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# BIFURCATION PHENOMENA IN THE 1:1 RESONANT HORN FOR THE FORCED VAN DER POL — DUFFING EQUATION

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This paper presents the 1:1 resonant horn bifurcation phenomena for the forced van der Pol — Duffing equation. It is shown that the transition to chaos in the case of small dissipation evolves in two parallel processes: a sequence of period-doubling bifurcations and the birth, growth and merging of homoclinic structures.



Recently much attention was paid to research of bifurcation phenomena within resonant horns and on their outlet [Aranson *et al.*, 1982; Ostlund *et al.*, 1983]. Both situations can be described rigorously (see, for example, Morozov & Shil'nikov [1983]; Afraimovich & Shil'nikov [1983]; Turaev & Shil'nikov [1986]) as well as by computer simulation, but certain details of the bifurcation process in concrete systems with large amplitude of external force are certainly of special scientific interest. One of the least studied is the 1:1 resonance.

One of the simplest systems where we can study the corresponding regularities, is the van der Pol — Duffing equation with harmonic external forcing:

$$\ddot{X} - \varepsilon(1 - X^2)\dot{X} + X^3 = 2A\cos(\omega t) \ . \qquad (1)$$

Different aspects of the rise and development of complicated dynamics in the given system were discussed in Ueda, [1979]; Van Buskirk & Jeffries [1985]; Holms [1979]; Dmitriev & Kislov [1989]. In this paper, we study the case of small dissipation: $\varepsilon = 0.006$. Forcing frequency $\omega$ is selected so that the system is in strict 1:1 resonance with the frequency of the self-oscillatory regime, established with $A = 0$; $\omega = \frac{2\pi}{T}$, where $T = 3.8876$. The magnitude $A$ – the forcing amplitude – was taken as bifurcation parameter.

For $A = 0$ the phase plane of the system is characterized by an unstable equilibrium state $(X = 0, \dot{X} = 0)$, which is encircled by a period-$T$-limit cycle. With small $A$, according to the Andronov-Vitt theory [Andronov & Vitt, 1930], there exist two resonant periodic trajectories (cycles) of period $T$: stable $L_1$, and saddle $L_2$. In addition, the equilibrium state is followed by an unstable cycle $L_3$ of period $T$. Along with it, all the trajectories, excluding $L_3$, $L_2$ and the stable separatrices of $L_2$ cycle, tend to $L_1$. As we increase the value of $A$, the vector field structure undergoes considerable changes, and approximately at $A = 11$ a chaotic oscillatory regime is observed in the system. It turns out that the transition to chaos in the given system has a number of essential features, conditioned by the effect of small dissipation. We consider that the most important of these peculiarities is the absence of a strict parameter borderline, that divides the vector field into regular and chaotic regimes.

Our research was performed by numerically integrating the map $F$ of the plane $(\dot{X}, X)$ over one period $T$. The fixed points of the map $F$ correspond to the cycles with period $T$ of system (1); the points of period in correspond to cycles with period $nT$. It is necessary to note that system (1) is symmetric with respect to the following transformation

$$(X \to -X, t \to -t + \pi/\omega) \ . \qquad (2)$$

We consider the periodic points of the map F to be symmetric, if they correspond to cycles which are symmetric with respect to the transformation (2). The above mentioned cycles $L_1$, $L_2$, $L_3$ are symmetric with respect to (2). We will designate the points of their intersections with the plane $t = 0$ as $O_1$, $O_2$, $O_3$ correspondingly.

Bifurcations of the fixed points of the map F do not happen until $A = 0.6$. Figure 1 shows the phase portrait for $A = 0.2$. It is evident that the longest part of the unstable manifold of the saddle point $O_2$ aims towards $O_1$. Homoclinic intersections of the stable and the unstable manifold is a characteristic feature of the given situation, as can be observed in Fig. 2, which shows the vicinity of the saddle point on a large scale. The birth of the homoclinic structure in this case can be explained by the fact that the dissipation is close to zero: the matter is, that the presence of homoclinic structures ("chaotic layers") is characteristic of conservative maps, and they cannot possibly disappear at small dissipation. It is evident from the figure, that the homoclinic structure here is very "narrow", and most trajectories of the $F$-map iterations tend to $O_1$. It should be mentioned, that, judging by Fig. 2, the stable and the unstable manifolds come into contact with $O_2$ for $A$ close to 0.2. So, because of Gavrilov & Shilnikov [1973], or $A = 0.2$, or for close values of $A$, the system possesses other stable periodic trajectories of rather large period besides $O_1$. They cannot, however, be revealed by numerical research.

For $A = 0.6$, $O_2$ and $O_3$ merge and disappear. The next bifurcation is the appearance of two symmetric trajectories of period 3. For $A = 2.64$ there appears one more period-3 symmetric trajectory of the saddle-knot type (one multiplicator is equal to 1, the other is smaller), which for $A > 2.64$ is divided into a saddle and a stable periodic trajectory (see Fig. 2, showing phase portrait for $A = 2.7$). As $A$ increases, period-3 saddle trajectory approaches $O_1$ and, for $A = 2.84$, comes as close to it as 0.03 by $X$ and 0.3 by $X$. At this moment the basin of attraction of $O_1$ is very small (it is limited by the stable manifolds of the period-3 saddle trajectory), and most trajectories aim to the fixed point of period 3. As $A$ increases further, the saddle trajectory comes off $O_1$ again, and at $A = 3.67$ it merges with the stable trajectory into a saddle-knot of period 3, which then disappears. The described sequence of bifurcations is characteristic of systems with small dissipation. In conservative systems the birth of secondary resonances near a stable (elliptic) fixed point with the change of parameter takes place permanently, but with the introduction of dissipation only a part of them survives. In our case the dissipation turns out to be sufficiently large to preserve only "the strongest" secondary resonance — 3 period one.

For $A = 4.7$, $O_1$ becomes a saddle point. It gives rise to a pair of stable fixed points $O_4$ and $O'_4$ (cycles of system (1), that go through $O_4$ and $O'_4$ at $t = 0$ are symmetric to each other with respect to the transformation (2)). Figure 4 shows a phase portrait at $A = 5$. With the growth of $A$ there appears a homoclinic intersection of the stable and the unstable manifolds of $O_1$ (see Fig. 5, illustrating a phase portrait for $A = 6$). At $A = 6.3$ there appears a pair of fixed cycles of the saddle-knot type, symmetric to each other with respect (2). At $A > 6.3$, the saddle-knots are divided into pairs of saddle and stable fixed points: $O_5$, $O'_5$ and $O_6$, $O'_6$ respectively. At $A = 6.296$, the stable points $O_4$ and $O'_4$ undergo a period-doubling bifurcation. Then, at $A = 7.395$, the stable cycles of period 2 come back into $O_4$ and $O'_4$, and the latter become stable again. For $A = 8.428$, $O_4$ and $O'_4$ merge with $O_5$ and $O'_5$ respectively and vanish. So, for $6.3 < A < 8.428$, the system has four stable modes, and hysteresis phenomena may be observed. Figures 6 and 7 show the phase portraits at $A = 6.5$ and $A = 7.4$, respectively. It can be seen, that as $A$ increases the reorganization of the stable and the unstable manifolds of saddle fixed points takes place. This is associated with the formation of homoclinic intersections and homoclinic contacts. Such homoclinic structures contain [Shil'nikov, 1967; Gavrilov & Shil'nikov, 1973] a limited number of saddle periodic trajectories and can also contain stable large-period trajectories. It's worth mentioning that in this case, for most initial conditions, phase trajectories tend to one of the stable fixed points, or to period-2 fixed points. Hence, despite the presence of homoclinics structures in the range of $A$ parameter variation, one cannot assert the establishing of chaotic fluctuations, but only a chaotic transient process.

For $A > 8.428$, there exist three fixed points: saddle $O_1$ and stable $O_6$ and $O'_6$ (phase portrait for $A = 9.5$ in Fig. 8). With an increase in $A$ there appear sequences of bifurcations such as period-doubling cascades: at $A = 9.961$, $O_6$ and $O'_6$ lose stability, period-2 trajectories, sprung from $O_6$ and $O'_6$, lose stability at $A = 10.996$, trajectories of period 4 at $A = 11.108$, trajectories of period 8 at $A = 11.121$ and so on. We see, that the distances between parameter bifurcation values decrease much faster than that with strongly dissipative systems, which is caused by a small dissipation. More important, in comparison with strongly dissipative systems, is the fact that, along with period-doubling, there proceeds a competing process of the

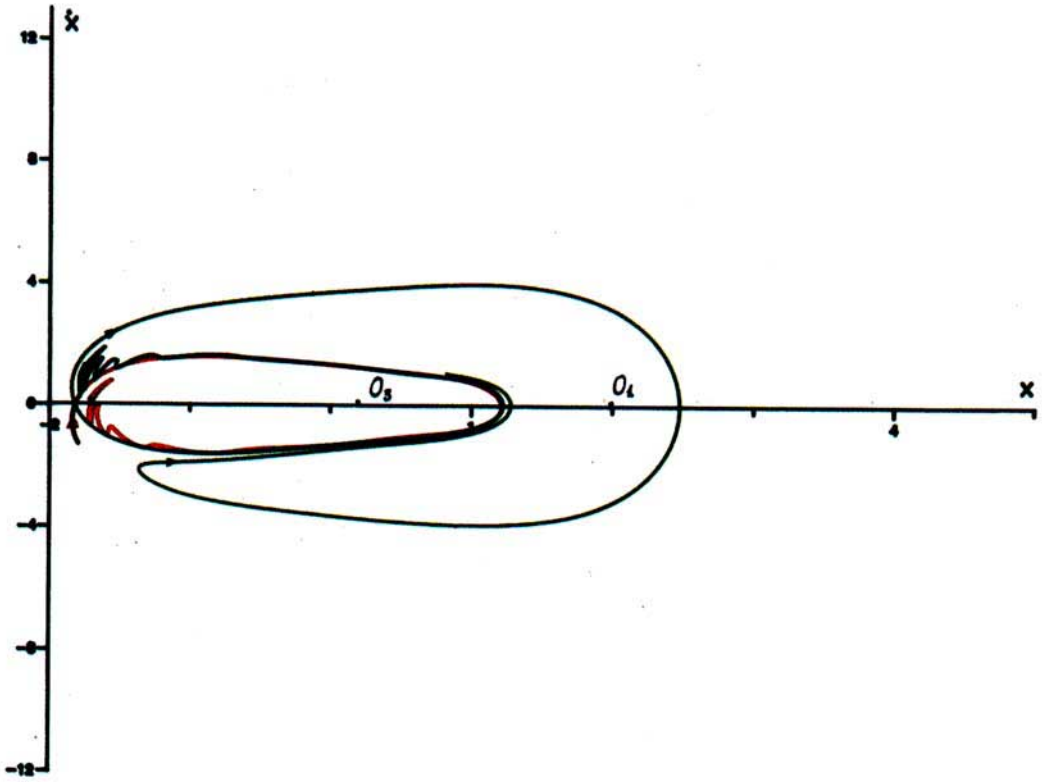


Fig. 1.

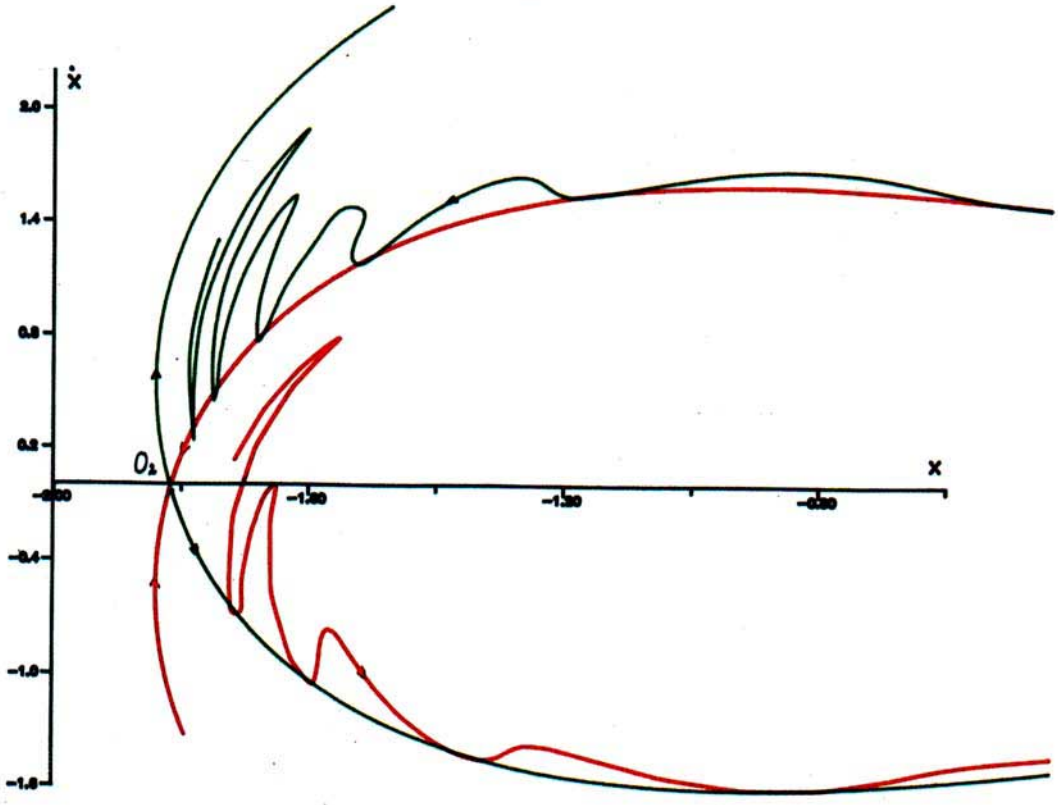


Fig. 2.

Figs. 1–2. Representative phase portraits in the 1:1 resonance horn: (1, 2) — A = 0, 2.

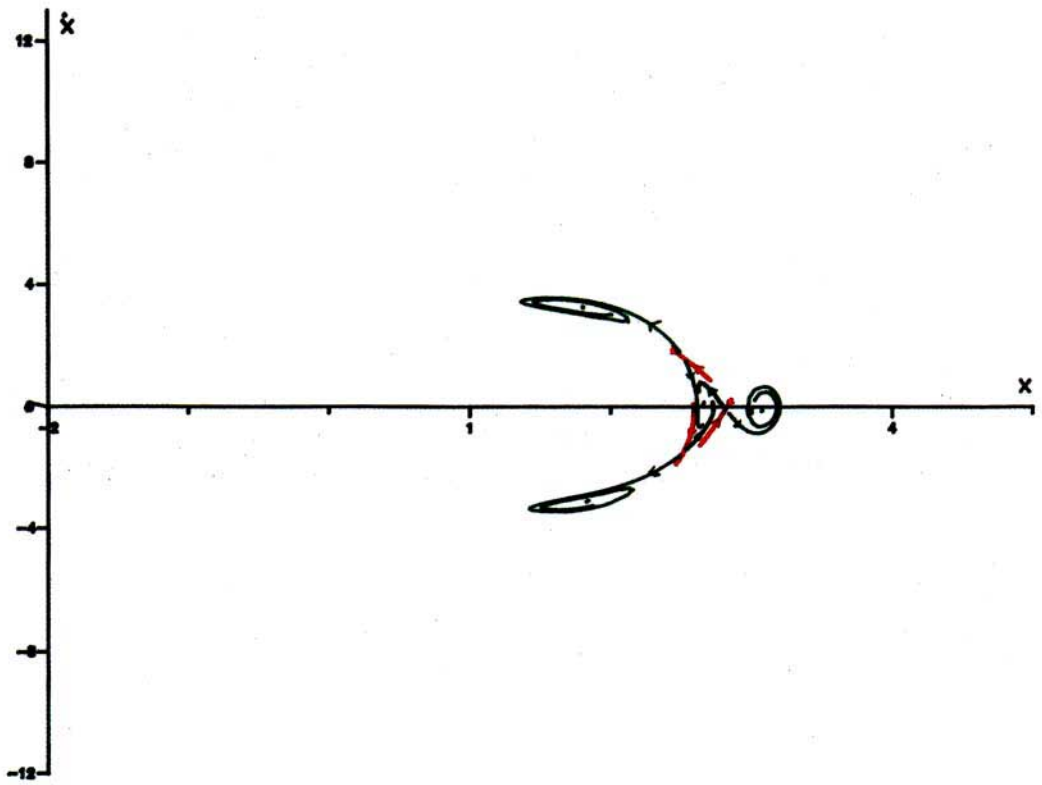


Fig. 3.

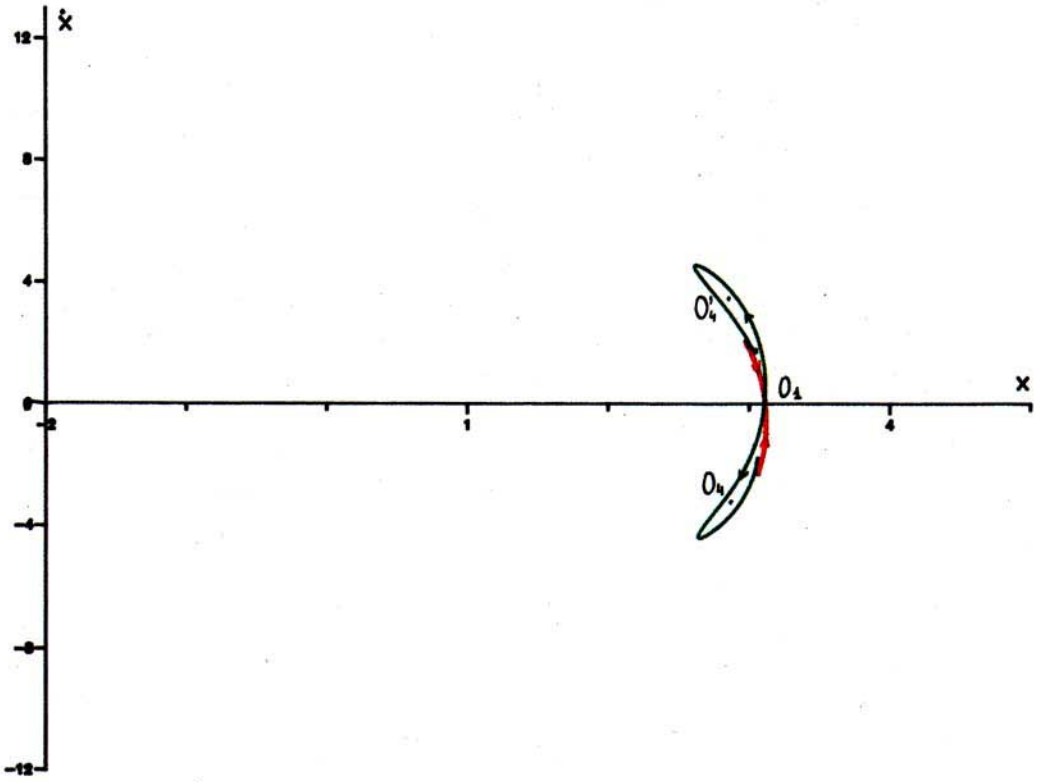


Fig. 4.

Figs. 3–4. Representative phase portraits in the 1:1 resonance horn: (3) — A = 2.7; (4) — A = 5.0.

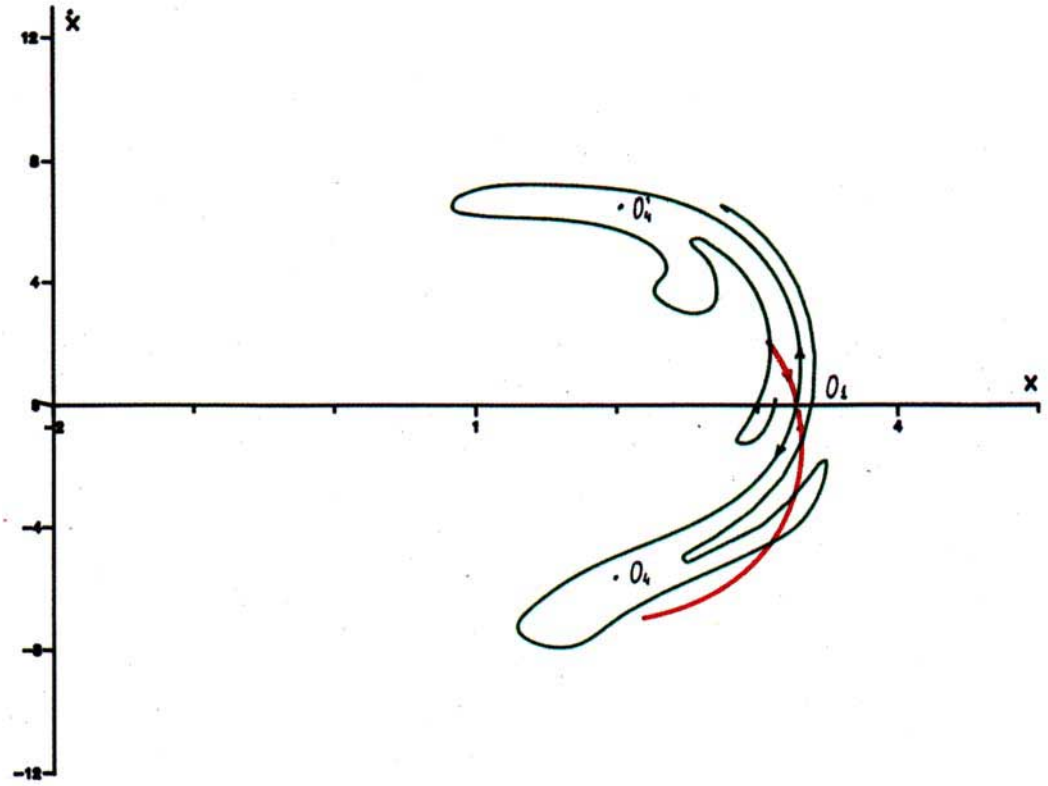


Fig. 5.

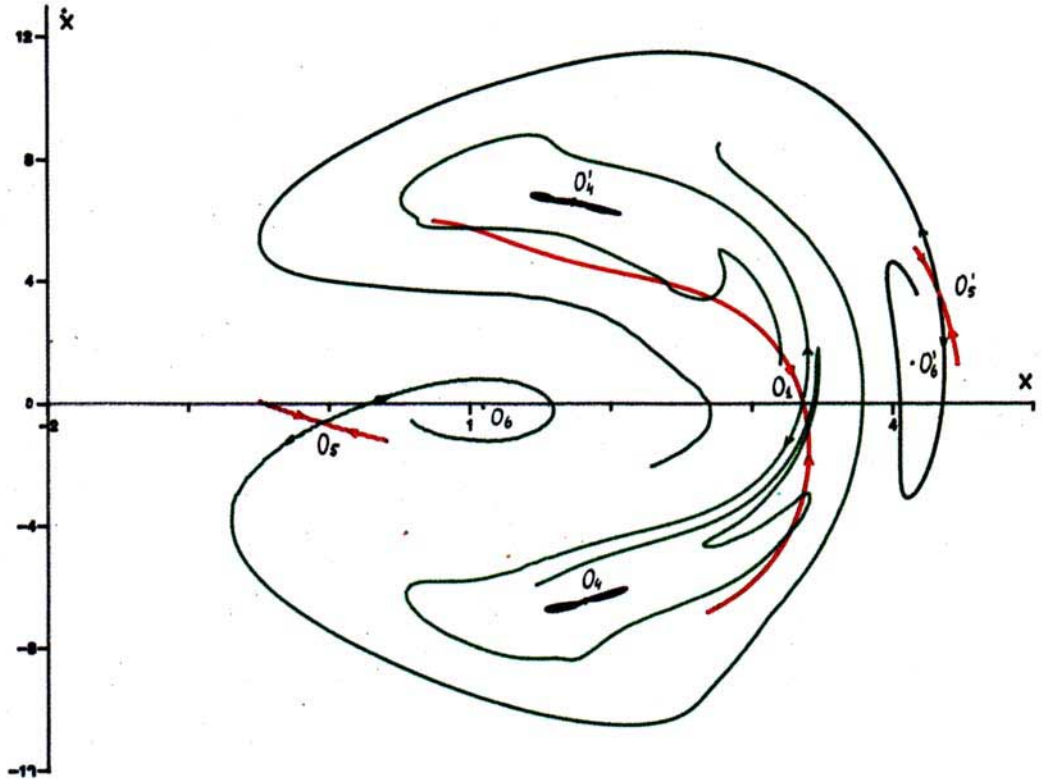


Fig. 6.

Figs. 5–6. Representative phase portraits in the 1:1 resonance horn: (5) — A = 6.0; (6) — A = 6.5.

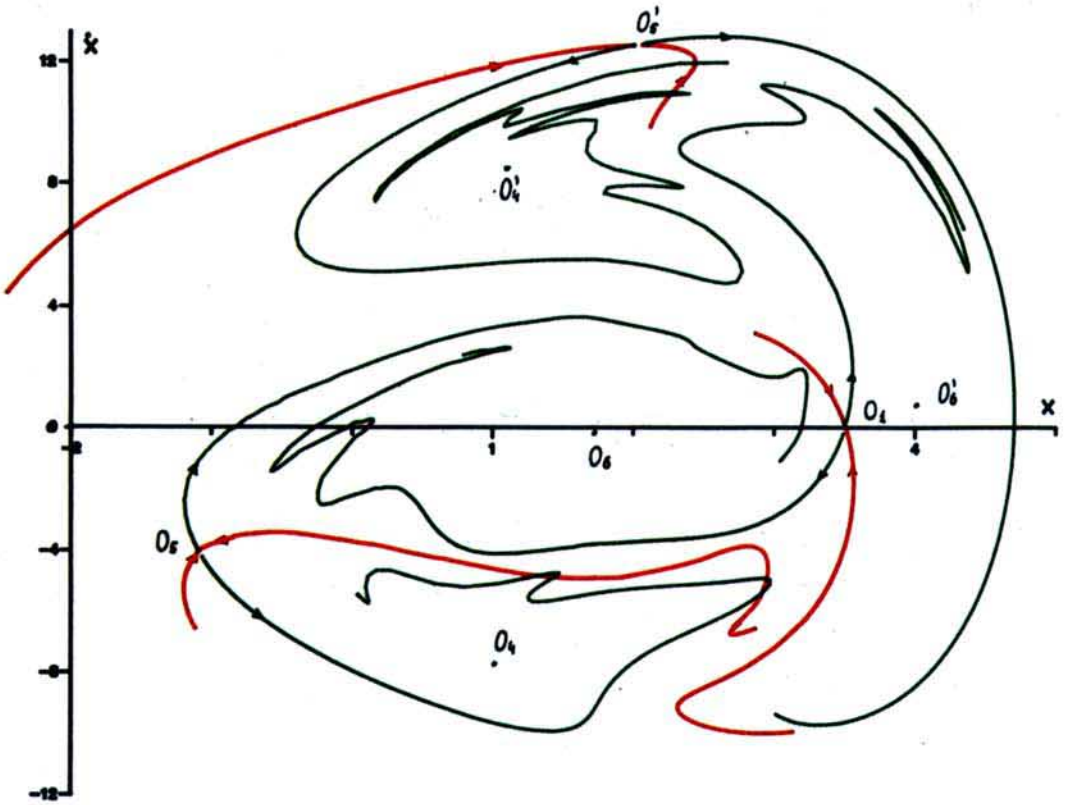


Fig. 7.

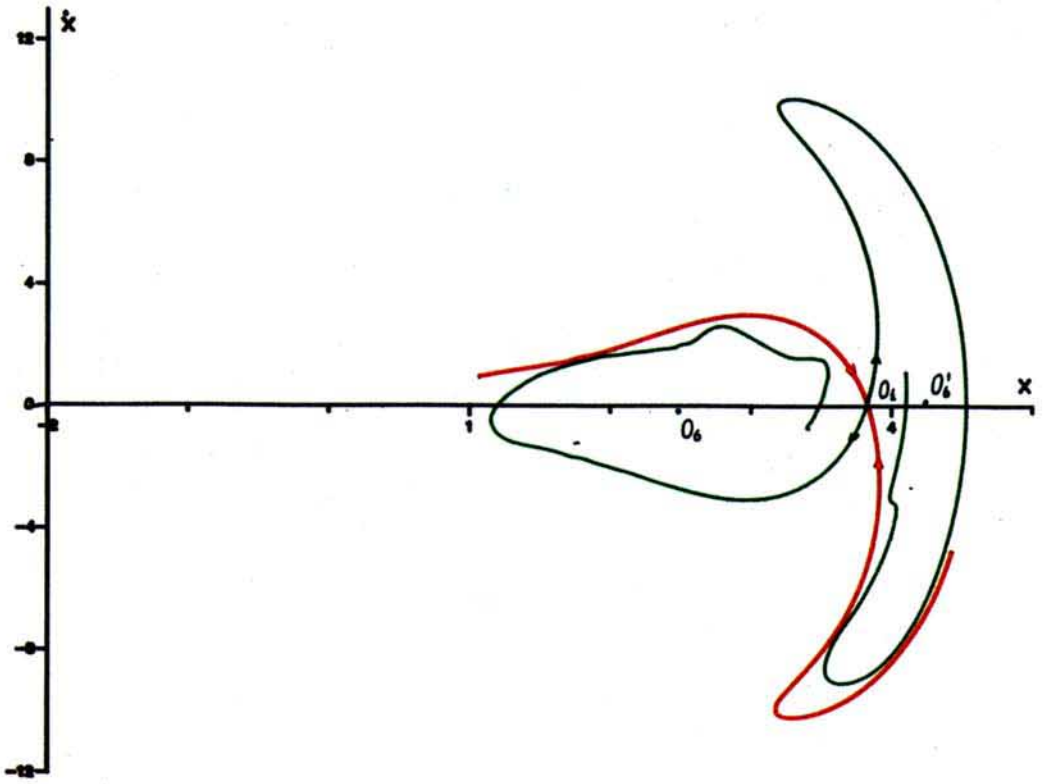


Fig. 8.

Figs. 7–8. Representative phase portraits in the 1:1 resonance horn: (7) — A = 7.4; (8) — A = 9.5.

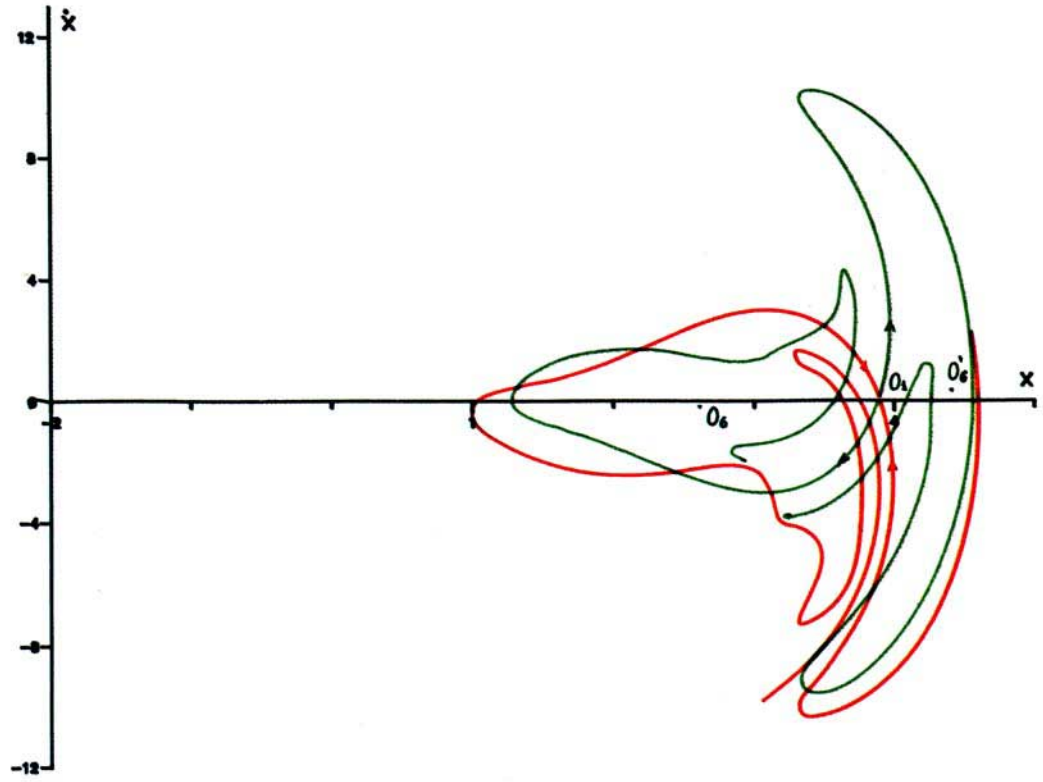


Fig. 9.

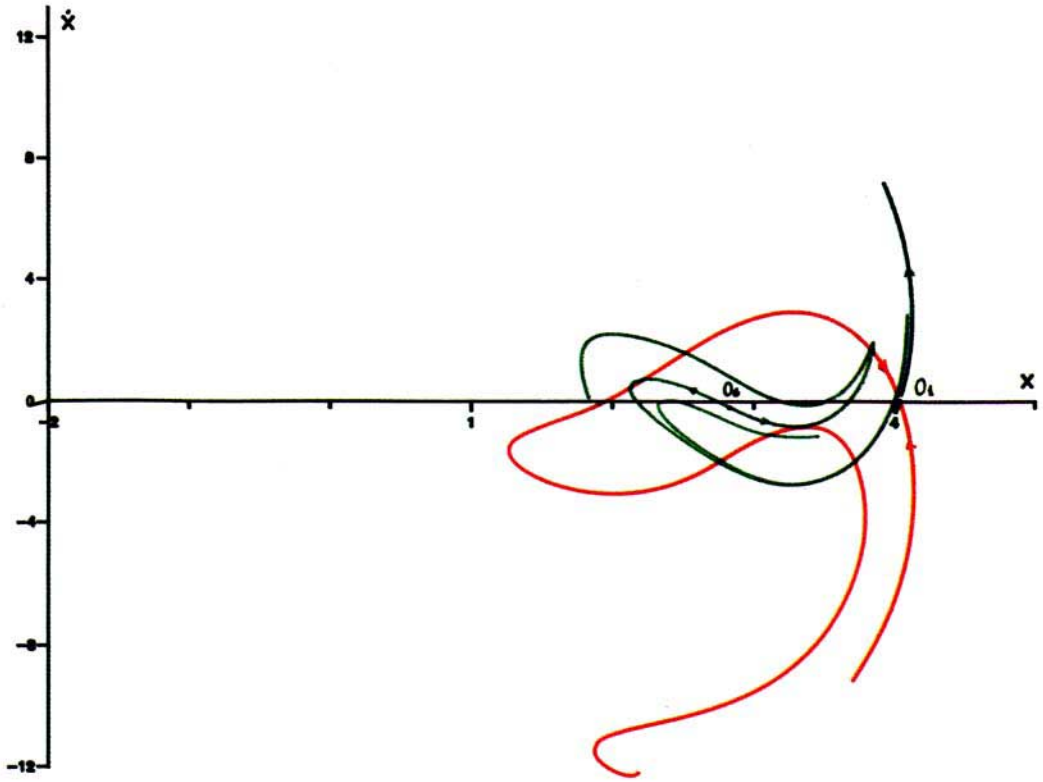


Fig. 10.

Figs. 9–10. Representative phase portraits in the 1:1 resonance horn: (9) — A = 10.0; (10) — A = 11.0.

transition to chaos through the birth of homoclinic structures.

Figure 8 illustrates that, for $A$ close to 9.5, the stable and the unstable manifolds come into contact with $O_1$. Then a nonattractive homoclinic structure is formed (see Fig. 9 with a phase portrait for $A = 10$). At $A = 11$ (after second doubling) a homoclinic structure becomes "wide", and due to the formation of heteroclinic trajectories it comprises cycles $O_6$ and $O'_6$, which have become saddle-type cycles (see Fig. 10, showing the intersection of the $O_6$ unstable manifold and the $O_1$ stable manifold). The appearance of homoclinic structures for saddle points which occur as a result of period-doubling might be expected because of small dissipation. It was quite unexpected, however, that homoclinic structures, referring to different saddle points, unite before the sequence of flip bifurcation is finished.

As can be seen from Fig. 10, for $A = 11$ the basin of attraction of period-4 stable trajectories is narrow and disjoint. So we may think, that for $A = 11$ the chaotic fluctuations are established in the system. Period-4 trajectory can be regarded as a "window" in the sphere of chaotic dynamics, along with, for example, long-period stable trajectories, which occurs as a result of homoclinic contacts (one of these contacts can be seen in Fig. 9). We think that the above reported process of transition to chaos through birth, growth and merging of homoclinic structures, which is parallel to doubling, is common for weak dissipative systems. It should be mentioned, that such a way of transition to chaos, in contrast to period-doubling in strong dissipative systems, cannot be characterized by a strict parameter borderline which divides the vector fields of regular behavior (as it is impossible to draw a strict borderline between the "wide" and the "narrow" homoclinic structures, or between the "wide" and the "narrow" basins of attraction).



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