International Journal of Bifurcation and Chaos, Vol. 6, No. 5 (1996) 949–968
Theme Issue (Part 1)

ON THE COMPLEX BIFURCATION SET FOR A SYSTEM WITH SIMPLE DYNAMICS

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Received October 20, 1995

Bifurcations of a heteroclinic contour composed of two equilibrium points of saddle-focus type and two heteroclinic orbits are considered. The case is selected where dynamics of the system is simple, i.e., no more than one periodic orbit is born at bifurcations in a small neighborhood of the contour. In spite of the simplicity of dynamic behavior, the structure of the bifurcation set corresponding to multi-round heteroclinic orbits is shown to be rather complicated. The complete bifurcation analysis is done under some conditions of a general position.

1. Introduction

We consider bifurcations of dynamical systems possessing a contour composed of two saddle-focus equilibrium points and two heteroclinic orbits connecting the equilibria (see Fig. 1). Such a contour can be considered as a generalization of a homoclinic loop with one saddle-focus (Fig. 2). According to the Shil'nikov theorem, saddle-focus homoclinic loops can be of two essentially different types. The homoclinic loops of the first type are associated with chaotic dynamics: in any neighborhood of this loop there exist non-trivial hyperbolic sets including infinitely many saddle periodic orbits, non-trivial recurrent orbits, etc. [Shil'nikov, 1965, 1970]. In fact, the complex behavior near such a loop is far to be exhausted by the presence of hyperbolic sets (see details in Ovsyannikov & Shil'nikov [1986, 1991]) and, till now, the homoclinic loops of a saddle-focus remain as one of the most complicated objects of the modern theory of dynamical systems.

The homoclinic loops of the second type belong to systems with simple dynamics: bifurcations of such loops can lead to the appearance of at most one periodic orbit [Shil'nikov, 1963, 1968]. The two types of homoclinic loops are distinguished by the so-called Shil'nikov condition: if the characteristic exponent[1] nearest to the imaginary axis has a non-zero imaginary part, then the presence of a homoclinic loop implies chaos; otherwise, dynamics is simple near the loop.

Analogously, the set of dynamical systems with a heteroclinic contour containing two saddle-foci can also be decomposed into two classes. The first class is composed of systems for which the Shil'nikov condition is fulfilled at least at one of the saddle-foci. The peculiarity of systems of this class is that either the system itself or a close system has non-trivial hyperbolic sets in a small neighborhood of the heteroclinic contour (this assertion follows obviously from the Shil'nikov theorem, since a homoclinic loop of any saddle-focus can be formed when the contour is split).

The second class consists of systems with simple dynamics, bifurcations of which can lead to the appearance of at most one periodic orbit in a small neighborhood of the contour. The main result of this paper is that the structure of the bifurcation

[1] We assign the term *characteristic exponents* for the roots of the characteristic equation of the system at the saddle-focus.

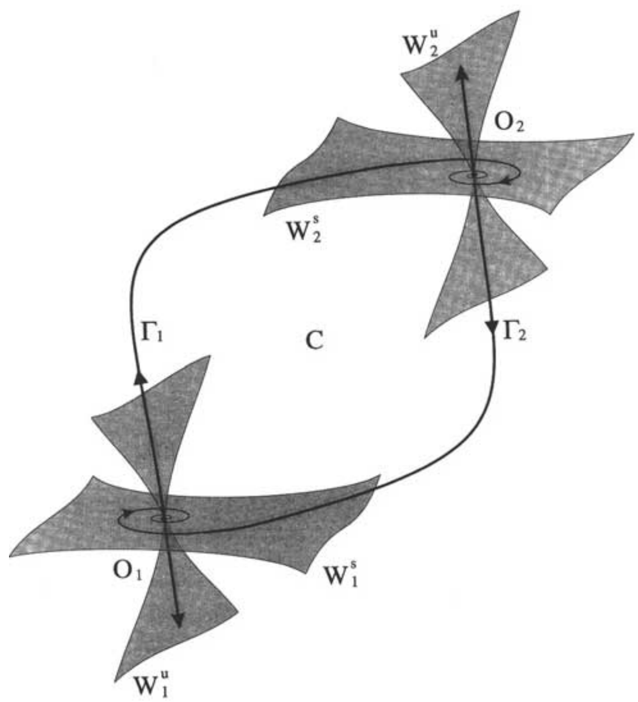

Fig. 1. The stable manifolds W_1^s and W_2^s of the saddle-foci O_1^s and O_2^s intersect with the unstable manifolds W_2^u and W_1^u along the orbits Γ_2 and Γ_1 respectively, forming a heteroclinic contour C.

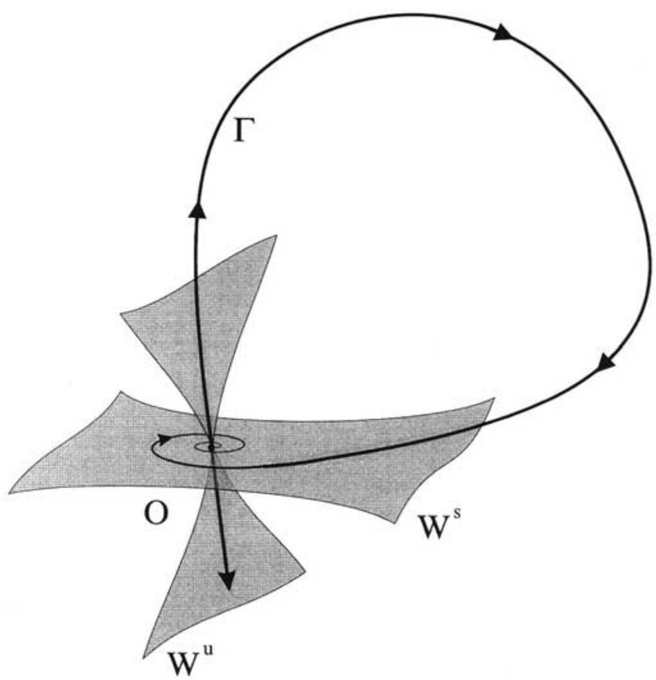

Fig. 2. The unstable manifold W^u of the saddle-focus O intersects the stable manifold W^s along the orbit Γ forming a homoclinic loop.

set corresponding to the formation of multi-round heteroclinic orbits of the saddle-foci turns out to be rather nontrivial even for systems of the second class. Specifically, for systems of the second class, we establish that the two-parameter bifurcation diagram contains a countable set of bifurcation curves corresponding to multi-round heteroclinic orbits. Moreover, these curves intersect the bifurcation sets corresponding to homoclinic loops at an infinite set the closure of which has cardinality of continuum (see Fig. 4 and Theorem 2.1).

Note an evident application of the obtained results: since heteroclinic orbits of an ODE describing plane travelling waves of an extended system correspond to the kink type solutions, our results could allow one to prove an existence of an infinite series of multi-kink solutions (and to investigate their structure) in some cases.

A more precise statement of the problem and the main result of this paper (Theorem 2.1) are given in Sec. 2. We reduce the problem to the three-dimensional case using the existence of an invariant smooth three-dimensional manifold in a small neighborhood of the heteroclinic contour. The proof of Theorem 2.1 is based on the study of the Poincaré map defined by the orbits belonging to the invariant manifold. This map is described in Sec. 3. The main property of the Poincaré map that we use in the proof is that the map is contractive. Some intermediate results implied by the contractivity of the map are proved in Sec. 4. The construction of the bifurcations sets is given in Sec. 5.

2. Statement of the Results

Let X_μ be a two-parameter family of three-dimensional dynamical systems. We suppose that the vector field of X_μ depends smoothly (C^r where $r \geq 1$) on phase variables and parameters. Let

(A) the system X_μ has two structurally stable equilibrium points O_1 and O_2 of the saddle-focus type; namely, the characteristic exponents of O_s are $(\lambda_s^1, \lambda_s^2) = -\alpha_s \pm i\omega_s$, $(\alpha_s > 0$, $\omega_s > 0)$ and $\gamma_s > 0$.

In this case the unstable manifolds W_1^u and W_2^u of O_1 and O_2 are one-dimensional. Each of them consists of three orbits: the saddle-focus itself and two *separatrices* leaving the saddle-focus in opposite directions. The stable manifolds W_1^s and W_2^s are two-dimensional, and all orbits of W_i^s have a shape of spirals tending to O_i as $t \to +\infty$.

Assume that

(B) at $\mu = 0$, the system X_0 has a contour C composed of the equilibrium points O_1, O_2 and by two separatrices $\Gamma_1 \subseteq W_1^u \cap W_2^s$, $\Gamma_2 \subseteq W_2^u \cap W_1^s$ (Fig. 1).

As we have mentioned, homoclinic loops of the saddle-foci O_1 and O_2 can arise at bifurcations of the system X_0. If at least one of the points O_1 or O_2 satisfy the Shil'nikov condition (in our case it is the inequality $\alpha_i < \gamma_i$), then in a small neighborhood of the loop there will exist non-trivial hyperbolic sets. Furthermore, as it follows from Ovsyannikov & Shil'nikov [1986, 1991] homoclinic tangencies may arise in the neighborhood of the loop which cause the appearance of infinitely degenerate periodic orbits [Gonchenko *et al.*, 1991, 1993]. It is clear that the complete description of bifurcations of system X_0 is inadmissible in this case.

Here we consider the opposite case where only one periodic orbit can appear at the bifurcations of the heteroclinic contour C. According to what has been said, we shall assume that the Shil'nikov conditions do not hold. Namely, we require

(C) the saddle values $(\gamma_i - \alpha_i)$ $(i = 1, 2)$ are both strictly less than zero.

We assume also that the two-parameter family X_μ is in a general position; i.e.

(D) in the space of dynamical systems the family X_μ is transverse to the codimension two bifurcational surface composed of systems having a heteroclinic contour close to C.

This condition guarantees that *splitting parameters* for the orbits Γ_1 and Γ_2 can be chosen to be the control parameters. It is convenient to denote the splitting parameter for the separatrix Γ_1 as μ_2 and the splitting parameter for the separatrix Γ_2 as μ_1. Specifically (see Fig. 3), if G_i^* is the point of intersection of the separatrix Γ_i with some cross section S_j constructed near the point O_j $(j = 3 - i)$, then μ_i is the distance[2] between G_i^* and the line of intersection of W_j^s with the same cross section.

We will show (Lemma 4.3) that there exists a small neighborhood U of C such that the system X_μ has no more than one periodic orbit in U for μ small enough (the analogous result was earlier proved in Chow *et al.* [1990]. The periodic orbit is attractive (its multipliers lie strictly inside the unit circle) and it bifurcates merging in a homoclinic loop of one of the saddle-foci.

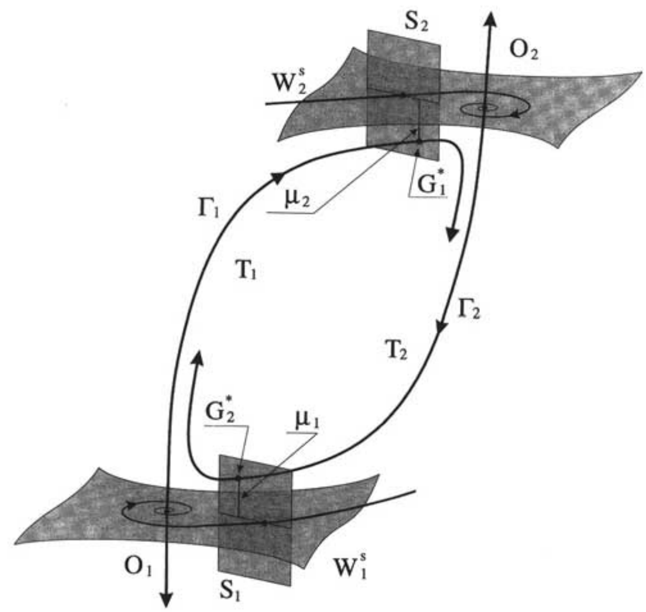

Fig. 3. Condition (D) guarantees that parameter μ_i $(i = 1, 2)$ can be chosen to be equal to the distance (taken with the sign) from the point G_j^* to the stable manifold W_i^s of the equilibrium state O_i where G_j^* is the first point of intersection of the separatrix Γ_j $(j = 3 - i)$ with cross section S_i constructed near O_i.

We denote the curve on the plane (μ_1, μ_2) that corresponds to the presence of a homoclinic loop of O_i as L_i. The curves L_i will be proved (see Eqs. (5.2)) to be the graphs of some smooth functions $\mu_1 = h_1(\mu_2)$ and $\mu_2 = h_2(\mu_1)$, respectively, $h_i(0) = h_i'(0) = 0$. In the region between L_1 and L_2 (i.e., in the region $\{\mu_1 > h_1(\mu_2),\ \mu_2 > h_2(\mu_1)\}$) the system X_μ has the unique periodic orbit and X_μ has no periodic orbits if μ does not belong to this region.

The bifurcation diagram (Fig. 4) for the family X_μ contains also the curves C_{12}^k and C_{21}^k $(k = 0, 1, 2, \ldots)$ such that for $\mu \in C_{ij}^k$ the system has an orbit of the heteroclinic connection which goes from the saddle-focus O_i, makes k rounds along U and enters the saddle-focus O_j. The curves C_{12}^k and C_{21}^k are given by equations $\mu_2 = h_{12}^k(\mu_1)$ and $\mu_1 = h_{21}^k(\mu_2)$, respectively, where h_{ij}^k are some smooth functions such that the derivatives of h_{ij}^k are bounded by a small constant, independent of k, which can be made arbitrarily small as μ tends to zero.

[2]Taken with a positive sign if G_i^* lies "above" W_j^s, and with negative sign if G_i^* lies "below" W_j^s.

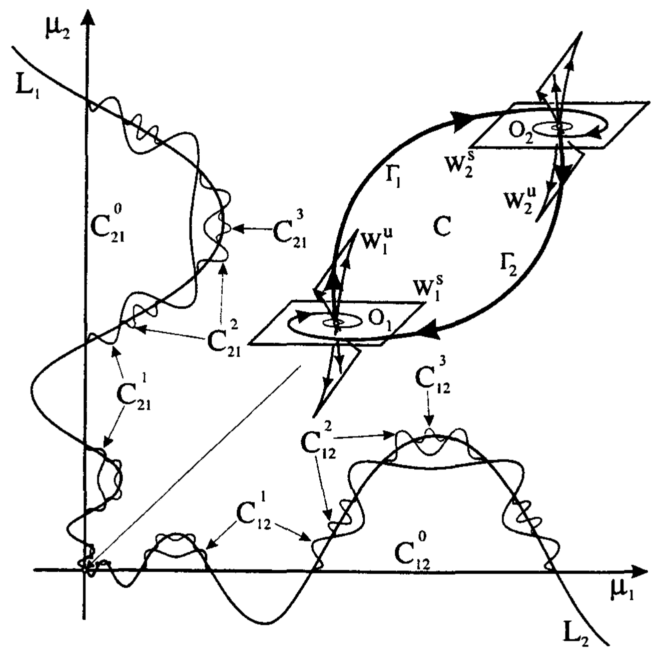

Fig. 4. The bifurcation diagram for the system X_μ. The curve L_i corresponds to a single-round homoclinic loop of the saddle-focus O_i. The curve C^k_{ij} corresponds to a heteroclinic orbit which goes from the saddle-focus O_i, makes k rounds along the contour and enters the saddle-focus O_j.

The following theorem gives a description of the bifurcational set corresponding to the multi-round saddle-foci connections.

Theorem 2.1. *The set of the curves C^k_{12} and C^k_{21} is organized according to the following inductive rule:*

1. $\mu_2 = 0$ *is the line* C^0_{12}, $\mu_1 = 0$ *is the line* C^0_{21}*;*
2. *The lines* C^0_{12} *and* C^0_{21} *intersect, respectively,* L_2 *and* L_1 *infinitely many times;*
3. *For each* $k = 0, 1, 2, \ldots$, *any two points of intersection of* L_1 *and* C^k_{21} (L_2 *and* C^k_{12}) *such that the inequality* $h_1(\mu_2) > h^k_{21}(\mu_2)$ (*respectively,* $h_2(\mu_1) > h^k_{12}(\mu_1)$) *holds between these points, are connected by a piece of the curve* C^{k+1}_{21} (*respectively,* C^{k+1}_{12}) *which, in turn, intersects* L_1 (*respectively,* L_2) *infinitely many times.*

The limit set of the points $L_i \cap C^k_{ji}$, ($k = 0, 1, 2, \ldots, i = 1, 2, j = 1, 2, i \neq j$), *has cardinality of continuum and corresponds to the presence of an orbit homoclinic to the saddle-focus* O_i *and an orbit which goes from the point* O_j *and winds to the homoclinic loop.*

Note that the analogous result holds for the multi-dimensional case. This can be proved by reduction to a global invariant three-dimensional "center" manifold the existence of which can be established under certain conditions. If X_μ is a two-parameter family of C^r-smooth ($r \geq 1$) $(n+m)$-dimensional ($n \geq 1$, $m \geq 2$) systems possessing two equilibrium states O_1 and O_2 with characteristic exponents $\lambda^m_s, \ldots, \lambda^1_s, \gamma^1_s, \ldots, \gamma^n_s$ ($s = 1, 2$) ordered so that

$$\mathrm{Re}\,\lambda^m_s \leq \cdots \leq \mathrm{Re}\lambda^1_s < 0 < \mathrm{Re}\,\gamma^1_s \leq \mathrm{Re}\gamma^n_s\,,$$

then Condition (A) is rewritten as

$$(\mathrm{A}') \qquad (\lambda^1_s,\, \lambda^2_s) = -\alpha_s \pm i\omega_s\,(\alpha_s > 0,\, \omega_s > 0)\,, \quad \gamma^1_s = \gamma_s > 0\,,$$

$$\mathrm{Re}\,\lambda^m_s \leq \cdots \leq \mathrm{Re}\lambda^3_s < -\alpha_s < 0 < \gamma_s < \mathrm{Re}\,\gamma^2_s \leq \cdots \leq \mathrm{Re}\,\gamma^n_s$$

Conditions (B)–(D) remain unchanged.

In this case the dimensions of the stable manifolds W^s_1 and W^s_2 are equal to m, and the dimensions of the unstable manifolds W^u_1 and W^u_2 are equal to n. In W^u_i there exists an $(n-1)$-dimensional strong-unstable invariant manifold W^{uu}_i. The main feature characterizing the strong-unstable manifold is that all orbits in it tend to O_i (as $t \to -\infty$) being tangent to the eigenspace corresponding to the eigenvalues $\gamma^2_i, \ldots \gamma^n_i$ whereas all orbits of $W^u_i \backslash W^{uu}_i$ are tangent, as $t \to -\infty$, to the eigendirection corresponding to the *leading* eigenvalue γ^1_i.

In W^s_i there exists an $(m-2)$-dimensional strong-stable manifold W^{ss}_i such that all orbits of $W^s_i \backslash W^{ss}_i$ have a shape of spirals which tend to O_i as $t \to \infty$ approaches the two-dimensional eigenspace corresponding to the pair of leading eigenvalues $(\lambda^1_i,\, \lambda^2_i)$.

Assume that

(E) the orbits Γ_1 and Γ_2 do not lie in the submanifolds W^{uu}_1, W^{ss}_2 and W^{uu}_2, W^{ss}_1, respectively.

This assumption means that the orbits Γ_1 and Γ_2 leave and enter the saddle-foci O_1 and O_2 along eigenspaces corresponding to the leading eigenvalues (Fig. 5).

The next (and final) assumption is necessary for the presence of the three-dimensional global invariant manifold (as well as Condition (E)). Denote by $E^{s+}_i \subset R^{m+1}$ and $E^{u+}_i \subset R^{n+2}$, ($i = 1, 2$) the eigenspaces of the linearization matrix of the

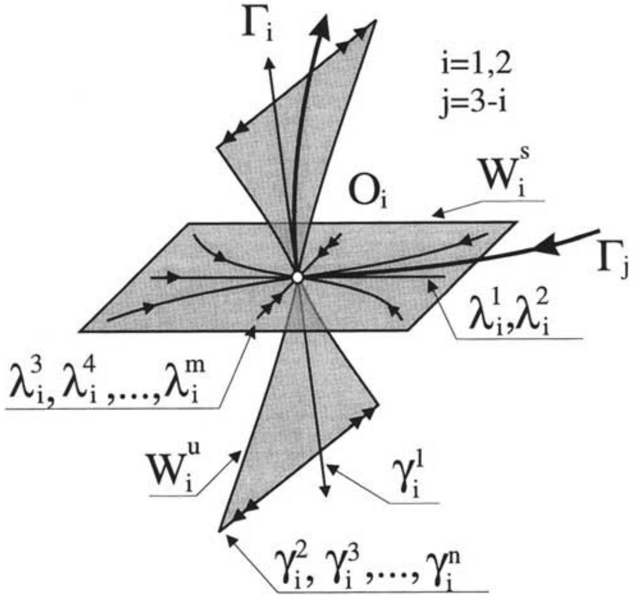

Fig. 5. Condition (E) means that, at $\mu = 0$, the separatrix Γ_i ($i = 1, 2$) leaves O_i along the eigen-direction corresponding to the leading characteristic exponent $\gamma_i^1 \equiv \gamma_i$. The separatrix Γ_j ($j = 3 - i$) enters O_i approaching the two-dimensional eigen-plane corresponding to the complex-conjugate leading characteristic exponents λ_i^1 and λ_i^2.

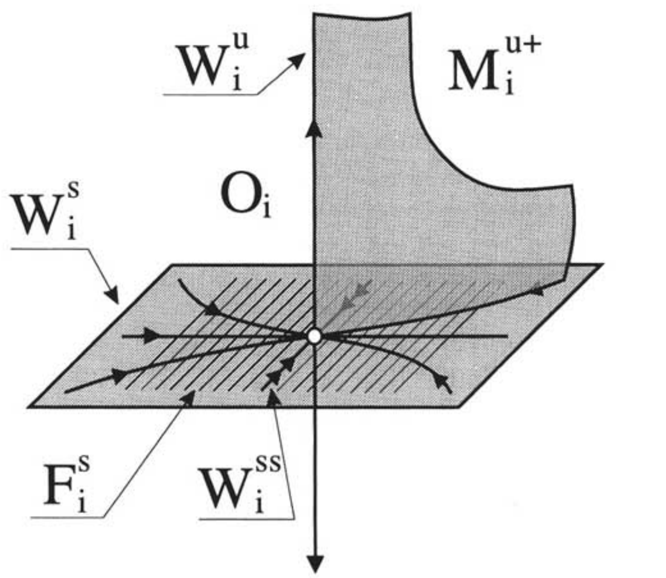

Fig. 7. There exists an invariant C^1-manifold M_i^{u+}, containing W_i^u, which is tangent at O_i to the eigenspace corresponding to the characteristic exponents $\lambda_i^2, \lambda_i^1, \gamma_1^1, \ldots, \gamma_1^n$. The manifold M_i^{s+} is not unique, but any two of them have a common tangents everywhere on W_i^u. The strong-stable submanifold W_i^{ss} is uniquely embedded into a smooth invariant codimension two foliation F_i^s on W_i^s.

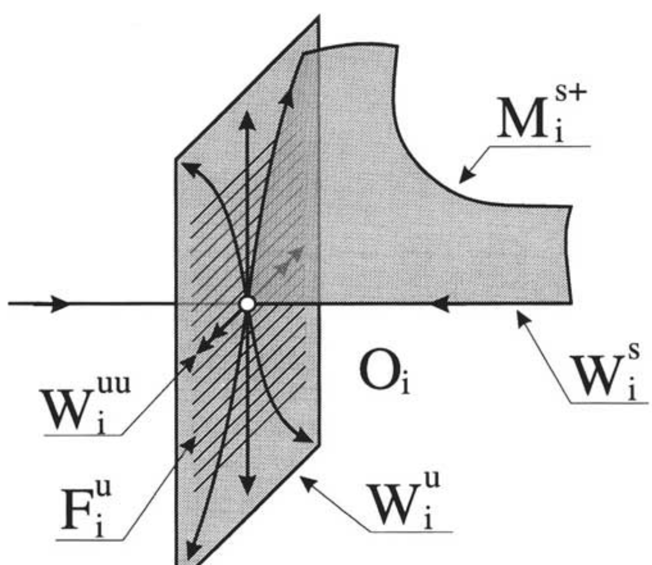

Fig. 6. There exists a C^1-smooth invariant manifold M_i^{s+}, containing W_i^s, which is tangent at O_i to the eigenspace corresponding to characteristic exponents $\lambda_i^m, \ldots, \lambda_i^1, \gamma_1^1$. The manifold M_i^{s+} is not uniquely defined, but any two of such manifolds have a common tangent everywhere on W_i^s. The strong-unstable manifold W_i^{uu} is uniquely embedded in a smooth invariant codimension one foliation F_i^u on W_i^u.

system X_0 at the point O_i which correspond to the eigenvalues $\lambda_i^m, \ldots, \lambda_i^1, \gamma_i^1$ and $\lambda_i^2, \lambda_i^1, \gamma_i^1, \ldots, \gamma_i^n$, respectively. It is well known [Hirsh *et al.*, 1977] that there exists an invariant C^1-smooth manifold M_i^{s+} tangent to E_i^{s+} at O_i (Fig. 6). The manifold M_i^{s+} contains W_i^s. It is not uniquely defined but any two of them have the same tangent at each point of W_i^s. Analogously, there exists an invariant C^1-manifold M_i^{u+} tangent to E_i^{u+} at O_i, containing W_i^u and possessing a uniquely defined tangent at each point of W_i^u (Fig. 7).

Moreover, it is known that the strong-unstable submanifold W_i^{uu} is uniquely embedded into the smooth invariant foliation F_i^u on the manifold W_i^u (Fig. 7), and the strong-stable manifold W_i^{ss} is uniquely embedded into the smooth invariant foliation F_i^s on W_i^s (Fig. 6). We require the following condition to be fulfilled.

(F) At each point of Γ_1 the manifolds M_2^{s+} and M_1^{u+} are transverse to the leaves of the foliations, respectively, F_1^u and F_2^s, and at each point of Γ_2 the manifolds M_1^{s+} and M_2^{u+} are respectively transverse to the leaves of F_2^u and F_1^s.

Note that Condition (F) must be verified only at one point on Γ_1 and at one point on Γ_2, because the manifolds M_i^{u+}, M_i^{s+} and the foliations F_i^u, F_i^s are invariant with respect to the flow defined by the system X_0. It should be noted also that the dimension of the manifold M_i^{u+} (M_i^{s+}) and the dimension of the leaves of the foliation F_i^s (respectively, F_i^u) complement each other. It means that Condition (F) is a condition of a general position as well as Conditions (A), (D) and (E).

By the methods of Turaev [1984], Shashkov [1991], Homburg [1993], and Sandstede [1994] where analogous results were proved for some different types of homoclinic loops and heteroclinic contours, the following theorem can be established (we postpone the proof for a forthcoming paper [Shashkov *et al.*, 1995])

Theorem 2.2. *If Conditions (A′), (B), (E), (F) are fulfilled, then there exists a small neighborhood U of the heteroclinic contour C such that, for all* μ *small enough, the system* X_μ *has a three-dimensional invariant* C^1*-smooth manifold* $\mathcal{M}_\mu^c$ *smoothly dependent on* μ *and such that any orbit not lying in* $\mathcal{M}_\mu^c$ *leaves U as t tends either to* $+\infty$ *or to* $-\infty$. *The manifold* $\mathcal{M}_\mu^c$ *is tangent at the point* O_i *to the eigenspace corresponding to the leading characteristic exponents* $(\lambda_i^1, \lambda_i^2, \gamma_i^1)$.

By this theorem the study of multi-dimensional systems satisfying Conditions (A′)–(F) is reduced to the study of the three-dimensional system on the invariant manifold $\mathcal{M}_\mu^c$. Evidently, the reduced system satisfies Conditions (A)–(D), therefore, the main Theorem 2.1 holds true for the multi-dimensional case.

3. Poincaré Map

According to the reduction Theorem 2.2 we may restrict ourselves to the three-dimensional case. In a neighborhood of the saddle-focus O_i there can be introduced local coordinates (x_i, y_i, z_i) such that the system X_μ takes the form

$$\begin{cases} \dot{x}_i = \gamma_i x_i + \cdots \\ \dot{y}_i = -\alpha_i y_i - \omega_i z_i + \cdots \\ \dot{z}_i = \omega_i y_i - \alpha_i z_i + \cdots \end{cases} \tag{3.1}$$

where dots stand for nonlinearities. Here $\gamma_i > 0$, $\alpha_i > 0$ and $\omega_i > 0$. By Condition (C) we have

$$-\alpha_i + \gamma_i < 0 \,. \tag{3.2}$$

The stable manifold W_i^s of O_i is a two-dimensional surface which, when $\mu = 0$, is tangent to the plane $x_i = 0$ at the point $O_i = (0, 0, 0)$. This means that W_i^s is locally the graph of a C^1-function

$$x_i = x_i^s(y_i, z_i, \mu) \tag{3.3}$$

where

$$x_i^s(0, 0, \mu) = 0 \,, \quad \left.\frac{\partial x_i^s(y_i, z_i, \mu)}{\partial(y_i, z_i)}\right|_{(y_i, z_i, \mu)=0} = 0 \,. \tag{3.4}$$

The unstable manifold W_i^u of O_i is locally the graph of a C^1-function

$$(y_i, z_i) = (y_i^u(x_i, \mu), z_i^u(x_i, \mu)) \tag{3.5}$$

where

$$\begin{gathered} (y_i^u(0, \mu), z_i^u(0, \mu)) = 0 \,, \\ \left.\frac{\partial(y_i^u(x_i, \mu), z_i^u(x_i, \mu))}{\partial x_i}\right|_{(x_i, \mu)=0} = 0 \,. \end{gathered} \tag{3.6}$$

The manifold W_i^u consists of three orbits: the saddle-focus O_i and two separatrices one of which is the orbit Γ_i connecting O_i and O_j ($j = 3 - i$) at $\mu = 0$. Without loss of generality we assume that the orbit Γ_i leaves O_i tangent to the positive x_i-axis.

In this case, if $\delta > 0$ and $x_i^- > 0$ are small enough, then the surface

$$S_i^{out} = \{(x_i, y_i, z_i) | x_i = x_i^- \,, \; \|y_i - y_i^-, z_i - z_i^-\| \le \delta\}, \tag{3.7}$$

is, for small μ, a cross section for the orbits close to Γ_i; here (x_i^-, y_i^-, z_i^-) is the point of the first intersection of Γ_i with the plane $x_i = x_i^-$.

At $\mu = 0$, the orbit Γ_i tends to O_j ($j = 3 - i$) as a spiral intersecting the plane $z_j = 0$ in a countable sequence of points accumulating at O_j. Take one of these points with the coordinates $(x_j^*, y_j^*, 0)$. The surface

$$S_j = \{(x_j, y_j, z_j) | z_j = 0 \,, \; \|x_j - x_j^*, y_j - y_j^*\| \le \delta\} \tag{3.8}$$

is a cross section for the orbits close to Γ_i if δ and μ are sufficiently small.

Thus, we have constructed four cross sections: S_1^{out} and S_2 to the separatrix Γ_1 and S_2^{out} and S_1 to the separatrix Γ_2. The surfaces S_1 and S_1^{out} lie in a small neighborhood of O_1, and the surfaces S_2 and S_2^{out} lie in a small neighborhood of O_2.

Let us introduce a coordinate u_i instead of the coordinate x_i on S_i such that

$$u_i = x_i - x_i^s(y_i, 0, \mu). \tag{3.9}$$

The intersection of the stable manifold W_i^s with S_i is the line $\{u_i = 0\}$ which divides S_i into two parts: $S_i^+ = \{u_i > 0\}$ and $S_i^- = \{u_i < 0\}$. We will also use the notation S_i^0 for the line $\{u_i = 0\}$.

On the cross section S_i^{out} we introduce coordinates (v_i^{out}, w_i^{out}) such that

$$v_i^{out} = y_i - y_i^u(x_i^-, \mu), \quad w_i^{out} = z_i - z_i^u(x_i^-, \mu). \tag{3.10}$$

The coordinates of the point of the first intersection of Γ_i with S_i^{out} are now $(v_i^{out} = 0, w_i^{out} = 0)$.

We study bifurcations in a small neighborhood U of the heteroclinic contour $C = O_1 \cup O_2 \cup \Gamma_1 \cup \Gamma_2$ which exists at $\mu = 0$. Any orbit that stays in U for all times intersects one of the cross sections S_i at least once. This intersection point cannot lie below W_i^s (in the region S_i^-), because the orbit would leave U in this case. If the intersection point belongs to W_i^s (i.e., if $u_i = 0$), the orbit is asymptotic to O_i. If, finally, the coordinate u_i of the intersection point is small and positive, the orbit will pass near O_i and intersect the cross section S_i^{out} (the smaller the starting value u_i, the closer is the intersection with S_i^{out} to the point $(v_i^{out} = 0, w_i^{out} = 0)$). After that, the orbit will pass near the orbit Γ_i and intersect the cross section S_j $(j = 3 - i)$ near the point $G_i^* = \Gamma_i \cap S_j$.

Thus, the flow near the heteroclinic contour C defines a pair of *the half-Poincaré maps*: $T_1 : S_1^+ \to S_2$ and $T_2 : S_2^+ \to S_1$. Moreover, dynamics near the contour is completely determined by behavior of their superposition $T_2 \circ T_1$ (or $T_1 \circ T_2$). Since $\lim_{u_i \to +0} T_i(u_i, y_i) = G_i^*$, we may assume $T_i|_{u_i=0} \equiv G_i^*$.

The main result of this section is given by the following lemma which, mainly, says that the maps T_i are contracting. If the system were at least C^2, this result would follow from Shil'nikov [1963]. We have to consider the C^1-smooth case because the invariant manifold in Theorem 2.2 is, in general, only C^1. Correspondingly, the reduced three-dimensional system may be only C^1 also, independently of the smoothness of the initial multidimensional system.

Lemma 3.1. *The half-Poincaré map* $T_i : (u_i, y_i) \mapsto (u_j, y_j)$ *can be written in the form*

$$T_i : \begin{cases} u_j - \mu_j = f_i(u_i, y_i, \mu) \\ y_j - y_j^*(\mu) = g_i(u_i, y_i, \mu) \end{cases} \tag{3.11}$$

where the functions f_i, g_i *are* C^1 *with respect to* (u_i, y_i, μ) *at all small* $u_i \geq 0$, *the function* y_j^* *depends smoothly on* μ, *and the following identities hold*

$$f_i(u_i = 0, y_i, \mu) \equiv 0, \quad g_i(u_i = 0, y_i, \mu) \equiv 0, \tag{3.12}$$

$$\left\| \frac{\partial(f_i, g_i)}{\partial(u_i, y_i, \mu)} \right\|_{u_i=0} \equiv 0. \tag{3.13}$$

Proof. The map T_i is the superposition of two maps: T_{ii} acting from S_i^+ to S_i^{out} and T_{ij} acting from S_i^{out} to S_j $(j = 3 - i)$. The flight time from S_i^{out} to S_j is bounded, therefore, the map $T_{ij} : (v_i^{out}, w_i^{out}) \mapsto (u_j, y_j)$ is a diffeomorphism and it can be written in the form

$$T_{ij} : \left\{ \begin{pmatrix} u_j - u_j^* \\ y_j - y_j^* \end{pmatrix} = A_i \begin{pmatrix} v_i^{out} \\ w_i^{out} \end{pmatrix} + \cdots \right. \tag{3.14}$$

where (u_j^*, y_j^*) are the coordinates of the point $G_i^* = \Gamma_i \cap S_j$ which depends smoothly on μ; dots stand for nonlinear terms. The matrix A_i is non-degenerate, i.e., $\det(A_i) \neq 0$.

By definition, the value u_j^* is the splitting parameter for the orbit Γ_i (see Fig. 3). Condition (B) that guarantees the existence of the heteroclinic contour C at $\mu = 0$ implies that $u_1^* = 0$ and $u_2^* = 0$ at $\mu = 0$. Condition (D) of the transversality of the two-parameter family X_μ to the codimension two bifurcational surface means that the Jacobian

$$\det \left\| \frac{\partial(u_1^*(\mu), u_2^*(\mu))}{\partial(\mu_1, \mu_2)} \right\|_{\mu=0}$$

is not equal to zero. Therefore, without loss of generality we may assume

$$u_1^*(\mu) \equiv \mu_1, \quad u_2^*(\mu) \equiv \mu_2. \tag{3.15}$$

Formula (3.14) for the map T_{ij} is now rewritten as

$$T_{ij} : \left\{ \begin{pmatrix} u_j - \mu_j \\ y_j - y_j^*(\mu) \end{pmatrix} = A_i \begin{pmatrix} v_i^{out} \\ w_i^{out} \end{pmatrix} + \cdots \right. \tag{3.16}$$

where $y_j^*(\mu)$ is a smooth function of $\mu = (\mu_1, \mu_2)$.

The study of the map T_{ii} is not so easy because the flight time from S_i to S_i^{out} is unbounded (it tends to infinity when the starting point tends to the stable manifold W_i^s). The regular method of the study of such kinds of maps near a saddle equilibrium state is based on the solution of the so-called *Shil'nikov problem* which can be formulated in our case as follows:

For given $\tau \geq 0$ and small x_i^1, y_i^0, z_i^0, to find an orbit $\{x_i(t), y_i(t), z_i(t)\}$, lying in a small neighborhood of the equilibrium state O_i, such that $x_i(\tau) = x_i^1, y_i(0) = y_i^0, z_i(0) = z_i^0$.

According to Shil'nikov [1967], the unique solution of this boundary-value problem exists for any $\tau \geq 0$ if x_i^1, y_i^0 and z_i^0 are sufficiently small. Moreover, the solution depends smoothly on the data τ, x_i^1, y_i^0, z_i^0 and μ. In other words:

There exist smooth functions X_i, Y_i and Z_i such that the orbit which, at $t = 0$, starts with a point (x_i^0, y_i^0, z_i^0) reaches a point (x_i^1, y_i^1, z_i^1) at $t = \tau$ if and only if

$$\begin{aligned} x_i^0 &= X_i(x_i^1, y_i^0, z_i^0, \tau, \mu), \\ y_i^1 &= Y_i(x_i^1, y_i^0, z_i^0, \tau, \mu), \\ z_i^1 &= Z_i(x_i^1, y_i^0, z_i^0, \tau, \mu). \end{aligned} \tag{3.17}$$

If we fix $x_i^1 = x_i^-$ and $z_i^0 = 0$, the last two equations of system (3.17) will give us the map $T_{ii} : S_i^+ \to S_i^{out}$ where τ should be expressed from the first equation of system (3.17) as a function of $(x_i^0, y_i^0) \in S_i^+$ and μ:

$$\begin{aligned} y_i^{out} &= Y_i(x_i^-, y_i^0, 0, \tau(x_i^0, y_i^0, \mu), \mu), \\ z_i^{out} &= Z_i(x_i^-, y_i^0, 0, \tau(x_i^0, y_i^0, \mu), \mu) \end{aligned} \tag{3.18}$$

where

$$x_i^0 = X_i(x_i^-, y_i^0, 0, \tau(x_i^0, y_i^0, \mu), \mu). \tag{3.19}$$

We will show (formula (3.35)) that $\frac{\partial X_i}{\partial \tau} \neq 0$. Therefore, the flight time τ can be found from Eq. (3.19) instead. Notice the relations which follow from Eq. (3.19):

$$\begin{aligned} \frac{\partial \tau}{\partial x_i^0} &= \left(\frac{\partial X_i}{\partial \tau}\right)^{-1}, \\ \frac{\partial \tau}{\partial y_i^0} &= -\left(\frac{\partial X_i}{\partial \tau}\right)^{-1} \frac{\partial X_i}{\partial y_i^0}, \\ \frac{\partial \tau}{\partial \mu} &= -\left(\frac{\partial X_i}{\partial \tau}\right)^{-1} \frac{\partial X_i}{\partial \mu}. \end{aligned} \tag{3.20}$$

By Eqs. (3.18) and (3.20)

$$\begin{aligned} \frac{\partial(y_i^{out}, z_i^{out})}{\partial x_i^0} &= \frac{\partial(Y_i, Z_i)}{\partial \tau}\frac{\partial \tau}{\partial x_i^0} = \frac{\partial(Y_i, Z_i)}{\partial \tau}\left(\frac{\partial X_i}{\partial \tau}\right)^{-1}, \\ \frac{\partial(y_i^{out}, z_i^{out})}{\partial y_i^0} &= \frac{\partial(Y_i, Z_i)}{\partial y_i^0} + \frac{\partial(Y_i, Z_i)}{\partial \tau}\frac{\partial \tau}{\partial y_i^0} = \frac{\partial(Y_i, Z_i)}{\partial y_i^0} - \frac{\partial(Y_i, Z_i)}{\partial \tau}\left(\frac{\partial X_i}{\partial \tau}\right)^{-1}\frac{\partial X_i}{\partial y_i^0}, \\ \frac{\partial(y_i^{out}, z_i^{out})}{\partial \mu} &= \frac{\partial(Y_i, Z_i)}{\partial \mu} + \frac{\partial(Y_i, Z_i)}{\partial \tau}\frac{\partial \tau}{\partial \mu} = \frac{\partial(Y_i, Z_i)}{\partial \mu} - \frac{\partial(Y_i, Z_i)}{\partial \tau}\left(\frac{\partial X_i}{\partial \tau}\right)^{-1}\frac{\partial X_i}{\partial \mu}. \end{aligned} \tag{3.21}$$

We will show that all the derivatives $\frac{\partial(y_i^{out}, z_i^{out})}{\partial(x_i^0, y_i^0, \mu)}$ have a limit as the flight time τ tends to infinity. Moreover

$$\lim_{\tau \to +\infty} \frac{\partial(y_i^{out}, z_i^{out})}{\partial(x_i^0, y_i^0)} = 0. \tag{3.22}$$

Note that the limit $\tau = +\infty$ corresponds to the starting point on the stable manifold W_i^s, or, what is the same, to the coordinate $u_i \equiv x_i^0 - x_i^s(y_i^0, \mu)$ of the starting point equal to zero. Thus, the presence (shown below) of a finite limit for the derivatives as $\tau \to +\infty$ means that the map T_{ii} remains smooth for all small $u_i \geq 0$. The same holds true for the superposition $T_i = T_{ij} \circ T_{ii}$ (since the map T_{ij} is smooth, see above). This is the first part of the statement of the lemma.

The second part of the lemma consists of identities (3.12), (3.13). Formulas (3.12) follow from the definition of μ_j and $y_j^*(\mu)$: as the coordinate u_i

of the starting point tends to zero, the image of the point by the map T_i tends to the point $G_i^* = \Gamma_i \cap S_j$ whose coordinates are, exactly, $(\mu_i, y_i^*(\mu))$. To prove identities (3.13), note that since the functions f_i, g_i vanish identically at $u_i = 0$, their derivatives with respect to y_i and μ also vanish (their existence is given by the first part of the lemma). As for the vanishing of the derivative with respect to u_i, it follows from the vanishing of derivatives of T_{ii} (formula (3.22)) and from the boundedness of the derivatives of T_{ij}.

To complete the proof of the lemma, we need, thus to obtain suitable estimates for the derivatives in Eqs. (3.21). Note that the solution of the Shil'nikov problem is found as a fixed point of some integral operator which is defined and uniformly contracting for all $\tau \geq 0$, including $\tau = +\infty$ (see Shil'nikov [1967]). Therefore, all derivatives in the right-hand side of formula (3.21) have a finite limit as $\tau \to +\infty$ and it is, therefore, sufficient for our purposes to prove that

$$\lim_{\tau\to+\infty} \frac{\partial(Y_i, Z_i)}{\partial y_i^0} = 0 \tag{3.23}$$

and

$$\lim_{\tau\to+\infty} \frac{\partial(Y_i, Z_i)}{\partial \tau}\left(\frac{\partial X_i}{\partial \tau}\right)^{-1} = 0 \tag{3.24}$$

We use the following result which can be extracted from Shil'nikov [1967]:

The functions X_i, Y_i and Z_i satisfy the estimates

$$\left|\frac{\partial X_i}{\partial x_i^1}\right| \leq Be^{-(\gamma_i-\xi)\tau}, \quad \left\|\frac{\partial(Y_i, Z_i)}{\partial(y_i^0, z_i^0)}\right\| \leq Be^{-(\alpha_i-\xi)\tau},$$
$$\left\|\frac{\partial X_i}{\partial(y_i^0, z_i^0, \mu)}\right\| \leq B, \quad \left\|\frac{\partial(Y_i, Z_i)}{\partial(x_i^1, \mu)}\right\| \leq B, \tag{3.25}$$

where B and ξ are positive constants; moreover, ξ can be made arbitrarily small by diminishing the size of the neighborhood of the equilibrium state where the considerations are carried out.

Note that Eq. (3.23) follows immediately from the upper right of inequalities (3.25). Therefore, it remains to prove only relation (3.24).

To find estimates for the derivatives of X_i, Y_i, Z_i with respect to τ we use the following trick. Note that if $\{x_i(t), y_i(t), z_i(t)\}$ is the orbit that starts with the point (x_i^0, y_i^0, z_i^0) at $t = 0$ and passes through the point (x_i^1, y_i^1, z_i^1) at $t = \tau$, then the evident identity follows from the definition of functions (X_i, Y_i, Z_i) as the solutions of the boundary-value problem (see Eq. (3.17)):

$$\begin{aligned} X_i(x_i(\tau+\Delta\tau), y_i^0, z_i^0, \tau+\Delta\tau, \mu) &= x_i^0, \\ Y_i(x_i^1, y_i(-\Delta\tau), z_i(-\Delta\tau), \tau+\Delta\tau, \mu) &= y_i^1, \\ Z_i(x_i^1, y_i(-\Delta\tau), z_i(-\Delta\tau), \tau+\Delta\tau, \mu) &= z_i^1. \end{aligned} \tag{3.26}$$

The differentiation of Eqs. (3.26) with respect to $\Delta\tau$ at $\Delta\tau = 0$ gives us the following identities:

$$\frac{\partial X_i}{\partial \tau} = -\frac{\partial X_i}{\partial x_i^1}\dot{x}_i|_{t=\tau} \tag{3.27}$$

and

$$\frac{\partial(Y_i, Z_i)}{\partial \tau} = \frac{\partial(Y_i, X_i)}{\partial(y_i^0, z_i^0)}(\dot{y}_i, \dot{z}_i)|_{t=0}. \tag{3.28}$$

Now, by (3.25) we have

$$\frac{\partial(Y_i, Z_i)}{\partial \tau} = O(e^{-(\alpha_i-\xi)\tau}) \to 0 \tag{3.29}$$

as $\tau \to +\infty$.

Note that all the time that the orbit $\{x_i(t), y_i(t), z_i(t)\}$ lies in a small neighborhood of the equilibrium state O_i, the following estimate holds

$$\frac{d}{dt}\left\|\frac{\partial(x_i(t), y_i(t), z_i(t))}{\partial x_i^0}\right\| \leq (\gamma_i+\varepsilon)\left\|\frac{\partial(x_i(t), y_i(t), z_i(t))}{\partial x_i^0}\right\| \tag{3.30}$$

due to the fact that the spectrum of the linearization matrix of the system at the point O_i lies to the left of the straight line $\mathrm{Re}(\cdot) = \gamma_i$ on the complex plane; here $\varepsilon > 0$ is some small constant. Inequality (3.30) implies that

$$\left|\frac{\partial x_i(t)}{\partial x_i^0}\right| \leq B_1 e^{(\gamma_i+\varepsilon)t} \tag{3.31}$$

for some constant B_1.

By differentiating the equality

$$x_i^0 = X_i(x_i^1, y_i^0, z_i^0, \tau, \mu) = X_i(x_i(\tau), y_i^0, z_i^0, \tau, \mu),$$

we have

$$\left.\frac{\partial X_i}{\partial x_i^1}\frac{\partial x_i}{\partial x_i^0}\right|_{t=\tau} = 1. \tag{3.32}$$

This means that

$$\frac{\partial X_i}{\partial x_i^1} \neq 0 \qquad (3.33)$$

and, by (3.31),

$$\left|\frac{\partial X_i}{\partial x_i^1}\right| \geq \frac{1}{B_1} e^{-(\gamma_i+\varepsilon)\tau} . \qquad (3.34)$$

Since the orbit $\{x_i(t), y_i(t), z_i(t)\}$ intersects the cross section $S_i^{out} : \{x_i = x_i^-\}$ transversely at $t = \tau$, it follows that $\dot{x}_i|_{t=\tau} \neq 0$. Therefore, by (3.33) and (3.27)

$$\frac{\partial X_i}{\partial \tau} \neq 0 \qquad (3.35)$$

and, due to (3.34), (3.29),

$$\frac{\partial(Y_i, Z_i)}{\partial \tau} \left(\frac{\partial X_i}{\partial \tau}\right)^{-1} = O(e^{(-\alpha_i+\gamma_i+\varepsilon+\xi)\tau}) . \qquad (3.36)$$

Condition (C) guarantees (see (3.2)) that $-\alpha_i + \gamma_i + \varepsilon + \xi < 0$ if ε and ξ are sufficiently small. Thus, estimate (3.36) implies relation (3.24), which gives the lemma. ■

The proven lemma establishes that the derivatives of the right-hand sides of the half-Poincaré maps T_i are small for sufficiently small values of u_i; i.e., these maps are *contracting* and the contraction constant q can be made arbitrarily small if the size of the neighborhood of the contour under consideration is taken to be small. We will show in the next section that the contractivity of the half-Poincaré maps imposes strong restrictions on the dynamics near the contour. Besides the contractivity, we will also use the following evident property of these maps:

The Spiral Property. If the starting point (u_i, y_i) on S_i^+ approaches the stable manifold W_i^s (i.e., if $u_i \to +0$), then its image (u_j, y_j) by the map T_i tends to the point $G_i^*(\mu_j, y_j^*) = \Gamma_i \cap S_j$ along a spiral-like curve (see Fig. 8).

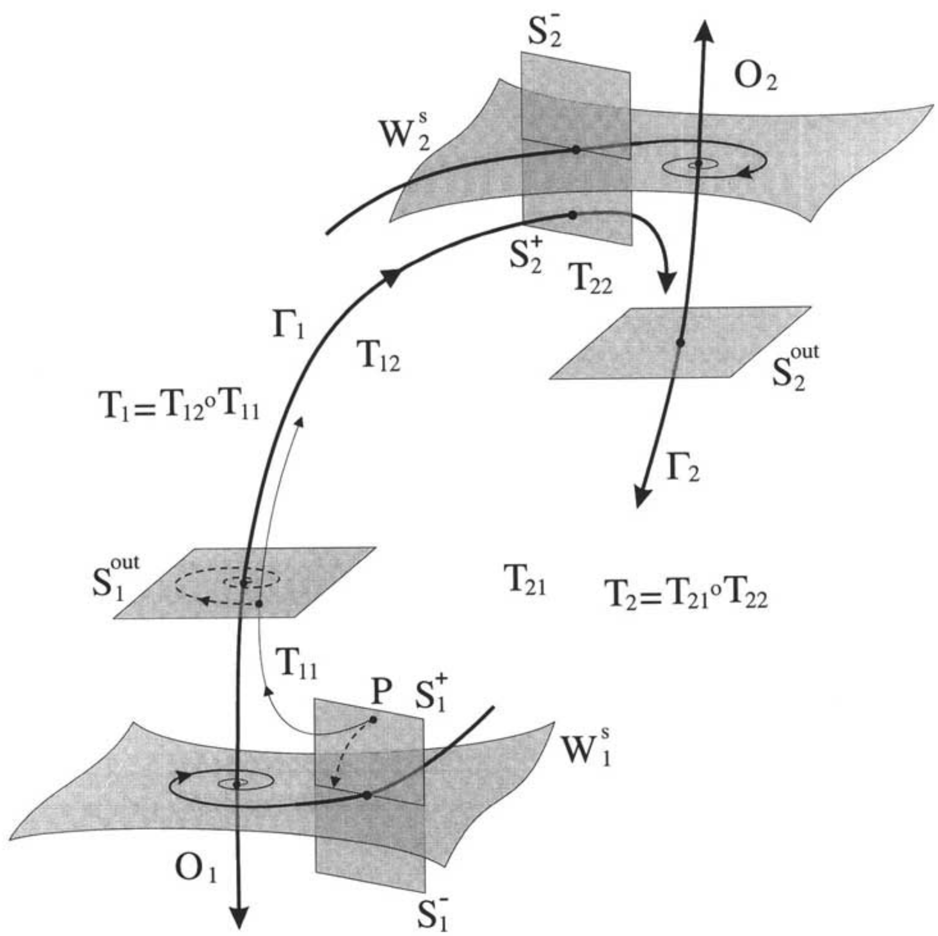

Fig. 8. When the initial point $P \in S_1$ tends to the stable manifold (staying in S_1^+), the image of the point under the action of the map T_{11} traces a spiral on the cross section S_1^{out} which is mapped onto a spiral on S_2 by the map T_{12}.

Note that the spiral property holds independently of the behavior of the coordinate y_i of the starting point and of the parameters μ (they are assumed to change in some continuous way). This property is a simple consequence of the fact that the characteristic exponents $(\lambda_i^1, \lambda_i^2)$ corresponding to the coordinates y_i and z_i have a non-zero imaginary part ($\mathrm{Im}\lambda_i^1 = \omega$, $\mathrm{Im}\lambda_i^2 = -\omega$). To prove the property, let us introduce the coordinates

$$v_i = y_i - y_i^u(x_i, \mu), \quad w_i = z_i - z_i^u(x_i, \mu) \qquad (3.37)$$

in a small neighborhood of O_i, where $(y_i, z_i) = (y_i^u(x_i, \mu), z_i^u(x_i, \mu))$ is the equation of the local unstable manifold W_i^u (see Eq. (3.5)). In the new coordinates W_i^u takes the form $(v_i = 0, w_i = 0)$. Since it is an invariant manifold, we have that $(\dot{v}_i = 0, \dot{w}_i = 0)$ at $(v_i = 0, w_i = 0)$. Thus, we can write (see (3.1))

$$\begin{aligned} \dot{v}_i &= -(\alpha_i + \cdots)v_i - (\omega_i + \cdots)w_i\,, \\ \dot{w}_i &= (\omega_i + \cdots)v_i - (\alpha_i + \cdots)w_i \end{aligned} \qquad (3.38)$$

where dots stand for terms vanishing at the origin.

For the polar angle $\varphi = \mathrm{Arctan}\,(w_i/v_i)$ and for the polar radius $\rho = \sqrt{v_i^2 + w_i^2}$ we have

$$\begin{aligned} \dot{\varphi} &= \omega + \cdots > \frac{\omega}{2} > 0\,, \\ \frac{\dot{\rho}}{\rho} &= -\alpha + \cdots < -\frac{\alpha}{2} > 0 \end{aligned} \qquad (3.39)$$

and

$$\begin{aligned} \varphi^1 &> \varphi^0 + \frac{\omega}{2}\tau\,, \\ \rho^1 &< \rho^0 e^{-\frac{\alpha}{2}\tau} \end{aligned} \qquad (3.40)$$

where $(\varphi = \varphi^0, \rho = \rho^0)$ corresponds to the starting point on S_i^+ and $(\varphi = \varphi^1, \rho = \rho^1)$ corresponds to its image on S_i^{out} by the map T_{ii}. If the starting point tends to W_i^s, the flight time τ tends to infinity. According to (3.40), the image of the starting point traces a spiral on S_i^{out}. Since the map T_{ij} is a diffeomorphism, this spiral is moved by T_{ij} in a spiral also, which was to be proved.

4. Some Lemmas

In this section we will prove a number of intermediate statements based on Lemma 3.1 of the previous section. As mentioned, identity (3.13) implies that, if the size of the cross section S_i is sufficiently small, the half-Poincaré map T_i is strongly contracting, i.e., for any two points $P_1 \in S_i^+ \cup S_i^0$ and $P_2 \in S_i^+ \cup S_i^0$

$$\mathrm{dist}(T_i(P_1), T_i(P_2)) \le q\,\mathrm{dist}(P_1, P_2)\,, \qquad (4.1)$$

where the constant q can be made arbitrarily small if the u_i-coordinate is small for the points P_1 and P_2.

The contractivity of the maps T_i imposes strong restrictions on the dynamics of the system X_μ in a small neighborhood U of the heteroclinic contour C. For instance, if there exists a periodic orbit, it must be attractive; i.e., all its multipliers must lie inside the unit circle because the Poincaré map near such an orbit is exponentially contracting as a composition of the exponentially contracting maps T_1 and T_2. Moreover, the periodic orbit must be *single-round* (i.e., homotopic to C in U). The last assertion follows directly from a more general statement:

Lemma 4.1. *If an orbit of X_μ stays in U for all positive times, then it tends either to one of the equilibrium states O_i or to an invariant set homotopic to C in U: a single-round periodic or homoclinic orbit, or the heteroclinic contour C itself* (*the latter if $\mu = 0$*).

Proof. The orbit under consideration intersects the cross sections S_1 and S_2 in a sequence of points $P_1, P_2, \ldots$,

$$T_1 P_{2k-1} = P_{2k} \in S_2,\; T_2 P_{2k} = P_{2k+1} \in S_1\,.$$

If the orbit does not tend to the saddle-foci O_1 and O_2, then the sequence is infinite. By the contractivity of the maps T_2 and T_1 the sequence $\mathrm{dist}(P_{k+2}, P_k)$ decreases as a geometric progression. Therefore, there exists a unique limit point $P^{(1)} = \lim P_{2k+1} \in S_1$ and a unique limit point $P^{(2)} = \lim_{k\to\infty} P_{2k} \in S_2$. Evidently, $P^{(2)} = T_1 P^{(1)}$ and $P^{(1)} = T_2 P^{(2)}$, i.e., $P^{(1)}$ is a fixed point of the Poincaré map $T_2 \circ T_1$. Hence, if $P^{(1)} \notin W^s_{1_{\mathrm{loc}}}$ and $P^{(2)} \notin W^s_{2_{\mathrm{loc}}}$, then the orbit passing through $P^{(1)}$ comes to $P^{(1)}$ again after one round in U. This means that it is a single-round periodic orbit. Otherwise, if $P^{(1)} \in W^s_{1_{\mathrm{loc}}}$ and $P^{(2)} \notin W^s_{2_{\mathrm{loc}}}$, or $P^{(2)} \in W^s_{2_{\mathrm{loc}}}$ and $P^{(1)} \notin W^s_{1_{\mathrm{loc}}}$, then it is a single-round homoclinic orbit. If, finally, $P^{(1)} \in W^s_{1_{\mathrm{loc}}}$ and $P^{(2)} \in W^s_{2_{\mathrm{loc}}}$, then $P^{(1)} = T_2(S_2^0) = G_2^*$ and $P^{(2)} = T_1(S_1^0) = G_1^*$ where $S_i^0 = S_i \cap W^s_{i_{\mathrm{loc}}}$ and G_i^* is the point of the first intersection of the separatrix Γ_i

with S_j. We got that $G_1^* \in W^s_{2loc}$ and $G_2^* \in W^s_{1loc}$ which means that $(\mu_1, \mu_2) = 0$ and the separatrices compose of the heteroclinic contour C. This completes the proof. ■

The proven statement implies, in particular, that the only recurrent orbit in U (except for the equilibrium states O_1 and O_2) can be a single-round periodic orbit. Note that this orbit is unique because it corresponds to the fixed point of the Poincaré map which is contracting and which cannot have more than one fixed point therefore. By the same reason, the periodic orbit cannot coexist with a homoclinic loop or with a heteroclinic contour. Note also that a homoclinic loop in U must also be single-round and unique: any point of intersection of the homoclinic loops with S_1 is a periodic point for the Poincaré map $T_2 \circ T_1$ and, since the contracting map cannot have more than one periodic point, it follows that the homoclinic orbit intersects S_1 only once. Analogously, if a heteroclinic contour exists in U, it can coexist neither with homoclinic nor with periodic orbits and it, moreover, must also be a single-round, i.e., the system can have a heteroclinic, contour only at $\mu = 0$.

Lemma 4.2. *Any orbit that stays in U for all times lies in the closure of the set $\Gamma_1 \cup \Gamma_2$.*

Proof. Denote as $\{\Gamma_i\}$ the set of points of intersection of the separatrix Γ_i with the cross sections S_1 and S_2. Note that $\{\Gamma_1\} \cap S_2$ contains, at least, the point G_1^* and it is therefore not empty. Analogously, $\{\Gamma_2\} \cap S_1 \neq \emptyset$. For any point $P \in S_i$ there can be defined a distance to the set $\{\Gamma_1\} \cup \{\Gamma_2\}$:

$$\mathrm{dist}\left(P, \{\Gamma_1\} \bigcup \{\Gamma_2\}\right) = \inf_{G \in (\{\Gamma_1\} \cup \{\Gamma_2\}) \cap S_i} \mathrm{dist}(P, G).$$

For any point $P \in S_i^+ \cup S_i^0$ the following inequality holds

$$\mathrm{dist}(T_i(P), \{\Gamma_1\} \cup \{\Gamma_2\}) \leq q\, \mathrm{dist}(P, \{\Gamma_1\} \cup \{\Gamma_2\}). \tag{4.2}$$

Indeed, let G be an arbitrary point in $\{\Gamma_1\} \cup \{\Gamma_2\}$. It follows from (4.1) that if $G \in S_i^+ \cup S_i^0$, then

$$\mathrm{dist}(T_i(P), T_i(G)) \leq q\, \mathrm{dist}(P, G). \tag{4.3}$$

If $G \in S_i^-$, then $\mathrm{dist}(P, S_i^0) < \mathrm{dist}(P, G)$, and

$$\begin{aligned}\mathrm{dist}(T_i(P), G_i^*) &= \mathrm{dist}(T_i(P), T_i(S_i^0)) \\ &\leq q\, \mathrm{dist}(P, S_i^0) < q\, \mathrm{dist}(P, G)\end{aligned} \tag{4.4}$$

where $G_i^* = \Gamma_i \cap S_j$ $(j = 3 - i)$. Inequalities (4.3), (4.4) imply (4.2).

Let some orbit stay for all negative times in the neighborhood U. Suppose that the orbit does not coincide with the equilibrium states O_1, O_2 or with separatrices Γ_1, Γ_2 (otherwise, the lemma is trivial). In this case a backward semi-orbit intersects the cross sections S_1 and S_2 infinitely many times. Let P_i be the sequential points of the intersection: $P_{2i} \in S_1$, $P_{2i+1} \in S_2$,

$$P_0 \xleftarrow{T_2} P_1 \xleftarrow{T_1} P_2 \xleftarrow{T_2} \dots$$

By (4.2)

$$\mathrm{dist}(P_0, \{\Gamma_1\} \cup \{\Gamma_2\}) \leq q^i\, \mathrm{dist}(P_i, \{\Gamma_1\} \cup \{\Gamma_2\}).$$

Since $q < 1$ and since i can be taken to be arbitrarily large whereas $\mathrm{dist}(P_i, \{\Gamma_1\} \cup \{\Gamma_2\})$ remains bounded for any i, it follows that

$$\mathrm{dist}(P_0, \{\Gamma_1\} \cup \{\Gamma_2\}) = 0. \tag{4.5}$$

which gives the lemma.

This lemma shows that the behavior of the orbits of the system X_μ in U is determined by the behavior of the separatrices. In particular, if there exists a periodic orbit in U, then at least one of the separatrices tends to it as $t \to +\infty$. Summarizing the results above we obtain the following list of possible types of behavior of orbits in U.

Lemma 4.3. *Let N be the set of all orbits lying in U entirely. At $\mu = 0$ the set N coincides with the heteroclinic contour C. If $\mu \neq 0$, then the set $N \backslash \{O_1 \cup O_2\}$ can consist of*

1. *no orbits (in this case both separatrices Γ_1 and Γ_2 leave U) (Fig. 9);*
2. *a single-round periodic orbit Π and two separatrices Γ_1 and Γ_2 tending to Π (Fig. 10);*
3. *a single-round periodic orbit Π and one of the separatrices that tends to K (the other leaves U) (Fig. 11);*
4. *a single-round periodic orbit Π, one of the separatrices that tends to Π and the other separatrix that* forms *a heteroclinic connection (Fig. 12);*
5. *a single-round homoclinic loop formed by one of the separatrices Γ_i (the other leaves U) (Fig. 13);*
6. *a single-round homoclinic loop formed by one of the separatrices Γ_i, and the other separatrix that forms a heteroclinic connection (Fig. 14);*

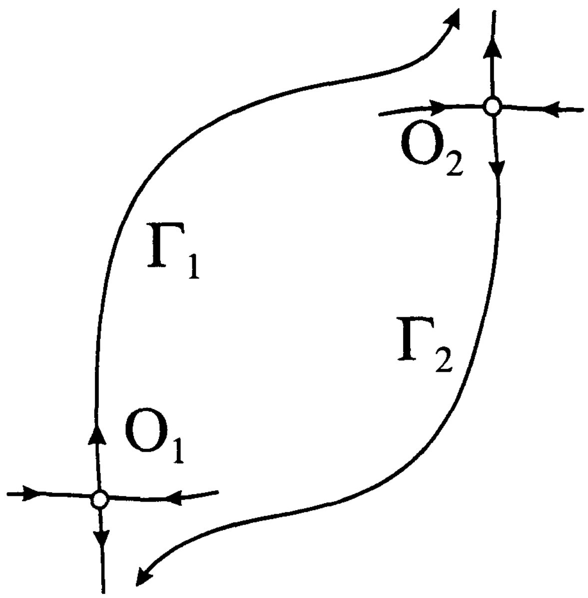

Fig. 9. The separatrices Γ_1 and Γ_2 leave the neighborhood U.

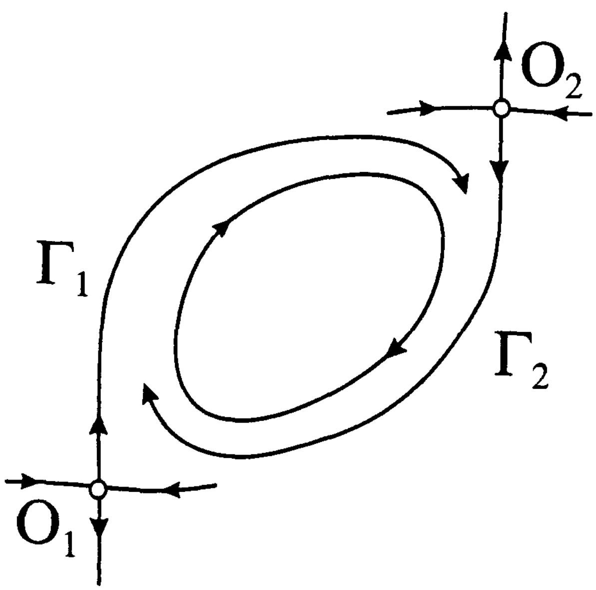

Fig. 10. The separatrices Γ_1 and Γ_2 tend to a single-round periodic orbit.

7. *a single-round homoclinic loop formed by one of the separatrices* Γ_i *and the other separatrix that tends to the loop* (*Fig. 15*);
8. *one orbit of a heteroclinic connection* (*the other separatrix leaves* U) (*Fig. 16*).

Proof. At $\mu = 0$ the closure of the set $\Gamma_1 \cup \Gamma_2$ is the heteroclinic contour C. Therefore, $N = C$ in this case by virtue of Lemma 4.2.

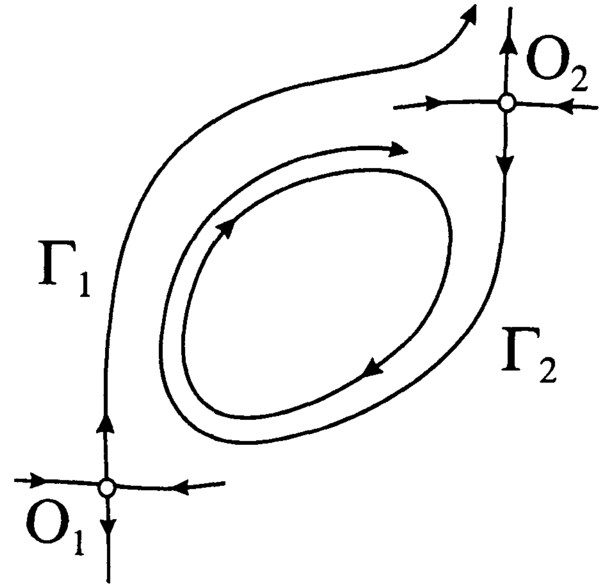

Fig. 11. The separatrix Γ_2 tends to a periodic orbit, and Γ_1 leaves U.

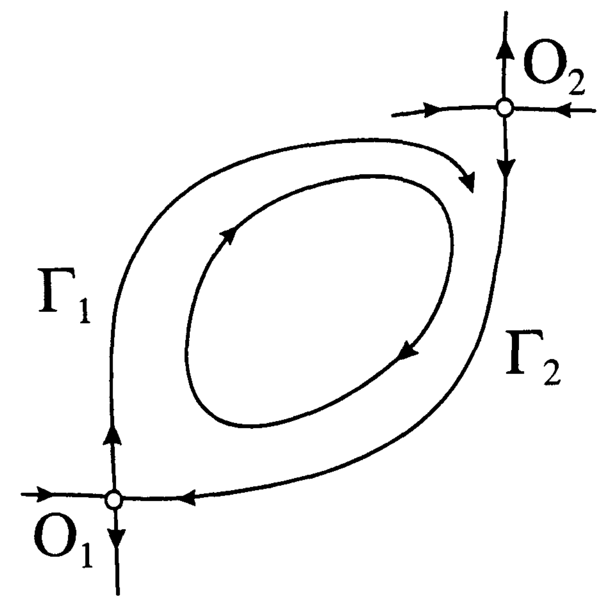

Fig. 12. The separatrix Γ_1 tends to a periodic orbit and the separatrix Γ_1 forms a heteroclinic connection (this is a schematical picture: in general, the connection may be multi-round).

Now let $\mu \neq 0$. If both separatrices leave U (in particular, this takes place if both $\mu_1 < 0$ and $\mu_2 < 0$), then $N \backslash \{O_1 \cup O_2\} = \emptyset$ by Lemma 4.2. This corresponds to item 1 of the present lemma.

If one of the separatrices (say, Γ_1) does not leave U, then, by Lemma 4.1, it may

1. tend to a single-round periodic orbit,

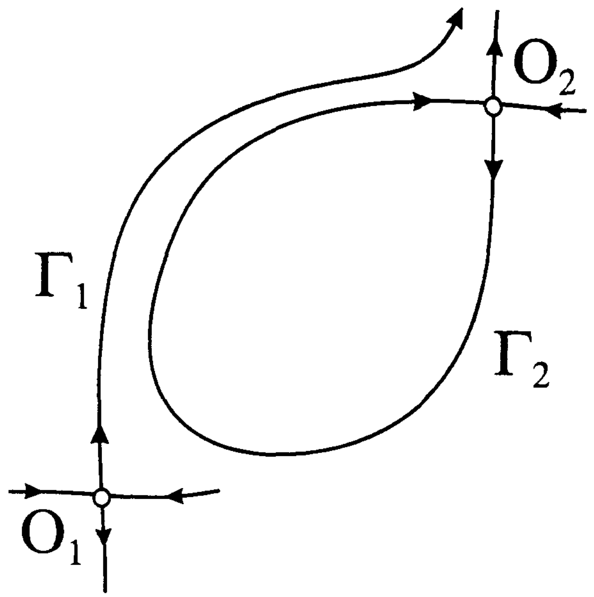

Fig. 13. The separatrix Γ_2 forms a single-round homoclinic loop, and the separatrix Γ_1 leaves the neighborhood U.

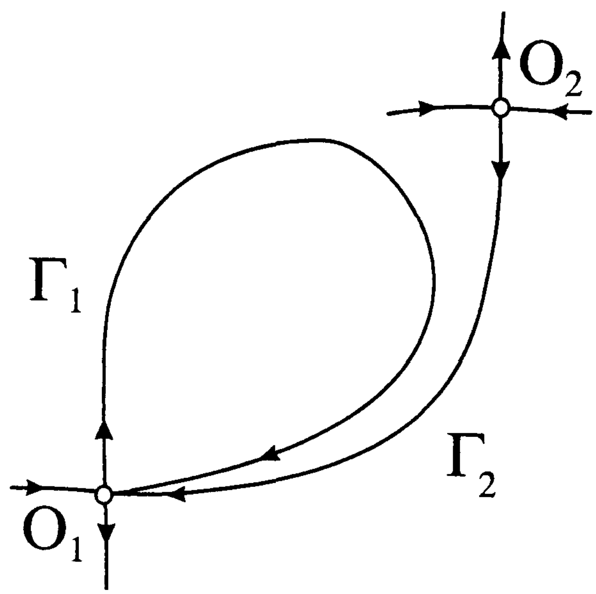

Fig. 14. The separatrix Γ_1 forms a homoclinic loop, and Γ_2 forms a heteroclinic connection.

2. tend to an equilibrium state and form a single-round homoclinic loop,
3. tend to an equilibrium state and form a heteroclinic connection,
4. tend to a single-round homoclinic loop formed by the other separatrix.

In the last case of this list the closure of the set $\Gamma_1 \cup \Gamma_2$ consists of the orbits O_1, O_2, Γ_1 and Γ_2. By

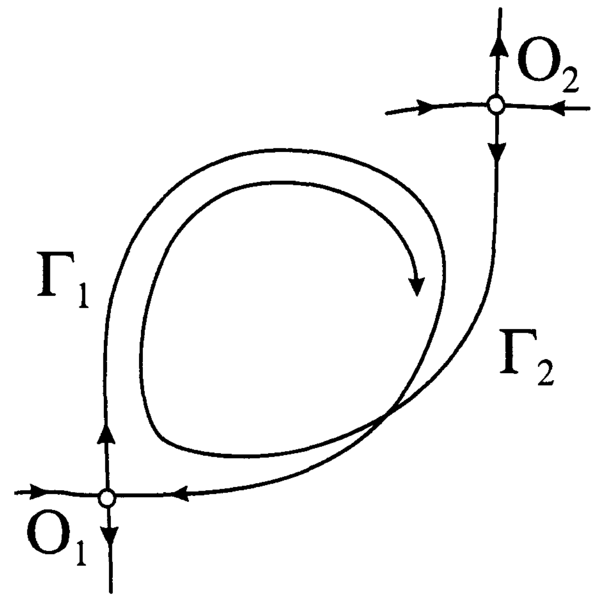

Fig. 15. The separatrix Γ_1 forms a homoclinic loop and the separatrix Γ_2 tends to the loop.

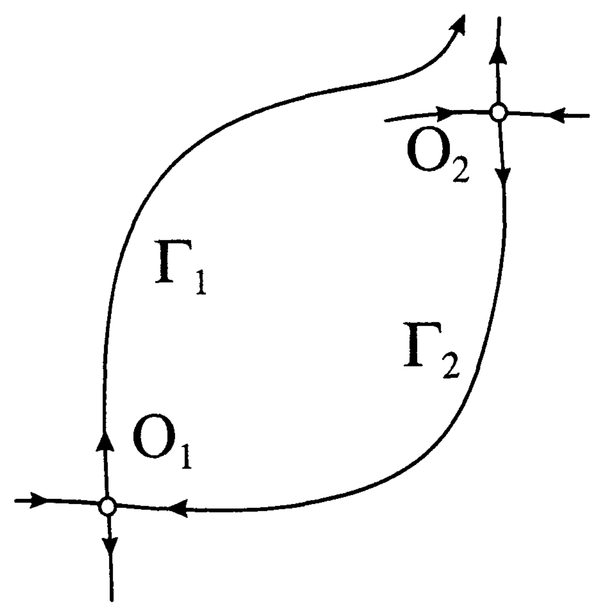

Fig. 16. The separatrix Γ_1 leaves the neighborhood U and the separatrix Γ_2 forms a heteroclinic connection.

Lemma 4.2, the set N equals to $\{O_1, O_2, \Gamma_1, \Gamma_2\}$; this corresponds to item 7 of the present lemma.

If the separatrix Γ_1 forms a heteroclinic connection (Case 3 of our list), the other separatrix may, by Lemma 4.1, tend to a single-round periodic orbit, or tend to an equilibrium state and form a homoclinic loop, or it may leave U (it cannot form another heteroclinic connection because there cannot be heteroclinic contours at $\mu \neq 0$). This corresponds, respectively, to items 4, 6 and 8 of the present lemma.

If the separatrix Γ_1 forms a homoclinic loop (case 2 of the list), the other separatrix may tend to the homoclinic loop (item 7 of the lemma), or tend to an equilibrium state and form a heteroclinic connection (item 6), or it may leave U (item 5).

If, finally, the separatrix Γ_1 tends to a periodic orbit Π (case 1 of the list), the other separatrix may also tend to Π (item 2 of the lemma), or it may form a heteroclinic connection (item 4), or it may leave U (item 3). We considered all possibilities and the lemma is proved. ■

Note that items 1–3 of the lemma correspond to the case where the system X_μ is structurally stable and it is structurally unstable in the other cases. Thus, we have

Corollary 4.1. *The bifurcation set of the family X_μ is composed of those μ for which:*

1. *the separatrix Γ_i forms a single-round homoclinic loop to the equilibrium state O_i (we denote the set of such μ's as L_i);*
2. *the separatrix Γ_i leaves O_i, makes k rounds in U and enters O_j; i.e., it forms a k-round heteroclinic connection (we denote the set of such μ's as C_{ij}^k, $i = 1, 2$, $j = 3 - i$, $k = 0, 1, \ldots$).*

In the next section we will show that all these sets are non-empty and study the structure of these sets.

5. The Construction of the Bifurcation Set

Consider the sequence $\{\Gamma_i\}$ of points at which the separatrix Γ_i intersects consequently the cross sections S_1 and S_2. The points of this sequence will be denoted as $G_i^1, G_i^2, \ldots$ (we used the notation G_i^* for the first point G_i^1 in the previous sections). By definition, the points $G_i^1, G_i^3, G_i^5, \ldots$ lie in S_j ($j = 3-i$) and the points $G_i^2, G_i^4, G_i^6, \ldots$ lie in S_i. Here,

$$G_i^{2s+1} = T_i(G_i^{2s}), \quad G_i^{2s} = T_j(G_i^{2s-1}).$$

The sequence $\{\Gamma_i\}$ may be infinite and in this case all points $\{G_i^1, G_i^2, \ldots\}$ lie in $S_1^+ \cup S_2^+$, or it may be finite and in this case the last point in the sequence belongs to $S_1^0 \cup S_1^-$ or to $S_2^0 \cup S_2^-$ and the other points lie in $S_1^+ \cup S_2^+$. In principle, one can imagine the case where the last point lies in $S_1^+ \cup S_2^+$, but sufficiently high, so that it does not belong to the neighborhood U. The following lemma shows that it is impossible for small μ.

Denote $\|\mu\| = \max(|\mu_1|, |\mu_2|)$.

Lemma 5.1. *The set $(\{\Gamma_1\} \cup \{\Gamma_2\}) \cap S_i$ ($i = 1, 2$) lies in the open disk with the radius $\|\mu\|$ and with the center at the point G_j^* ($j = 3-i$); i.e., $\text{dist}(P, G_j^*) < \|\mu\|$ for any $P \in (\{\Gamma_1\} \cup \{\Gamma_2\}) \cap S_i$.*

Proof. The points $G_j^* \equiv G_j^1$ themselves belong to these disks. Let some point G_i^s ($s \geq 1$) belong to such a disk. For more definiteness, assume that the point G_i^s lies in S_1. Then,

$$\text{dist}(G_i^s, S_1^0) < \text{dist}(G_2^*, S_1^0) + \|\mu\| < 2\|\mu\|$$

(by definition, $\text{dist}(G_2^*, S_1^0) = \mu_2 \leq \|\mu\|$). If the point G_i^s is not the last in the sequence $\{\Gamma_i\}$, then, by the contractivity of the map T_1 (formula (4.1)), we have

$$\begin{aligned}\text{dist}(G_i^{s+1}, G_1^*) &= \text{dist}(T_1(G_i^s), T_1(S_1^0)) \\ &\leq q\,\text{dist}(G_i^s, S_1^0) < 2q\|\mu\| < \|\mu\|\end{aligned}$$

if q is taken less than 1/2. In other words, the next point G_i^{s+1} also belongs to the open disk with the radius $\|\mu\|$ but, now, with the center at the point G_1^*. This, by induction, gives the lemma. ■

According to this result, in the region $\{\mu_2 > 0, \mu_2 \geq \mu_1\}$ none of the points of $\{\Gamma_1\} \cup \{\Gamma_2\}$ can lie on S_2^0. Therefore, in this region neither the separatrix Γ_1, nor the separatrix Γ_2 can belong to W_2^s. Thus, in this region on the parameter plane, there is no parameter values corresponding to heteroclinic orbits going from O_1 to O_2, or to a homoclinic loop of O_2. Analogously, in the region $\{\mu_1 > 0, \mu_1 \geq \mu_2\}$ there is no parameter values corresponding to heteroclinic orbits going from O_2 to O_1, or to a homoclinic loop of O_1.

We see that in the region $\{\mu_2 > 0, \mu_2 \geq \mu_1\}$ there may exist only the bifurcation sets L_1 and C_{21}^k ($k = 1, 2, \ldots$) corresponding, respectively, to a single-round homoclinic loop of O_1 and to heteroclinic orbits connecting O_2 and O_1. In the region $\{\mu_1 > 0, \mu_1 \geq \mu_2\}$ there may exist only the bifurcation sets L_2 and C_{12}^k ($k = 1, 2, \ldots$).

As we mentioned in the previous section, no bifurcations happen in the region $(\mu_1 < 0, \mu_2 < 0)$

because there both separatrices leave the neighborhood U. The rest of the parameter plane is divided by the line $\mu_1 = \mu_2$ into two parts, and we may restrict ourselves by the study of the part $\{\mu_2 > 0,\ \mu_2 \geq \mu_1)$ due to the symmetry of the problem.

The bifurcational curve L_1 is easily found. The corresponding homoclinic loop exists if (and only if)

$$G_1^2 = T_2(G_1^*) \in S_1^0.$$

By Lemma 3.1, this is equivalent to the equality

$$\mu_1 + f_2(\mu_2, y_2^*(\mu), \mu) = 0 \tag{5.1}$$

Since f_2 tends to zero along with its first derivatives as $\mu_2 \to +0$, this implicit equality is uniquely resolved at all small $\mu_2 > 0$ and it can be rewritten as

$$\mu_1 = h_1(\mu_2) \tag{5.2}$$

where h_1 is a smooth function defined at all small $\mu_2 > 0$, and such that $h_1 \to 0$, $h_1' \to 0$ as $\mu_2 \to +0$.

Moreover, if $\mu_1 < h_1(\mu_2)$, then the point $G_1^2 = T_2(G_1^*)$ lies below the line $S_1^0 = S_1 \cap W_{1_{\rm loc}}^s$. Therefore, the separatrix Γ_1 leaves U in this case. If, otherwise, $\mu_1 > h_1(\mu_2)$, then the point G_1^2 lies above the line S_1^0. In this case, if $\mu_1 - h_1(\mu_2)$ is small, the separatrix Γ_1 tends to a single-round periodic orbit which is born from the loop, according to Shil'nikov [1963]. Actually, the separatrix tends to the periodic orbit for all μ_1 of the interval $(h_1(\mu_2), \mu_2)$. This follows from Lemma 4.1 because the separatrix cannot leave U when μ belongs to this interval: it cannot intersect S_2^0 by virtue of Lemma 5.1 and it cannot intersect S_1^0 since in this case a multi-round homoclinic loop would exist which is forbidden by the contractivity of the Poincaré map (see the previous section).

Note the important property of the curve L_1.

Lemma 5.2. *The graph of the function $\mu_1 = h_1(\mu_2)$, defining the bifurcational curve L_1, intersects the line C_{21}^0: $\mu_1 = 0$ infinitely many times; moreover, the sets $\{\mu_2 | h_1(\mu_2) > 0\}$ and $\{\mu_2 | h_1(\mu_2) < 0\}$ are both non-empty and consist of an infinite number of intervals.*

Proof. Fix $\mu_1 = 0$ and let the parameter μ_2 tend to zero from the positive side. The coordinate u_2 of the first intersection point $G_1^* \equiv G_1^1$ of the separatrix Γ_1 with S_2 is identically equal to μ_2. Therefore, $u_2 \to +0$ which means that $G_1^1 \to S_2^0$ as $\mu_2 \to +0$. According to the *spiral property* of the map T_2 (see Sec. 3), the point $G_1^2 = T_2(G_1^1)$ winds as a spiral to the point $G_2^* \equiv G_2^1$ which belongs to S_1^0 because $\mu_1 = 0$. Thus, the point $G_1^2 = T_2(G_1^1)$ intersects, as $\mu_2 \to +0$, the line S_1^0 infinitely many times moving from S_1^+ to S_1^- and back. Since the condition $G_1^2 = T_2(G_1^1) \in S_1^0$ corresponds to the presence of the homoclinic loop, this completes the proof.

We have studied how the separatrix Γ_1 behaves in the region $\{\mu_2 > 0,\ \mu_2 > \mu_1\}$. The behavior of Γ_2 is less trivial in this region. As we mentioned, when μ lies in the region $\{\mu_2 > 0,\ \mu_2 > \mu_1\}$, the separatrix Γ_2 may form a k-round ($k = 0, 1, 2, \ldots$) heteroclinic connection, i.e., it may make k rounds along U and enter the equilibrium state O_1. For instance, at $\mu_1 = 0$, the orbit Γ_2 forms a zero-round connection. Further efforts are aimed at the construction of the bifurcational sets C_{21}^k ($k = 0, 1, 2, \ldots$) corresponding to such connections.

Suppose the orbit Γ_2 makes k ($k = 1, 2, 3, \ldots$) rounds along U and intersects the cross section S_1 at some point. We have that Γ_2 intersects the cross sections S_1 and S_2 consequently at the points G_2^1, $G_2^2, \ldots, G_2^{2k}$, G_2^{2k+1} ($G_2^s \in S_1$ if $s = 1, 3, 5, \ldots, 2k+1$, and $G_2^s \in S_2$ if $s = 2, 4, \ldots, 2k$). ■

Lemma 5.3. *For the coordinates (u^s, y^s) of the intersection points G_2^s, ($s = 1, 2, \ldots, 2k+1$) the following estimate holds*

$$\left\| \frac{d(u^s, y^s)}{d\mu} \right\| < \beta\,, \tag{5.3}$$

where the constant β is independent of k and of s.

Proof. We have

$$G_2^{s+1} = T_i(G_2^s) \tag{5.4}$$

where $i = 1$ if $s = 1, 3, 5, \ldots, 2k+1$, ($G_2^s \in S_1$), and $i = 2$ if $s = 2, 4, 6, \ldots, 2k$, ($G_2^s \in S_2$).

Formula (3.11) for the map T_i allows one to rewrite Eq. (5.4) in the form

$$\begin{cases} u^{s+1} = \mu_j + f_i(u^s, y^s, \mu)\,, \\ y^{s+1} = y_j^* + g_i(u^s, y^s, \mu)\,, \end{cases} \tag{5.5}$$

By virtue of Lemma 3.1, the derivatives of the functions f_i and g_i can be made arbitrarily small if the neighborhood U of the contour is taken to be small.

Therefore, we can write

$$\left\|\frac{d(u^{s+1},\, y^{s+1})}{d\mu}\right\| \le \left\|\frac{\partial(\mu_j,\, y_j^*)}{\partial\mu}\right\| + \left\|\frac{\partial(f_i,\, g_i)}{\partial\mu}\right\| + \left\|\frac{\partial(f_i,\, g_i)}{\partial(u^s,\, y^s)}\right\| \left\|\frac{d(u^s,\, y^s)}{d\mu}\right\| \le K + q\left\|\frac{d(u^s, y^s)}{d\mu}\right\| . \tag{5.6}$$

where K and q are positive constants and q can be taken to be arbitrarily small. By induction, from inequality (5.6) we get a uniform boundedness for the norm

$$\left\|\frac{d(u^s,\, y^s)}{d\mu}\right\| . \tag{5.7}$$

The lemma is proved. ■

Lemma 5.3 is used in the proof of the following result:

Lemma 5.4. *If the set C_{21}^k ($k = 1, 2, \ldots$) of the parameter values corresponding to a k-round heteroclinic connection is not empty, then*

1. *the set C_{21}^k is given by equation $\mu_1 = h_{21}^k(\mu_2)$ where h_{21}^k is a smooth function defined on some open set $\mathcal{D}_{21}^k$ of values of μ_2 and the derivative of h_{21}^k is bounded by a small constant independent of k;*
2. *the connected components of the domain $\mathcal{D}_{21}^k$ are open intervals such that the corresponding connected components of the curve C_{21}^k are ended at points which belong to the set $\{\bigcup_{s=0}^{k-1} C_{21}^s \cap L_1\} \bigcup \{\mu = (0, 0)\}$;*
3. *the inequalities $h_{21}^s(\mu_2) < h_1(\mu_2)$, ($s = 0, 1, \ldots, k-1$) hold everywhere on $\mathcal{D}_{21}^k$.*

Proof. Suppose that for some value of μ the separatrix Γ_2 forms a k-round heteroclinic connection. In this case, Γ_2 intersects the cross sections S_1 and S_2 successively at the points $G_2^1, G_2^2, \ldots, G_2^{2k}, G_2^{2k+1}$. Here $G_2^2, G_2^4, \ldots, G_2^{2k} \in S_2^+$, $G_2^1, G_2^3, G_2^5, \ldots, G_2^{2k-1} \in S_1^+$, and $G_2^{2k+1} \in S_1^0$. By Eq. (3.11)

$$u^{2k+1} = \mu_1 + f_2(u^{2k},\, y^{2k},\, \mu)\,,$$

and, since $G_2^{2k+1} \in S_1^0$, it follows that $u^{2k+1} = 0$, i.e., we have

$$\mu_1 + f_2(u^{2k}, y^{2k}, \mu) = 0\,. \tag{5.8}$$

Note that the norm $\|\frac{df_2(u^{2k}, y^{2k}, \mu)}{d\mu}\|$ is small, since

$$\left\|\frac{df_2(u^{2k},\, y^{2k},\, \mu)}{d\mu}\right\| \le \left\|\frac{\partial f_2}{\partial\mu}\right\| + \left\|\frac{\partial f_2}{\partial(u^{2k}, y^{2k})}\right\| \left\|\frac{d(u^{2k}, y^{2k})}{d\mu}\right\| , \tag{5.9}$$

and the norms $\|\frac{\partial f_2}{\partial\mu}\|$ and $\|\frac{\partial f_2}{\partial(u^{2k}, y^{2k})}\|$ are small by virtue of Lemma 3.1 and the norm $\|\frac{d(u^{2k}, y^{2k})}{d\mu}\|$ is uniformly bounded by virtue of Lemma 5.3.

One can apply the Implicit Function Theorem to Eq. (5.8) which gives that if Eq. (5.8) is fulfilled for some μ, then in a small neighborhood of this point on the parameter plane the implicit relation (5.8) is resolved and take the desired form $\mu_1 = h_{21}^k(\mu_2)$ where the norm $|\frac{\partial h(\mu_l)}{\partial\mu_l}|$ is small. This gives item 1 of the lemma.

Let us write the condition $\mu \in C_{21}^k$ in the form

$$(T_2 \circ T_1)^k G_2^1 \in S_1^0\,. \tag{5.10}$$

Obviously, for the end point of a connected component of the curve C_{21}^k, formula (5.10) remains valid and, moreover, at least one of the points G_2^s ($s = 1, 2, 3, \ldots, 2k$) lies on the line S_1^0 (if $s = 1, 3, 5, \ldots, 2k-1$) or on the line S_2^0 (if $s = 2, 4, 6, \ldots, 2k$). By Lemma 5.1, in the region $\{\mu_2 \ge 0,\, \mu_2 \ge \mu_1\}$ which we consider here, the points G_2^s cannot lie on S_2^0 (unless $\mu_1 = 0$).

Thus, at the end point, at least one of the points $G_2^1, G_2^3, G_2^5, \ldots, G_2^{2k-1}$ lies on the line S_1^0, i.e., the separatrix Γ_2 is an s-round connection for some $s \le k-1$. The latter statement means that

$$(T_2 \circ T_1)^s G_2^1 \in S_1^0\,. \tag{5.11}$$

Formulas (5.10) and (5.11) imply that $(T_2 \circ T_1)^{k-s} (S_1^0) \in S_1^0$, i.e., the separatrix Γ_1 forms a homoclinic loop.

Thus, we have proved that if the end point is not the point $\mu = (0, 0)$, then it corresponds to the presence of an s-round heteroclinic connection ($s < k$) formed by the separatrix Γ_2 and a homoclinic loop formed by the separatrix Γ_1. This gives item 2 of the lemma.

To prove item 3 consider a sequence of points $G^1, G^2, G^3, \ldots$ defined by the rule: $G^1 = G_2^*$ (it is the first point of intersection of the separatrix Γ_2 with cross section S_1) and $G^{s+1} = T_2(G^s)$ if $s = 2, 4, 6, \ldots$, and $G^{s+1} = T_1(G^s)$ if $s = 1, 3, 5, \ldots$

(the process is stopped if $G^s \in S_1^-$ or $G^s \in S_2^-$). By definition, if the separatrix Γ_2 does not form a heteroclinic connection, this sequence coincides with the sequence $\{G_2^s\}$ of the points where Γ_2 intersects S_1 and S_2. If the heteroclinic connection exists, then a point of this sequence lies on S_1^0 and the sequence is continued by the points of intersection of the separatrix Γ_1 with S_1 and S_2.

Let (u^{2s}, y^{2s}) be the coordinates of the point $G^{2s} \in S^2$ and let (u^{2s+1}, y^{2s+1}) be the coordinates of the point $G^{2s+1} \in S^1$. By Eq. (3.11)

$$u^{2s+1} = \mu_1 + f_2(u^{2s}, y^{2s}, \mu) . \qquad (5.12)$$

We showed that the norm $\left\|\frac{df_2(u^{2s}, y^{2s}, \mu)}{d\mu}\right\|$ is uniformly small with respect to s, therefore

$$\frac{du^{2s+1}}{d\mu_1} \geq 1 - \left\|\frac{df_2(u^{2k}, y^{2k}, \mu)}{d\mu}\right\| > 0 . \qquad (5.13)$$

This means that for fixed $\mu_2 > 0$ the coordinate u^{2s+1} is an *increasing* function of μ_1.

Suppose now that there is a value μ_2^*, belonging to the domain $\mathcal{D}_{21}^k$ of the function h_{21}^k, such that

$$h_{21}^s(\mu_2^*) \geq h_1(\mu_2^*) \qquad (5.14)$$

for some $s = 0, 1, \ldots, k-1$. For the parameter value $(\mu_1 = h_{21}^s(\mu_2^*),\ \mu_2 = \mu_2^*)$ the sequence $G^1, G^2, \ldots$ is infinite. Indeed, since the separatrix Γ_2 forms an s-round heteroclinic connection, the point G^{2s+1} belongs to S_1^0. The image $T_1 S_1^0$ is the point G_1^* where the separatrix Γ_1 intersects S_2. Therefore, the point G^{2s+2} in our sequence coincides with G_1^* (by definition, $G^{2s+2} = T_1 G^{2s+1}$) and the successive points in the sequence are the iterations of the point G_1^*. We proved that the sequence of these iterations is infinite in the region $\mu_2 \geq \mu_1 \geq h_1(\mu_2)$[3], but it is the region to which the parameter value $(\mu_1 = h_{21}^s(\mu_2^*),\ \mu_2 = \mu_2^*)$ belongs by virtue of (5.14)).

In particular, $u^{2k+1} \geq 0$ for the given value of μ. Since u^{2k+1} is an increasing function of μ_1, we get that $u^{2k+1} > 0$ for $\mu_1 > h_{21}^s(\mu_2^*)$. This implies that the separatrix Γ_2 cannot form a k-round heteroclinic connection for $\mu_1 > h_{21}^s(\mu_2^*)$.

If $\mu_1 < h_{21}^s(\mu_2^*)$, the value u^{2s+1} must be less than zero (since it is equal to zero at $\mu_1 = h_{21}^s(\mu_2^*)$ and it must decrease when μ_1 decreases). This means that the sequence $G^1, G^2, \ldots$ cannot contain more than $(2s+1)$ points. In particular, we get that the separatrix Γ_2 cannot form a k-round heteroclinic connection for $\mu_1 < h_{21}^s(\mu_2^*)$ (because $k > s$). We arrive at the contradiction: for the given μ_2^* there is no value μ_1 corresponding to the k-round heteroclinic connection but μ_2^* was supposed to belong to the domain of the function h_{21}^k. Thus, we must conclude that inequality (5.14) does not hold. The lemma is proved.

Now we can prove Theorem 2.1. Evidently, the bifurcation set C_{21}^0 is the line $\mu_1 = 0$. We have also established (see Eq. (5.2)) that the bifurcation set L_1 is a smooth curve $\mu_1 = h_1(\mu_2)\mu_2 > 0$ intersecting the line $\mu_1 = 0$ infinitely many times (Lemma 5.2).

To complete the proof of the theorem it is necessary to construct bifurcation sets $C_{21}^k : \{\mu_1 = h_{21}^k(\mu_2)\}$, $(k = 1, 2, 3, \ldots)$. Let, for some k, the following conditions be fulfilled:

1. the domain $\mathcal{D}_{21}^k$ of the function h_{21}^k is not empty.
2. The graphs of the functions $\mu_1 = h_1(\mu_2)$ and $\mu_1 = h_{21}^k(\mu_2)$ intersect each other in infinitely many points; moreover, the sets $\{\mu_2 | h_1(\mu_2) > h_{21}^s\}$ and $\{\mu_2 | h_1(\mu_2) < h_{21}^s\}$ are non-empty and both consist of a countable number of intervals.

(These conditions are fulfilled for $k = 0$). Let us show that these conditions are fulfilled for the value $(k+1)$ (this, by induction, would give the theorem). Let $P = (\mu_1^p, \mu_2^p)$ and $Q = (\mu_1^q, \mu_2^q)$ be successive points of intersection of the curves L_1 and C_{21}^s (which are given by the equations $\mu_1 = h_1(\mu_2)$ and $\mu_1 = h_{21}^s(\mu_2)$, respectively). Assume that $\mu_2^p < \mu_2^q$. It follows from item 3 of Lemma 5.4 that if $h_1(\mu_2) < h_{21}^k(\mu_2)\}$ on the interval (μ_2^p, μ_2^q), then this interval does not intersect with the domains of functions $h_{21}^{k'}$ for $k' = k+1, k+2, \ldots$). Therefore, below we will consider only the points P and Q such that $h_1(\mu_2) > h_{21}^s(\mu_2)$ on the interval (μ_2^p, μ_2^q).

Let $\mu_1 = h_1(\mu_2)$, $(\mu \in L_1)$ and let μ_2 tend to $\mu_2^p + 0$ (to $\mu_2^q - 0$). In this case, the u-coordinate of the point G_2^{2k+1} of intersection of the separatrix Γ_2 with the cross section S_1 tends to $+0$.

According to the spiral property, the image $G_2^{2k+2} = T_1 G_2^{2k+1}$ winds along a spiral curve to the point $G_1^* = \Gamma_1 \cap S_2$. Since $G_1^* \notin S_2^0$ here, the map T_2 is a diffeomorphism near the point G_1^*. Consequently, the point $G_2^{2k+3} = T_2 G_2^{2k+2}$ traces a spiral on S_1 winding to the point $T_2 G_1^*$ (Fig. 17). We take parameters on the line L_1 corresponding to a single-round homoclinic loop formed by the separatrix

[3]If $\mu_2 > h_1(\mu_2)$, then the separatrix Γ_1 tends to a single-round periodic orbit, and if $\mu_2 = h_1(\mu_2)$, Γ_1 forms a loop and the iterations of G_1^* form the infinite periodic sequence $G_1^*, T_2G_1^*, G_1^*, T_2G_1^*, G_1^* \ldots$.

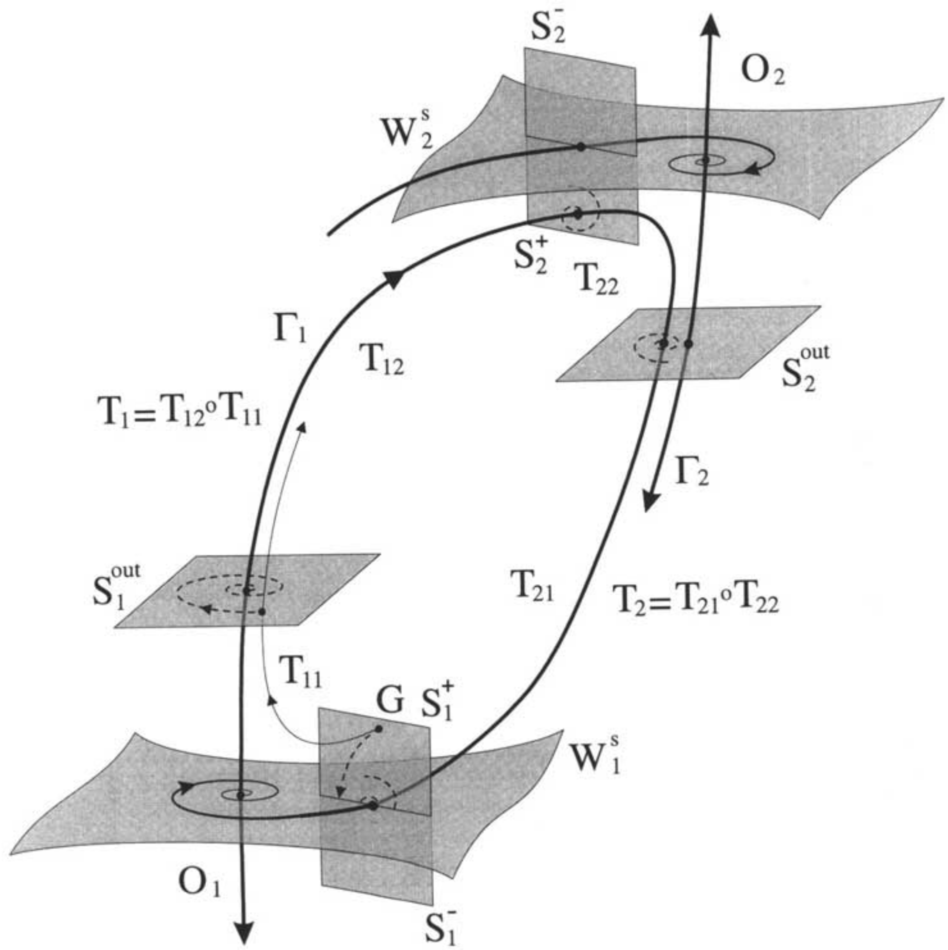

Fig. 17. When the initial point $G \in S_1$ tends to the stable manifold (staying in S_1^+), the image of the point under the action of the map T_{11} traces a spiral on the cross section S_1^{out} which is mapped onto a spiral on S_2 by the map T_{12}. The map T_2 transfers this spiral to the cross section S_1. The image point winds now to the point on S_1^0 where the separatrix Γ_1, that forms a homoclinic loop for the given parameter value, intersects S_1. Thus the image point intersects S_1^0 infinitely many times.

Γ_1. Therefore, the point $T_2G_1^*$ lies in S_2^0. Accordingly, the spiral behavior of the point G_2^{2k+3} means that this point infinitely many times intersects the line S_1^0 moving from S_1^+ to S_1^- and then to S_1^+ again.

When the point G_2^{2k+3} lies on S_1^0, this means the presence of the $(k+1)$-round heteroclinic connection. Thus, we find that on the arc of the curve L_1 which corresponds to $\mu_2 \in (\mu_2^p, \mu_2^q)$ there exists infinitely many points of intersection with the bifurcational set C_{21}^{k+1}, i.e., the domain $\mathcal{D}_{21}^{k+1}$ of the function h_{21}^{k+1} is not empty and intersects with the interval (μ_2^p, μ_2^q).

Let us show that the function h_{21}^{k+1} is defined for all $\mu_2 \in (\mu_2^p, \mu_2^q)$. Indeed, it follows from item 2 of Lemma 5.4 that the end points of the domain of h_{21}^{k+1} correspond to the end points on the graph of h_{21}^{k+1} which are some points of intersection of the graph of the function h_1 ($\mu \in L_1$) with the graph of the function h_{21}^s ($\mu \in C_{21}^s$) for some $s = 0, 1, \ldots, k$.[4]

By construction, the points P and Q are successive points of intersection of the curves L_1 with C_{21}^k. This means that on the interval (μ_2^p, μ_2^q) there is no other point of intersection $L_1 \cap C_{21}^k$. Also, we have that the interval (μ_2^p, μ_2^q) lies in the domain $\mathcal{D}_{21}^k$ of the function h_{21}^k. According to item 3 of Lemma 5.4, everywhere on the domain must be $h_{21}^s < h_1$ for all $s < k$. Thus, the equality $h_{21}^s = h_1$ is impossible on the interval (μ_2^p, μ_2^q) and there are no intersection points of $L_1 \cap C_{21}^s$ ($s = 0, 1, \ldots, k-1$) for $\mu_2 \in (\mu_2^p, \mu_2^q)$.

[4]Evidently, the point $\mu = (0, 0)$ is not the end points of the graph of h_{21}^{k+1} because $0 \notin (\mu_2^p, \mu_2^q)$.

Thus the function h_{21}^{k+1} is defined for all $\mu_2 \in (\mu_2^p, \mu_2^q)$. We have also already shown that the graph $\mu_1 = h_{21}^{k+1}(\mu_2)$ $(\mu \in C_{21}^{s+1})$ intersects $\mu_1 = h_1(\mu_2)$ $(\mu \in L_1)$ infinitely many times and the sets $\{\mu_2 | h_1(\mu_2) > h_{21}^{s+1}(\mu_2)\}$ and $\{\mu_2 | h_1(\mu_2) < h_{21}^{s+1}(\mu_2)\}$ are not empty and they contain both an infinite number of intervals. The theorem is proved. ∎

Acknowledgments

The authors are grateful to L. P. Shil'nikov for useful discussions. This research was supported by the EC-Russia Collaborative Project ESPRIT P 9282-ACTCS and by the ISF grant R98300.

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