}\bar{y}_2 + o(\gamma_2^{-k_2}), \end{aligned} \tag{5.14}$$

[3] The notation $o(\lambda^k)$ means $\lambda^k$ times a function (of all variables and parameters) which uniformly tends to zero as $k \to \infty$; recall that $\lambda$ and $\gamma$ may vary as parameters vary.

where the functions $\phi_{1,2}$ are $o(\lambda_1^{k_1})$ and independent of $y$. Analogously, the map $T_{21}T_{02}^{k_2}$ that takes a point $(x_2, y_2) \in \sigma_{2k_2}$ into a point $(\bar{x}_1, \bar{y}_1) \in \sigma_{1k_1}$ is given by

$$\begin{aligned} \bar{x}_1 - x_1^+ &= a_2\lambda_2^{k_2}x_2 + b_2(y_2 - y_2^-) + o(y_2 - y_2^-) + o(\lambda_2^{k_2}), \\ \gamma_1^{-k_1}\bar{y}_1 &= c_2\lambda_2^{k_2}x_2 + \sum_{s=0}^{m-1}\mu_s(y_2 - y_2^-)^s \\ &\quad + d_2(y_2 - y_2^-)^{m+1} + o((y_2 - y_2^-)^{m+1}) + o(\lambda_2^{k_2}) + o(\gamma_1^{-k_1}) \end{aligned} \tag{5.15}$$

[see (5.7)]. We will now change the variable $y_1$ to $y_1 + d_1^{-1}(c_1\lambda_1^{k_1}x_1 + \phi_2(x_1))$. Then the second lines in (5.14) and (5.15) will change to

$$d_1(y_1 - y_1^-) = \gamma_2^{-k_2}\bar{y}_2 + o(\gamma_2^{-k_2}),$$

and, respectively,

$$\begin{aligned} \gamma_1^{-k_1}\bar{y}_1 &= c_2\lambda_2^{k_2}x_2 + \sum_{s=0}^{m}\hat{\mu}_s(y_2 - y_2^-)^s + d_2(y_2 - y_2^-)^{m+1} \\ &\quad + o((y_2 - y_2^-)^{m+1}) + o(\lambda_2^{k_2}) + o(\gamma_1^{-k_1}), \end{aligned}$$

where $\hat{\mu}_s = \mu_s + O(\lambda_1^{k_1})$ $(s = 0, \dots, m-1)$ and $\hat{\mu}_m = O(\lambda_1^{k_1})$.

It is easy to see that since $d_1 \neq 0, d \neq 0$ one can find constants $C_j = O(|\lambda_{3-j}|^{k_{3-j}} + |\gamma_j|^{-k_j})$ and $K_j = O(|\gamma_{3-j}|^{-k_{3-j}} + |\lambda_j|^{k_j})$ $(j = 1, 2)$ such that, after the following shift of the origin:

$$x_{j,new} = x_j - x_j^+ + C_j, \quad y_{j,new} = y_j - y_j^- + K_j, \tag{5.16}$$

we obtain $(\bar{x}_{2,new}, y_{1,new}) = 0$ at $(x_{1,new}, \bar{y}_{2,new}) = 0$, and $\bar{x}_{1,new} = 0$, $\partial^m\bar{y}_{1,new}/\partial y_{1,new}^m = 0$ at $(x_{2,new}, y_{2,new}) = 0$. Thus, after this transformation, the maps $T_{12}T_{01}^{k_1}$ and $T_{21}T_{02}^{k_2}$ will be written as

$$\begin{aligned} d_1\bar{x}_2 &= D\lambda_1^{k_1}x_1 + O(\gamma_2^{-k_2}\bar{y}_2) + o((|\lambda_1|^{k_1} + |\gamma_2|^{-k_2})x_1), \\ d_1 y_1 &= \gamma_2^{-k_2}\bar{y}_2 + o(\gamma_2^{-k_2}\bar{y}_2) + o(|\gamma_2|^{-k_2}x_1), \end{aligned} \tag{5.17}$$

and, respectively,

$$\begin{aligned} \bar{x}_1 &= b_2 y_2 + o(y_2) + O(\lambda_2^{k_2}x_2), \\ \gamma_1^{-k_1}\bar{y}_1 &= c_2\lambda_2^{k_2}x_2 + \sum_{s=0}^{m-1}\tilde{\mu}_s y_2^s + d_2 y_2^{m+1} + o(y_2^{m+1}) + o(\lambda_2^{k_2}x_2), \end{aligned} \tag{5.18}$$

where the modified parameters $\tilde{\mu}_s$ are such that $\tilde{\mu}_s = \mu_s + o(1)_{k_{1,2}\to+\infty}$.

By virtue of (5.8), by an arbitrarily small change of $\nu$ we can make $\theta := |\ln|\lambda_2\gamma_2|/\ln|\lambda_1\gamma_1||$ rational without changing the values of $\mu$. Therefore, we may from the very beginning assume that at $\nu = 0$ the system has a heteroclinic cycle with a tangency of the order $m$, and the value $\theta_0 := \theta(0)$ is rational. Further we always assume that

$$k_1 = \theta_0 k_2 \tag{5.19}$$

in (5.17), (5.18). We will also assume that both $k_1$ and $k_2$ are even, so $\lambda_j^{k_j}$ and $\gamma_j^{k_j}$ are positive. It follows from (5.19), (5.6), (5.1) that

$$\gamma_2^{-k_2} \ll \lambda_1^{k_1} \ll \gamma_1^{-k_1} \ll \lambda_2^{k_2}. \tag{5.20}$$

Now, let us introduce new, rescaled coordinates $(X_1, Y_1, X_2, Y_2)$ and parameters $(B, E_0, \dots, E_{m-1})$ by the following rule:

$$\begin{aligned} x_1 &= b_2 d_1 \gamma_1^{-k_1/m} \gamma_2^{-k_2/m} X_1, \quad x_2 = D b_2 \lambda_1^{k_1} \gamma_1^{-k_1/m} \gamma_2^{-k_2/m} X_2, \\ y_1 &= \gamma_1^{-k_1/m} \gamma_2^{-k_2(1+1/m)} Y_1, \quad y_2 = d_1 \gamma_1^{-k_1/m} \gamma_2^{-k_2/m} Y_2, \\ \tilde{\mu}_s &= d_1^s \left(\gamma_1^{k_1/m} \gamma_2^{k_2/m)}\right)^{-(m+1-s)} E_s \quad (s = 0, \dots, m-1), \\ \theta &= \beta_0 + \frac{\ln(-B/(D b_2 c_2))}{k_2 |\ln|\lambda_1 \gamma_1||}; \end{aligned} \tag{5.21}$$

recall that $D b_2 c_2 < 0$ by our assumption (5.5), so the new parameter $B$ should be positive. After the rescaling, the maps $T_{12} T_{01}^{k_1}$ [given by (5.17)] and $T_{21} T_{02}^{k_2}$ [given by (5.18)] are rewritten as follows [we take into account (5.19), (5.20)]:

$$\begin{aligned} \bar{X}_2 &= X_1 + o(1)_{k_{1,2} \to +\infty} \\ Y_1 &= \bar{Y}_2 + o(1)_{k_{1,2} \to +\infty}, \qquad \text{and} \\ \bar{X}_1 &= Y_2 + o(1)_{k_{1,2} \to +\infty}, \\ \bar{Y}_1 &= -B X_2 + \sum\nolimits_{s=0}^{m-1} E_s Y_2^s + d\, Y_2^{m+1} + o(1)_{k_{1,2} \to +\infty}, \end{aligned}$$

where $d := d_2 d_1^{m+1} \neq 0$. It follows that with our choice of $k_{1,2}$ the map $\mathcal{T} := T_{21} T_{02}^{k_2} T_{12} T_{01}^{k_1}$ on $\sigma_{1k_1}$ can be written as

$$\bar{X} = Y + o(1)_{k_{1,2} \to +\infty}, \quad \bar{Y} = -BX + \sum_{s=0}^{m-1} E_s Y^s + d\, Y^{m+1} + o(1)_{k_{1,2} \to +\infty},$$

where $(X, Y)$ are the coordinates which were previously denoted as $(X_1, Y_1)$. By denoting the right-hand side of the equation for $\bar{X}$ as the new $Y$-variable, we finally write the map $\mathcal{T}$ in the following form

$$\bar{X} = Y, \quad \bar{Y} = -BX + \sum_{s=0}^{m-1} E_s Y^s + d\, Y^{m+1} + o(1)_{k_{1,2} \to +\infty}. \tag{5.22}$$

The transformation to coordinates $(X, Y)$ was of class $C^\rho$ with respect to the original coordinates and $C^{\rho-m}$ with respect to the parameters (more precisely: the $m$th derivatives with respect to the coordinates are $C^{\rho-m}$ with respect to both the coordinates and parameters). The $o(1)$-terms in (5.22) are functions of $(X, Y)$ and $(B, E_0, \dots, E_{m-1})$ which tend to zero as $k_{1,2} \to +\infty$. The derivatives of these terms up to the order $\rho$ with respect to $(X, Y)$ and $(B, E_0, \dots, E_{m-1})$ also tend to zero as $k_{1,2} \to +\infty$, provided the number of differentiations with respect to the parameters $(B, E_0, \dots, E_{m-1})$ does not exceed $(\rho - m)$. In any case, since $\rho \geq m+1$, we have that the derivatives of these terms with respect to $(X, Y)$ up to the order $m$ tend to zero, all with at least one derivative with respect to $(B, E_0, \dots, E_{m-1})$.

Next we show that if the $o(1)$-terms are sufficiently small (i.e. $k_{1,2}$ are sufficiently large), then a map of form (5.22) has, at certain uniformly (with respect to $k$) bounded values of $E$ and $B > 0$ and in a bounded region of $(X, Y)$, a fixed point near which the map is smoothly conjugate to (5.9). This will give us the lemma: by (5.21), (5.16), any bounded values of $(X, Y)$ correspond to $(x_1, y_1) \to (x_1^+, y_1^-)$ as $k_{1,2} \to +\infty$, i.e. if the values of $(X, Y)$ stay bounded (uniformly for all $k_{1,2}$ large enough) at the fixed point, then the latter belongs to the domain of definition of the first-return map $T_{21}T_{02}^{k_2}T_{12}T_{01}^{k_1}$ at all sufficiently large $k_{1,2}$; and any bounded values of $(E_0, \dots, E_{m-1})$ and $B > 0$ correspond to $\nu_k \to 0$ as $k_{1,2} \to +\infty$.

We start with shifting the origin of the $(X, Y)$ coordinates to a fixed point. Namely, we may take any $Y^*$ as the coordinate $X = Y = Y^*$ of the fixed point of (5.22); then the second equation defines the corresponding value of $E_0 = Y^*(1+B) - \sum_{s=1}^{m-1} E_s (Y^*)^s - d\ (Y^*)^{m+1} + o(1)$. After shifting the origin to the fixed point, the map takes the form

$$\begin{aligned} \bar{X} &= Y, \\ \bar{Y} &= -\hat{B}X + 2Y + \sum\nolimits_{s=1}^{m} \hat{E}_s Y^s + O(Y^{m+1}) + O(|XY| + X^2) \cdot o(1)_{k\to+\infty}, \end{aligned} \tag{5.23}$$

where the coefficients $\hat{E}_s$, $s = 1, \dots, m-1$, are related to $E_1, \dots, E_{m-1}$ by a diffeomorphism:

$$\hat{E}_1 = E_1 - 2 + \sum_{j=2}^{m-1} jE_j (Y^*)^{j-1} + (m+1)d\ (Y^*)^m + o(1)_{k\to+\infty},$$

$$\hat{E}_s = E_s + \sum_{j=s+1}^{m-1} E_j C_j^s (Y^*)^{j-s} + C_{m+1}^s d\ (Y^*)^{m+1-s} + o(1)_{k\to+\infty}\ (s = 2, \dots, m-1),$$

and

$$\hat{E}_m = (m+1)d\ Y^* + o(1)_{k\to+\infty}, \quad \hat{B} = B + o(1)_{k\to+\infty}.$$

As we see, bounded values of $\hat{B}, \hat{E}_1, \dots, \hat{E}_m$ give us bounded values of $B, E_0, \dots, E_{m-1}$ as well.

We will further fix $\hat{B} = 1$ and will take $(\hat{E}_1, \dots, \hat{E}_m)$ small (tending to zero as $k \to +\infty$), so the map will limit, as $k \to +\infty$, to

$$\bar{X} = Y, \quad \bar{Y} = -X + 2Y.$$

The fixed point of this map at $(X, Y) = 0$ is parabolic, i.e. it has a double multiplier $(+1)$. It is well-known [72,73] that the map near such point can be formally embedded into a flow, i.e. it can be approximated by the time-1 shift of an autonomous flow up to terms of any given order. Below we finish the proof of the Lemma by showing that parameters $(\hat{E}_1, \dots, \hat{E}_m)$ can be chosen in such a way that the autonomous flow whose time-1 shift approximates map (5.23) up to the order $m$ coincides with the conservative flow (5.10) in certain coordinates. In order to do this, we recall first the normal form theory for maps with a parabolic fixed point.

In general, we can always bring the linear part of the map to the Jordan form, i.e. we can write such map in the form

$$\begin{aligned} \bar{u} &= u + v + \sum_{2\le i+j\le m} \alpha_{ij} u^i v^j + o(|u|^m + |v|^m), \\ \bar{v} &= v + \sum_{2\le i+j\le m} \beta_{ij} u^i v^j + o(|u|^m + |v|^m). \end{aligned} \tag{5.24}$$

By denoting

$$z = v + \sum_{2\le i+j\le m} \alpha_{ij} u^i v^j + o(|u|^m + |v|^m) \equiv \bar{u} - u, \tag{5.25}$$

the map takes the form

$$\bar{u} = u + z, \quad \bar{z} = z + \sum_{2\le i+j\le m} \phi_{ij} u^i z^j + o(|u|^m + |z|^m). \tag{5.26}$$

Here

$$\phi_{ij} = \beta_{ij} + \sum_{i<k\le i+j} C_k^i \alpha_{k,i+j-k} + G_{ij}, \tag{5.27}$$

where $G_{ij}$ stand for polynomial functions of the coefficients $\alpha_{i'j'}$ and $\beta_{i'j'}$ with $i'+j' < i+j$; each monomial in $G_{ij}$ has degree 2 or higher. Let us make the polynomial coordinate transformation $(u, z) \mapsto (U, Z)$, where

$$U = u + \sum_{0\le j\le n-2} A_j u^{n-j} z^j, \quad Z = z + \sum_{0\le j\le n-2} A_j (\bar{u}^{n-j} \bar{z}^j - u^{n-j} z^j). \tag{5.28}$$

The map will retain its form (5.26) with the coefficients $\phi_{ij}$ unchanged for $i + j < n$, and $\phi_{ij,new} = \phi_{ij} + \sum_{0\le l\le j-2} (2^{j-l} - 2) C_{n-l}^{j-l} A_l$ at $i + j = n$. It is easy to see that, by choosing $A_l$ appropriately, one can make $\phi_{ij}$ with $j \ge 2$, $i = n - j$ assume any given values. In particular, we can make $\phi_{n-j,j} = 0$ for all $2 \le j \le n$. Then, by making such transformations consecutively, with $n$ running from 2 to $m$, one can make all the coefficients $\phi_{ij}$ with $j \ge 2$ vanish; the new coefficients $\phi_{ij}$ with $j = 0, 1$ will not vanish and will differ from the old ones by nonlinear (i.e. quadratic and higher order) expressions which depend only on the coefficients $\phi_{i'j'}$ with $i' + j'$ strictly less than $i + j$, i.e. these coefficients are still given by (5.27) with some new $G_{ij}$.

Since the number of coordinate transformations we apply is finite, it follows that for any fixed $m$ one must arrive to a similar conclusion for any map (with a fixed point at zero) whose linear part is sufficiently close to that of (5.24). Namely, for any sufficiently small $\varepsilon_{1,2}$, transformation (5.25) accompanied by a finite sequence of polynomial transformations of form (5.28) with certain smoothly dependent on $\varepsilon$ coefficients $A$ brings the map

$$\begin{aligned} \bar{u} &= u + v + \sum_{2\le i+j\le m} \alpha_{ij} u^i v^j + o(|u|^m + |v|^m), \\ \bar{v} &= (1 + \varepsilon_1) v + \varepsilon_2 u + \sum_{2\le i+j\le m} \beta_{ij} u^i v^j + o(|u|^m + |v|^m), \end{aligned} \tag{5.29}$$

to the normal form

$$\bar{u} = u + z, \quad \bar{z} = (1 + \varepsilon_1)z + \varepsilon_2 u + \sum_{i=2}^{m} (\phi_{i0} u^i + \phi_{i-1,1} u^i z) + o(|u|^m + |z|^m), \qquad (5.30)$$

where

$$\phi_{i0} = \beta_{i0} + G_{i0} + H_{i0}, \quad \phi_{i1} = \beta_{i1} + (i + 1)\alpha_{i+1,0} + G_{i1} + H_{i1}. \qquad (5.31)$$

Here $G_{ij}$ is a polynomial of $\alpha_{i'j'}$ and $\beta_{i'j'}$ with $i' + j' < i + j$, all monomials of $G_{ij}$ are of degree at least 2; the function $H_{ij}$ is a linear function of $\alpha_{i'j'}$ and $\beta_{i'j'}$ with $i' + j' = i + j$; the coefficients of $G_{ij}$ and $H_{ij}$ depend on $\varepsilon_{1,2}$ as well, and $H_{ij}$ vanish at $\varepsilon_1 = \varepsilon_2 = 0$.

Now, let us find, given any $m$, for any sufficiently small $\varepsilon_{1,2}$, a pair of functions $\psi_{0,1}(u)$ such that after an appropriate choice of the coordinates $(u, v)$, map (5.29) becomes $o(|u|^m + |v|^m)$-close to the time-1 map by a flow of the form

$$\dot{u} = v, \quad \dot{v} = \psi_0(u) + v\psi_1(u). \qquad (5.32)$$

Obviously, the flow has to have an equilibrium at zero and the eigenvalues of the corresponding linearization matrix have to be logarithms of the eigenvalues of $\begin{pmatrix} 1 & 1 \\ \varepsilon_2 & 1 + \varepsilon_1 \end{pmatrix}$, the linearization matrix of (5.29). Thus, we put

$$\begin{aligned} &\psi_0(0) = 0, \quad \psi_1(0) = \ln(1 + \varepsilon_1 - \varepsilon_2), \\ &\psi_0'(0) = -\ln\Big(1 + \frac{\varepsilon_1}{2} + \sqrt{\varepsilon_2 + \frac{\varepsilon_1^2}{4}}\Big) \cdot \ln\Big(1 + \frac{\varepsilon_1}{2} - \sqrt{\varepsilon_2 + \frac{\varepsilon_1^2}{4}}\Big). \end{aligned} \qquad (5.33)$$

The time-$t$ map of (5.32) can be found by expanding it in powers of the initial conditions $(u(0), v(0))$. We omit this computation. The result is that the time-1 map of (5.32) has a form which can be brought, by an $O(\varepsilon_{1,2})$-close to identity linear transformation, to form (5.29) with

$$\begin{aligned} \alpha_{ij} &= \frac{1}{j+1}\left(\psi_{i+j,0}\left[\frac{C_{i+j}^i}{j+2} + O(\varepsilon_{1,2})\right] + \psi_{i+j-1,1}\left[\frac{C_{i+j-1}^i}{j} + O(\varepsilon_{1,2})\right]\right) + \tilde{\alpha}_{ij}, \\ \beta_{ij} &= \psi_{i+j,0}\left[\frac{C_{i+j}^i}{j+1} + O(\varepsilon_{1,2})\right] + \psi_{i+j-1,1}\left[\frac{C_{i+j-1}^i}{j} + O(\varepsilon_{1,2})\right] + \tilde{\beta}_{ij}, \end{aligned} \qquad (5.34)$$

where $\tilde{\alpha}_{ij}$ and $\tilde{\beta}_{ij}$ are nonlinear functions of the coefficients $\psi_{i'j'}$ with $i' + j' < i + j$ (we denote $\psi_j(u) = \sum_i \psi_{ij} u^i$). By the definition of the normal form, two maps of form (5.29) can be made $o(|u|^m + |v|^m)$-close by means of a smooth coordinate transformation if their normal forms (5.30) coincide up to the order $m$. Thus, we find from formulas (5.31), (5.33), (5.34) that a map of form (5.29) can indeed, after an appropriate coordinate transformation, be made $o(|u|^m + |v|^m)$-close to the time-1 map by a flow of form (5.32) with the coefficients defined by

$$\psi_{i0} = \beta_{i0} + \tilde{\psi}_{i0}, \quad \psi_{i1} = \beta_{i1} + (i + 1)(\alpha_{i+1,0} - \beta_{i+1,0}) + \tilde{\psi}_{i1}, \qquad (5.35)$$

where $\tilde{\psi}_{ij}$ is a polynomial function of $\alpha_{i'j'}$ and $\beta_{i'j'}$ with $i' + j' \le i + j$; the monomials that include $\alpha_{i'j'}$ or $\beta_{i'j'}$ with $i' + j' < i + j$ are of degree 2 or higher, while the terms that include $\alpha_{i'j'}$ or $\beta_{i'j'}$ with $i'+j'=i+j$ are linear and the corresponding coefficients are small of order $O(\varepsilon_{1,2})$. Note that it follows from (5.33) that formulas (5.35) remain also valid at $j=1$, if we formally put $\beta_{10}=\varepsilon_2$, $\beta_{01}=\varepsilon_1$, $\alpha_{10}=\alpha_{01}=0$.

Returning to map (5.23), by putting $X=u$ and $Y=u+v$, we bring the map at $\hat{B}=1$ to form

$$\bar{u}=u+v, \quad \bar{v}=v+\beta_0(u)+v\beta_1(u)+O(\|E\|v^2)+o(|u|^m+|v|^m),$$

where $\|E\|$ stands for the norm of the vector $(\hat{E}_1,\ldots,\hat{E}_m)$, and the coefficients of the polynomials $\beta_j=\sum_{i=1}^{m-j}\beta_{ij}u^i$ $(j=0,1)$ satisfy

$$\beta_{i0}=\hat{E}_i+o(1)_{k\to+\infty}, \quad \beta_{i1}=(i+1)\hat{E}_{i+1}+o(1)_{k\to+\infty}.$$

This is a map of form (5.29). Therefore, by making normal form transformations described above, we can, at $\hat{E}_1=\varepsilon$ sufficiently small, make this map $o(|u|^m+|v|^m)$-close to the time-1 map of the flow (5.32) with

$$\psi_{i0}=\hat{E}_i+o(1)_{k\to+\infty}+o(\|E\|), \quad \psi_{i1}=o(1)_{k\to+\infty}+o(\|E\|);$$

see (5.35). It follows that at sufficiently large $k$ we can always choose the values of $\hat{E}_1,\ldots,\hat{E}_m$ in such a way that $\psi_0(u)\equiv s\psi_1(u)$, where we take $s=1$ if $\psi_1'(0)\le 0$, and $s=-1$ if $\psi_1'(0)>0$.

As we see, for an appropriate choice of the parameters $E_i$ map (5.23) can be made $o(|u|^m+|v|^m)$-close to the time-1 map of the flow

$$\dot{u}=v, \quad \dot{v}=\psi_1(u)(s+v),$$

which takes the desired form (5.10) after the change $(u,v)\to(su,sv)$; here $\Psi(u)=-\psi_1(su)$, so the requirement $\Psi'(0)\ge 0$ is ensured by our choice of $s$. □

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