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Remarkable charged particle dynamics near magnetic field null lines

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ABSTRACT

The study of charged-particle motion in electromagnetic fields is a rich source of problems, models, and new phenomena for nonlinear dynamics. The case of a strong magnetic field is well studied in the framework of a guiding center theory, which is based on conservation of an adiabatic invariant—the magnetic moment. This theory ceases to work near a line on which the magnetic field vanishes—the magnetic field null line. In this paper, we show that the existence of these lines leads to remarkable phenomena which are new both for nonlinear dynamics in general and for the theory of charged-particle motion. We consider the planar motion of a charged particle in a strong stationary perpendicular magnetic field with a null line and a strong electric field. We show that particle dynamics switch between a slow guiding center motion and the fast traverse along a segment of the magnetic field null line. This segment is the same (in the principal approximation) for all particles with the same total energy. During the phase of a guiding center motion, the magnetic moment of particle's Larmor rotation stays approximately constant, i.e., it is an adiabatic invariant. However, upon each traversing of the null line, the magnetic moment changes in a random fashion, causing the particle to choose a new trajectory of the guiding center motion. This results in a stationary distribution of the magnetic moment, which only depends on the particle's total energy. The jumps in the adiabatic invariant are described by Painlevé II equation.

The existence of adiabatic invariants—approximate conservation laws for systems with slow and fast motions—plays an important role in different physical theories. One of such theories is guiding center theory of motion of charged particles in a strong magnetic field. This theory is based on the adiabatic invariance of magnetic moment for the particle motion. Basic assumption of the guiding center approach is that the magnetic field is strong and vanishes nowhere. We show that, for a planar motion in strong perpendicular magnetic fields, if the magnetic field vanishes on some line (magnetic field null line) and the strong electric field is present, then the particle gets involved in a peculiar process of capture and release by the null line, which leads to large chaotic oscillations of the particle magnetic moment. Such a behaviour has not been previously reported in charged particles dynamics or in nonlinear dynamics in general.

I. INTRODUCTION

Classical guiding center theory[6,9] is a basic tool for description of charged particles motion in strong electromagnetic fields. In this theory, the particle motion is represented as a slow motion of the guiding center and a fast rotation in the Larmor circle around the guiding center. Averaging over this fast rotation eliminates the phase of the particle in a Larmor circle from Hamiltonian equations of motion. The canonical conjugate to this phase variable is the magnetic moment of the particle. This magnetic moment is, therefore, the first integral of the averaged equations of motion and the approximate first integral (adiabatic invariant) of the exact equations of motion. The frequency of the Larmor rotation is proportional to the magnitude of the magnetic field. If the magnetic field vanishes somewhere, then the Larmor frequency vanishes, and averaging over Larmor motion does not describe particle dynamics. Guiding center theory ceases to work. There are, however, field configurations with a magnetic field null line, notably, in a space plasmas, that have been intensively studied.[7,8] Formation of such null lines are closely related to magnetic reconnection,[1] which can shape a magnetic field configuration with a prolonged null line and a scalar potential maximizing (or minimizing) at the line.[3,11]

In this paper, we address how the emergence of such null lines influences the dynamics of a charged particle.

II. EQUATIONS OF MOTION

We consider the planar motion of a charged particle in strong, stationary magnetic and electric fields. Let (x, y) be Cartesian coordinates in the plane of motion. The magnetic field is directed perpendicular to this plane and has the intensity $\frac{1}{\varepsilon}B(x, y)$, where $\varepsilon > 0$ is a small parameter. Let the magnetic field vanish on a smooth curve $\mathcal{L}$ (see Fig. 1). The equations of motion are

$$\varepsilon\ddot{x} = \frac{e}{mc}B\dot{y} - \frac{\partial V}{\partial x}, \quad \varepsilon\ddot{y} = -\frac{e}{mc}B\dot{x} - \frac{\partial V}{\partial y}, \tag{1}$$

where $\frac{1}{\varepsilon}V(x, y)$ is the electrostatic potential; m, e, and c are the mass, charge of the particle, and the speed of light, respectively. In what follows, we assume that the units of dimensions are chosen such that $m = e = c = 1$. Thus, we have a system with a small parameter multiplying higher derivatives (i.e., a singularly perturbed system).

Dynamics outside a vicinity of $\mathcal{L}$ constitute the fast rotation with a slowly varying radius (the Larmor radius r_L) around the guiding center, whose slow motion is given by[6,9]

$$\dot{\xi} = -\frac{\mu}{B}\frac{\partial B}{\partial \eta} - \frac{1}{B}\frac{\partial V}{\partial \eta}, \quad \dot{\eta} = \frac{\mu}{B}\frac{\partial B}{\partial \xi} + \frac{1}{B}\frac{\partial V}{\partial \xi}. \tag{2}$$

Here, ξ and η are the coordinates of the guiding center, and $B = B(\xi, \eta)$ and $V = V(\xi, \eta)$. The parameter μ is the magnetic moment of the particle: the ratio of its kinetic energy to the value of the magnetic field at the guiding center. The magnetic moment is an adiabatic invariant of the system; away from $\mathcal{L}$, it determines the Larmor radius, $r_L = (2\varepsilon\mu/B)^{1/2}$. We will show that the passage near $\mathcal{L}$ leads to a jump of order 1 in the value of this adiabatic invariant. Thus, we will from the very beginning consider the case $\mu \sim 1$.

As the particle energy, H, is conserved, the guiding center moves along the lines of constant $H = (\mu B + V)\varepsilon^{-1}$. We consider the situation where the level lines of H intersect the null line. Note that the guiding center cannot cross the null line once it reaches it, because both $\dot{\xi}$ and $\dot{\eta}$ change sign when B changes sign in Eq. (2). Therefore, the motion of the particle along the level lines of H is interrupted by the periods where the particle stays close to $\mathcal{L}$. In order to analyze the capture process to the vicinity of $\mathcal{L}$ and the motion along it, we introduce new coordinates (r, s) near $\mathcal{L}$ as follows:

$$x = X(s) + rY'(s), \quad y = Y(s) - rX'(s),$$

where $x = X(s)$, $y = Y(s)$ is the equation of $\mathcal{L}$ in terms of the arclength parameter s (so r measures the deviation from $\mathcal{L}$; see Fig. 1). We will use the expansions

$$B = b(s)r + O(r^2), \quad V = v_0(s) + v_1(s)r + O(r^2). \tag{3}$$

By changing the sign of r, we can always assume that $b(s) > 0$. Note that in typical applications, the potential V has an extremum on $\mathcal{L}$, i.e., $v_1(s) \equiv 0$.[3,11]

One can check that, for small r, the guiding center equations (2) give

$$\frac{d}{dt}r^2 \sim -\frac{2}{b(s)}{v_0}'(s). \tag{4}$$

Therefore, in a finite time, the particle arrives at the null line near some point s_a (the point of absorption) such that ${v_0}'(s_a) > 0$. As B vanishes at this point while μ remains constant in the guiding center approximation, the particle's kinetic energy μB drops near the absorption point. This implies that the point s_a is, independently of μ, determined by the total energy H of the particle

$$\varepsilon H = v_0(s_a). \tag{5}$$

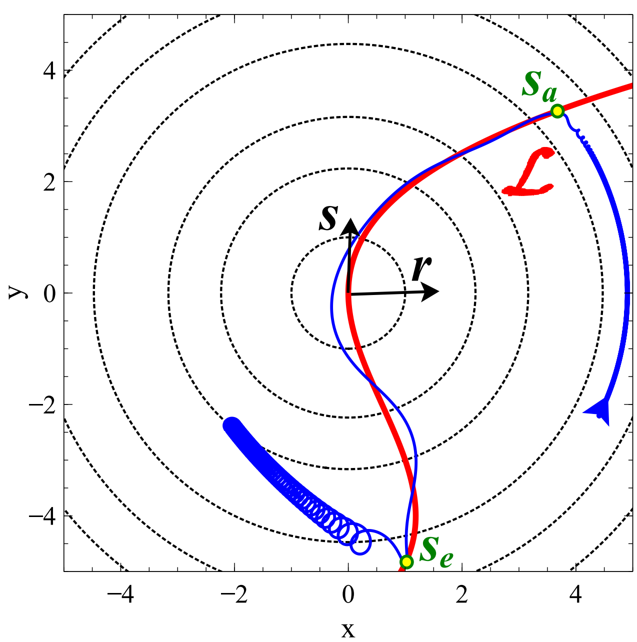

FIG. 1. Schematic of the system: dashed lines show level lines of the electrostatic potential, the red line is the magnetic field null line, the blue lines show particle trajectories, and the black arrows illustrate the local coordinate system.

III. DYNAMICS CLOSE TO THE NULL LINE. CHANGE OF THE ADIABATIC INVARIANT

Following motions are described by Eq. (1) written in the coordinates (r, s). As r is small near $\mathcal{L}$, we only retain the terms of the lowest order in r. In this approximation, the variables (r, s) can be treated as Cartesian coordinates, and the 3D vector potential of the magnetic field has the s-component equal to $A_s = b(s)r^2/(2\varepsilon)$, and the other two components are zero. Let p_r, p_s be the momenta of the particle corresponding to r and s. The dynamics near $\mathcal{L}$ are described (to the main order in r) by the Hamiltonian

$$E = \frac{1}{2}\left[p_r^2 + \left(\mathcal{P}_s - b(s)\frac{r^2}{2\varepsilon}\right)^2\right] + \frac{1}{\varepsilon}v_0(s) + \frac{1}{\varepsilon}v_1(s)r, \qquad (6)$$

where the canonical momentum $\mathcal{P}_s$ is given by $\mathcal{P}_s = p_s + A_s = p_s + b(s)r^2/(2\varepsilon)$.

At a finite distance r from the null line, the value of $\mathcal{P}_s$ is of order ε^{-1} and is positive. At small r, we have from (6) that $\frac{d}{dt}\mathcal{P}_s \sim -\varepsilon^{-1}v_0'(s)$, so $\mathcal{P}_s$ starts to decrease with time when the particle approaches $\mathcal{L}$ near the absorption point, where $v_0'(s) > 0$. In the system defined by (6), the variables s and $\mathcal{P}_s$ evolve much slower than (r, p_r), as long as $\mathcal{P}_s \gg \varepsilon^{-1/3}$. In this regime, we should consider, as a zero order approximation, the dynamics of r, p_r described by

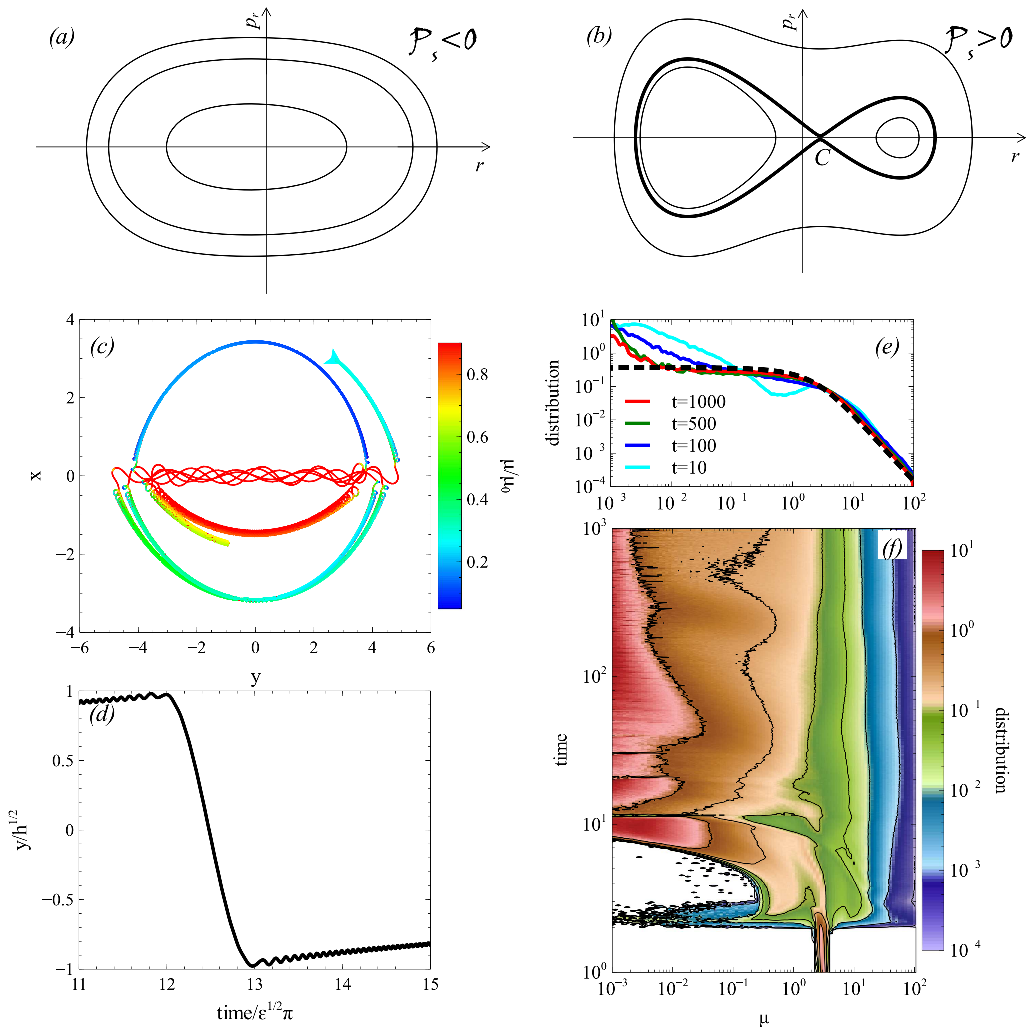

FIG. 2. Panels (a) and (b) show phase portraits of Hamiltonian (6) for frozen $\mathcal{P}_s$. Panels (c) and (d) show results of integration of Eqs. (1) with $B(x, y) = x/\varepsilon$ (i.e., the magnetic field null line is the axis $x = 0$), $V(x, y) = (x^2 + y^2)/(2\varepsilon)$, $\varepsilon = 10^{-3}$. In this case, the guiding center trajectories have equations $\mu x + (x^2 + y^2)/2 = h$ = const, μ = const, i.e., they are arcs of circles (different circles for different values of the magnetic moment μ). These arcs intersect the axis $x = 0$ at the same two points $x = 0, y = \pm\sqrt{2h}$. Motion along the axis $x = 0$ is (approximately) described by the equation $\varepsilon\ddot{y} + y = 0$, with initial conditions $y = \sqrt{2h}, \dot{y} = 0$. Panel (c) shows trajectory in the (x, y) plane with colors indicating the value of the magnetic moment μ (normalized to some typical value μ_0; values $\mu > \mu_0$—close to $B = 0$—are replaced by μ_0). Panel (d) shows the time dependence of the y-coordinate of a trajectory for a fragment of motion near the null line $x = 0$ and illustrates the jump in the adiabatic invariant. Panels (e) and (f) show the temporal evolution of the distribution of μ: 1D distributions are in panel (e) and the 2D time vs μ space is shown in panel (f). The initial distribution peaks around $\mu \sim 3$ (all trajectories are calculated for $2h = 25$ and $\varepsilon = 10^{-3}$). The black dashed line in (e) shows the theoretical prediction $f(\mu) = 4/(\pi\sqrt{2h}) \cdot \left[1 - \chi \arcsin(1 + \chi^2)^{-1/2}\right]$ with $\chi = \mu/\sqrt{2h}$. This distribution is obtained from the formula $f(\mu) = C\,T(\mu, h)$, where T is the time of the guiding center motion from the ejection to the absorption point at the magnetic field null line, and C is the normalization constant such that $\int_0^\infty f(\mu)d\mu = 1$.

the Hamiltonian (6) with frozen s and $\mathcal{P}_s$. At positive $\mathcal{P}_s$, this is the motion in a double-well potential [see Fig. 2(b)].

The oscillations close to a minimum of the potential represent the Larmor rotation. The slow evolution of $(s, \mathcal{P}_s)$ is governed by equations obtained by averaging over the fast oscillations of (r, p_r). The standard fact of the averaging theory is that the action I of the fast oscillations is an adiabatic invariant. The action is the area bounded by the corresponding phase curve in the plane (r, p_r), divided by 2π, i.e., $I = \frac{1}{2\pi}\oint_\gamma p_r dr$. While $(s, \mathcal{P}_s)$ slowly change, the closed path γ, along which (r, p_r) oscillate, changes in such a way that the value of I remain nearly constant. When γ is close to an elliptic point corresponding to a minimum of the double-well potential [and the corresponding oscillations of (r, p_r) around this point are considered in the linear approximation], the averaged equations are equivalent to the guiding center approximation, and the adiabatic invariant equals $|\mu|$. Therefore, the value of I remains close to $|\mu|$ as long as there is a separation of scales in the system defined by (6). This means that the area inside the curve γ remains bounded all the time when $\mathcal{P}_s \gg \varepsilon^{-1/3}$. One can check that under these conditions, γ must stay close to the elliptic point and the corresponding oscillations of (r, p_r) will remain in the linear regime. Therefore, the guiding center approximation remains valid up to the moment when $\mathcal{P}_s$ decreases to the values of order of $\varepsilon^{-1/3}$. After that, there is no separation of motions into fast and slow ones. As $I \sim 1$, one infers that $p_r \sim \varepsilon^{-1/3}$ and $r \sim \varepsilon^{1/3}$ when the guiding center approximation starts to break. We, therefore, scale the variables and time as follows (the "hat" marks the new variables):

$$r = \varepsilon^{1/3}\hat{r}, \quad p_r = \varepsilon^{-1/3}\hat{p}_r, \quad s - s_a = \varepsilon^{1/3}\hat{s}, \quad \mathcal{P}_s = \varepsilon^{-1/3}\hat{\mathcal{P}}_s,$$
$$t - t_0 = \varepsilon^{2/3}\hat{t}, \quad E - \frac{1}{\varepsilon}v_0(s_a) = \varepsilon^{-2/3}\hat{E}, \tag{7}$$

where t_0 is the moment of time when $\mathcal{P}_s = 0$. In the principal approximation, $s = s_a$ at $t = t_0$. The system for the rescaled variables is given, in the limit $\varepsilon \to 0$, by the rescaled Hamiltonian,

$$\hat{E} = \frac{1}{2}\left[\hat{p}_r^2 + \left(\hat{\mathcal{P}}_s - b_a\frac{\hat{r}^2}{2}\right)^2\right] + v'_{0,a}\hat{s} + v_{1,a}\hat{r}, \tag{8}$$

where $v'_{0,a} = v'_0(s_a)$, $v_{1,a} = v_1(s_a)$, $b_a = b(s_a)$. The corresponding equations of motion are

$$\frac{d}{d\hat{t}}\hat{r} = \hat{p}_r, \quad \frac{d}{d\hat{t}}\hat{p}_r = \left(\hat{\mathcal{P}}_s - b_a\frac{\hat{r}^2}{2}\right)b_a\hat{r} - v_{1,a}, \tag{9}$$
$$\frac{d}{d\hat{t}}\hat{s} = \left(\hat{\mathcal{P}}_s - b_a\frac{\hat{r}^2}{2}\right), \quad \frac{d}{d\hat{t}}\hat{\mathcal{P}}_s = -v'_{0,a}.$$

Thus, $\hat{\mathcal{P}}_s = -v'_{0,a}\hat{t}$ and

$$\frac{d^2}{d\hat{t}^2}\hat{r} = \left(-v'_{0,a}\hat{t} - b_a\frac{\hat{r}^2}{2}\right)b_a\hat{r} - v_{1,a}, \tag{10}$$

which is a nonhomogeneous Painlevé II equation.[2]

As we see, $\hat{\mathcal{P}}_s$ decreases monotonically from very large positive to very large negative values. In the nonrescaled variables, this means that the system gets into a regime where $\mathcal{P}_s \ll -\varepsilon^{-1/3}$ after a time interval on the order of $\varepsilon^{2/3}$. In this regime, the separation of scales between the (relatively) slow motion of $(s, \mathcal{P}_s)$ and the fast motion of (r, p_r) reemerges, and the dynamics of (r, p_r) are again described by the Hamiltonian (6) with slowly varying s and $\mathcal{P}_s$, like at the end of the guiding center motion regime. However, $\mathcal{P}_s$ is negative now, so, for frozen s and $\mathcal{P}_s$, this is the motion in a single-well potential, as shown in Fig. 2(a). The oscillations around the minimum of this potential correspond to fast oscillations of the particle around the null line $\mathcal{L}$, on top of the slower drift along $\mathcal{L}$ (the evolution of the s variable).

The action I_0 of small oscillation near the elliptic point in the (r, p_r) plane [Fig. 2(a)] at a given value of $(s, \mathcal{P}_s)$ is an adiabatic invariant. By analyzing Painlevé II equation (10), one finds the relation between I_0 and I. Note that scaling (7) does not change areas in (r, p_r) plane, therefore I and I_0 coincide, for the case of small oscillations, with the adiabatic invariants of the asymptotic limit of Eq. (10) at large negative and, respectively, large positive times. For Painlevé II equation, there are connection formulas that relate behavior of solutions at $\hat{t} \to -\infty$ and $\hat{t} \to +\infty$.[4,5] For homogeneous Painlevé II equation, $(v_{1,a} = 0)$, these formulas were used to obtain explicit connecting formulas for the adiabatic invariants in the small oscillations limit;[10] similar formulas can be obtained in the general nonhomogeneous case. We do not present the connecting formulas here as they are rather involved analytical expressions. Importantly, they give I_0 as a function of I and φ, the phase of Larmor rotation. Since φ rotates fast, the change in the adiabatic invariant is, essentially, random. In the nonrescaled time t, the adiabatic invariant changes over a short interval (on the order of $\varepsilon^{2/3}$). Therefore, we conclude that falling on the null line at the end of the guiding center motion regime is accompanied by a sudden, random jump of the order of 1 in the adiabatic invariant.

IV. DRIFT ALONG THE NULL LINE

The consequent drift along $\mathcal{L}$ is described by the system for $(s, \mathcal{P}_s)$ variables, obtained by averaging over the fast (r, p_r) oscillations. This is a Hamiltonian system that depends on I_0 as a parameter. In the limit of small (hence, harmonic) oscillations

$$I_0 = \frac{\frac{1}{2}\left(p_r^2 + |\mathcal{P}_s|b(s)r^2/\varepsilon\right)}{(|\mathcal{P}_s|b(s)/\varepsilon)^{1/2}}.$$

Since I_0 is on the order of 1, this gives

$$r \sim (\varepsilon/|\mathcal{P}_s|)^{1/4}, \quad p_r \sim (|\mathcal{P}_s|/\varepsilon)^{1/4}. \tag{11}$$

As $|\mathcal{P}_s| \gg \varepsilon^{-1/3}$, these estimates imply that the terms $b(s)\frac{r^2}{2\varepsilon}$, p_r^2, and $v_1(s)r/\varepsilon$ can be neglected in Eq. (6). Thus, the drift along the null line is governed, to the main order, by the Hamiltonian,

$$\mathcal{E} = \frac{1}{2}\mathcal{P}_s^2 + v_0(s)/\varepsilon. \tag{12}$$

In other words, it is the motion in the potential v_0/ε

$$\varepsilon\ddot{s} + \frac{\partial v_0(s)}{\partial s} = 0. \tag{13}$$

Initial values are $s \approx s_a$ and $\mathcal{P}_s = -C\varepsilon^{-1/3}$, with some constant $C \gg 1$. In particular, $\mathcal{P}_s(0) = o(\varepsilon^{-1/2})$, so Eq. (12) implies (in the principal approximation)

$$\dot{s} = \mathcal{P}_s = -\varepsilon^{-1/2}\left(v_0(s_a) - v_0(s)\right)^{1/2}.$$

Hence, a finite length segment of the magnetic field null line is traversed during the time of the order of $\varepsilon^{1/2}$. As $\mathcal{P}_s \sim \varepsilon^{-1/2}$, it follows from (11) that

$$r \sim \varepsilon^{3/8}, \quad p_r \sim \varepsilon^{-3/8}$$

in this regime. The frequency of oscillations of (r, p_r) is $(|b\mathcal{P}_s|/\varepsilon)^{1/2} \sim \varepsilon^{-3/4}$.

When the particle moving along the null line arrives at the point of ejection, $s = s_e$, where $v_0(s_e) = v_0(s_a) = \varepsilon H$ [see Eq. (5)], it can leave a neighborhood of the null line and change to the guiding center mode of motion. This happens as the value of $|\mathcal{P}_s|$ drops to its initial value $O(\varepsilon^{-1/3})$ at $s = s_e$, so the Painlevé II approximation again becomes valid. Thus, the motion in the rescaled coordinates can again be described by system (9), where $v'_{0,a}$, b_a, and $v_{1,a}$ are replaced by $v'_{0,e} = v'_0(s_e)$, $b_e = b(s_e)$, and $v_{1,e} = v_1(s_e)$, respectively. Since $v'_{0,e}$ is negative, $\mathcal{P}_s$ starts growing and achieves large positive values, $\sim 1/\varepsilon$, i.e., the particle switches to Larmor rotation around a guiding center. According to (4), the guiding center departs from the null line with a nonzero speed [as $v'_0(s_e) < 0$]. Again, the switch in the mode of motion leads to a random jump in the adiabatic invariant, so the new phase of the guiding center motion will proceed with a new value of the magnetic moment μ.

V. STATIONARY DISTRIBUTION OF THE MAGNETIC MOMENT

In principle, the two consecutive jumps of the adiabatic invariant (at the point of absorption by $\mathcal{L}$ and at the point of ejection) can compensate each other in some special resonant cases. However, in general, such a compensation does not occur, and the changes in the magnetic moment between different phases of the guiding center motion can be treated as a random process, as confirmed by our numerical experiments [see Fig. 2(c)].

This strong scattering of the magnetic moment (μ) results in a rapid evolution of any initial distribution of μ toward the stationary distribution. For a fixed energy level, $H = h =$ const, the stationary distribution is obtained from the invariant Liouville's measure by integrating over the fast Larmor rotation. It follows that its density is proportional to the time of the guiding center motion from the ejection to the absorption point of the magnetic field null line. To check this conclusion, we numerically integrated 10^6 trajectories with the same h and with narrow initial distribution of μ. Figures 2(e) and 2(f) show that the distribution rapidly evolves to the theoretically predicted one. (The part of the distribution with small μ evolves slower than that with large μ, because the equilibration happens due to the jumps of the adiabatic invariant at the moments of absorption and ejection by the null line and the time interval from the ejection to the absorption is longer for the particles with small μ.)

VI. CONCLUSION

The presence of the magnetic field null line creates new types of particle dynamics in the strong electromagnetic field, which is of interest both for the theory of charged-particle motion and for the general theory of nonlinear dynamics. The particle switches between finite time intervals corresponding to the Larmor rotation [of radius $\sim (\varepsilon\mu)^{1/2}$ and frequency $\sim \varepsilon^{-1}$, where ε^{-1} is the magnitude of the external field] about the guiding center that moves towards the null line, and short (of duration $\sim \varepsilon^{1/2}$) intervals of strongly accelerated motion backwards along the null line. The motion along the null line is controlled by the electrostatic potential [see Eq. (13) and Fig. 2(d)] and is accompanied by oscillations across the null line with frequency $\sim \varepsilon^{-3/4}$ and amplitude $\sim \varepsilon^{3/8}$.

For each Larmor rotation phase, the guiding center follows a different path, determined by the magnetic moment μ. As μ increases, the path approaches closer to the null line [cf. blue and red lines in Fig. 2(c)], but the direction of this motion is opposite to that in the null line capture phase. The paths start at the ejection point and end at the absorption point on the null line, which remain unchanged from one Larmor phase to another and depend only on the total energy of the particle. The jumps in the value of magnetic moment between the two consecutive Larmor phases can be treated as a random process, whose characteristics can be determined from the analysis of the Painlevé II equation that describes the absorption and ejection of the particle by the null line.

This randomness, if we consider a multiparticle situation where the interaction between the particles can be neglected, will lead to a stationary distribution of particles and their magnetic moments, which depends only on the initial distribution of the particles energy.

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