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ON SOME MATHEMATICAL TOPICS IN CLASSICAL SYNCHRONIZATION. A TUTORIAL

ANDREY SHILNIKOV
Department of Mathematics and Statistics,
Georgia State University, Atlanta, GA 30303-3083, USA
ashilnikov@mathstat.gsu.edu

LEONID SHILNIKOV
Institute for Applied Mathematics & Cybernetics,
Nizhny Novgorod, Russia 603005

DMITRY TURAEV
Ben Gurion University of the Negev, Beer Sheva, Israel

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A few mathematical problems arising in the classical synchronization theory are discussed; especially those relating to complex dynamics. The roots of the theory originate in the pioneering experiments by van der Pol and van der Mark, followed by the theoretical studies by Cartwright and Littlewood. Today, we focus specifically on the problem on a periodically forced stable limit cycle emerging from a homoclinic loop to a saddle point. Its analysis allows us to single out the regions of simple and complex dynamics, as well as to yield a comprehensive description of bifurcational phenomena in the two-parameter case. Of a particular value is the global bifurcation of a saddle-node periodic orbit. For this bifurcation, we prove a number of theorems on birth and breakdown of nonsmooth invariant tori.

Keywords: Synchronization; saddle-node; homoclinic bifurcation.

1. Introduction. Homoclinic Loop Under Periodic Forcing

The following two problems are the enduring ones in the classic theory of synchronization: the first is on the behavior of an oscillatory system forced by a periodic external force and the second is on the interaction between two coupled oscillatory systems. Both cases give a plethora of dynamical regimes that occur at different parameter regions. Here, a control parameter can be the amplitude and frequency of the external force, or the strength of the coupling in the second problem.

In terms of the theory of dynamical systems the goal is to find a synchronization region in the parameter space that corresponds to the existence of a stable periodic orbit, and next describe the ways synchronization is lost on the boundaries of such a region.

Since a system under consideration will be high dimensional (the phase space is of dimension three in the simplest case and up), one needs the whole arsenal of tools of the contemporary dynamical systems theory.

However, our drill can be simplified substantially if the amplitude of the external force (or the coupling strength) μ is sufficiently small. So when $|\mu| \ll 1$, a two-dimensional invariant torus replaces the original limit cycle. The behavior of the

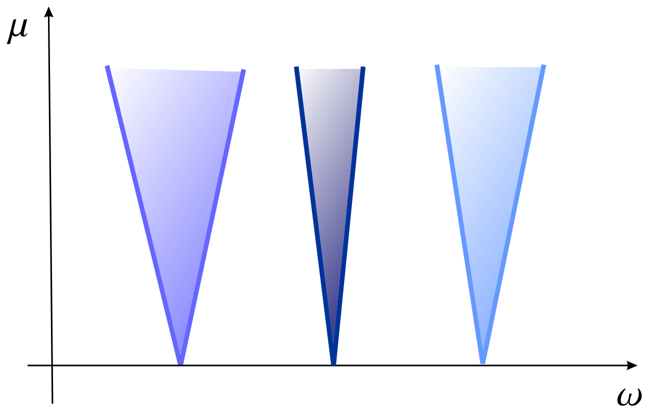


Fig. 1. Arnold's tongues in the (ω, μ)-parameter plane.

trajectories on this torus is given by the following diffeomorphism of the circle:

$$\theta_{n+1} = \theta_n + \omega + \mu f(\theta_n) \mod 2\pi\,, \tag{1}$$

where f is 2π-periodic in θ, and ω measures the difference in frequencies. The typical (ω, μ)-parameter plane of (1) looks like one shown in Fig. 1.

In theory, each rational value ω on the axis $\mu = 0$ is an apex for a synchronization zone known as Arnold's tongue. Within each Arnold's tongue the Poincaré rotation number:

$$r = \frac{1}{2\pi} \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N} (\theta_{n+1} - \theta_n)$$

stays rational but becomes irrational outside of it. The synchronous regime or phase-locking is observed at rational values of the rotation number corresponding to the existence of the stable periodic orbits on the torus, followed by the beatings — quasi-periodic trajectories existing at irrational ω.

Note that in applications long periodic orbits existing in narrow Arnold's tongues corresponding to high order resonances are hardly distinguishable from quasi-periodic ones densely covering the torus.

The above is also true for the van der Pol equation:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \omega_0^2 x = \mu A \sin \omega t \tag{2}$$

in the "quasi-linear" case, i.e. when $|\mu| \ll 1$ and $A^2 < \frac{4}{27}$, as shown by van der Pol [1927], Andronov and Vitt [1930], Krylov and Bogolubov [1947], as well as by Afraimovich and Shilnikov [1974a, 1974b] who proved the persistence of the torus after the saddle-node periodic orbit vanishes on the boundary of a principal resonance zone.

The case of a non-small μ, which is a way more complicated than the quasi-linear one, has a long history. In 1927 van der Pol and van der Mark [1927] published their new results on experimental studies of the sinusoidally driven neon bulb oscillator. Although they put the primary emphasis upon the effect of division of the frequency of oscillations in the system, they also noticed that: "often an irregular noise is heard in the telephone receiver before the frequency jumps to the next lower value." This might mean that they run across the coexistence of stable periodic regimes with different periods (often rather long: in the experiments, these periods were 100–200 times larger than that of the driving force) as well as, in modern terminology, a complex dynamics (though considered as a side-product then). The latter may indicate, if not the existence of a strange attractor in the phase space of the system, then, at least, the abundance of saddle orbits comprising a nontrivial set in charge for the complex transient process.

These experiments drew Cartwright and Littlewood' attention. In 1945 they published some astonishing results of their analysis carried out for van der Pol equation (2) with $\mu \gg 1$ [Cartwright & Littlewood, 1945]. Namely, they pointed out the presence of two kinds of intervals for amplitude values A in the segment $(0; 1/3)$: the intervals of the first kind corresponded to a trivial (periodic) dynamics in the equation; whereas in the intervals of the second kind, besides both coexisting stable periodic orbits, there was a nontrivial non-wandering set consisting of unstable orbits and admitting a description in terms of symbolic dynamics with two symbols. Thus, they were the first who found that a three-dimensional dissipative model might have countably many periodic orbits and a continuum of aperiodic ones. The elaborative presentation of these results was done by Littlewood sometime later [Littlewood, 1957a, 1957b].

Levinson [1949] presented an explanation of these results for the following equation:

$$\varepsilon\ddot{x} + \dot{x}\,\mathrm{sign}(x^2 - 1) + \varepsilon x = A \sin t\,, \tag{3}$$

with $\varepsilon \ll 1$. Since this equation is piecewise linear, the study of the behavior of its trajectories can be essentially simplified, making the analysis quite transparent.

The idea of "typicity" of the complex behavior of trajectories for a broad class of nonlinear equations was voiced by Littlewood [1957a, 1957b] in the explicit form. Today, it is interesting to note that the very first paper [Cartwright & Littlewood, 1945] contained a prophetic statement about the topological equivalence of the dynamics of the van der Pol equations with different values of A corresponding to complex nontrivial behavior. In other words, the dynamical chaos was viewed by Cartwright and Littlewood as a robust (generic) phenomenon. When Levinson pointed out these results to Smale, the latter found that they might admit a simple geometric interpretation, at least at the qualitative level. This led Smale to his famous example (dated by 1961) of a diffeomorphism of the horseshoe with a nontrivial hyperbolic set conjugated topologically with the Bernoulli subshift on two symbols. The commencement of the modern theory of dynamical chaos, as we all know it, was thus proclaimed.

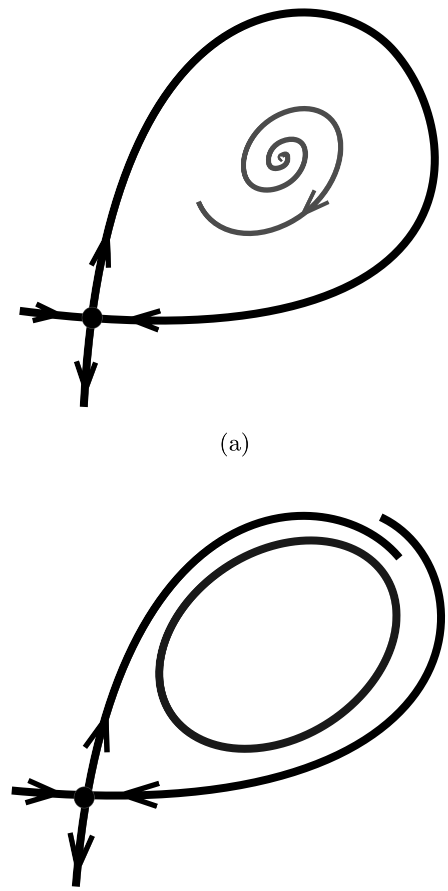
The study of equations of Levinson's type was proceeded by works of Osipov [1975] and Levi [1981] in 70–80s. They produced a complete description for nonwandering sets as well as proved their hyperbolicity in the indicated intervals of parameter A values.

The analysis in the works mentioned above was done only for a finite number of intervals of values of A (the intervals of hyperbolicity). The total length of the remaining part tends to zero as $\mu \to \infty$ (or $\varepsilon \to 0$). Thus, it turns out that dynamical features occurring within the intervals of non-hyperbolicity could be indeed neglected in the first approximation. Note however that it is those intervals which correspond to the transition from simple to complex dynamics along with appearances and disappearances of stable periodic orbits as well as onset of homoclinic tangencies leading to the existence of the Newhouse intervals of structural instability, and so forth ... In systems which are not singularly-perturbed all these effects have to be subjected to a scrupulous analysis.

A suitable example in this sense can be an autonomous system

$$\dot{x} = X(x, \mu),$$

which is supposed to have a stable periodic orbit L_μ that becomes a homoclinic loop to a saddle equilibrium state as $\mu \to 0^+$. One may wonder what happens as the system is driven periodically by a small force of amplitude of order μ? This problem was studied in a series of papers by Afraimovich and Shilnikov [1974a, 1974b, 1977, 1991]. Below, we overview a number of obtained results that are of momentous value for the synchronization theory.



(a)

(b)

Fig. 2. Homoclinic bifurcation leading to the birth of the stable limit cycle, plane.

For the sake of simplicity we confine the consideration to two equations. Suppose that the autonomous system

$$\dot{x} = \lambda x + f(x, y, \mu),$$
$$\dot{y} = \gamma y + g(x, y, \mu)$$

has an equilibrium state of the saddle type at the origin with a negative saddle value

$$\sigma = \lambda + \gamma < 0. \tag{4}$$

Let this saddle have a separatrix loop at $\mu = 0$, see Fig. 2(a).

As well known [Andronov & Leontovich, 1963; Andronov *et al.*, 1971] that the system shall have a single periodic orbit bifurcating off the loop for small μ. Let that be so when $\mu > 0$, see Fig. 2(b). In general, the period of the new born cycle is of order $|\ln \mu|$. The last observation makes this problem and that on van der Pol equation having, at $A = 0$, a relaxation limit cycle of period $\sim 1/\mu$ resembling: in

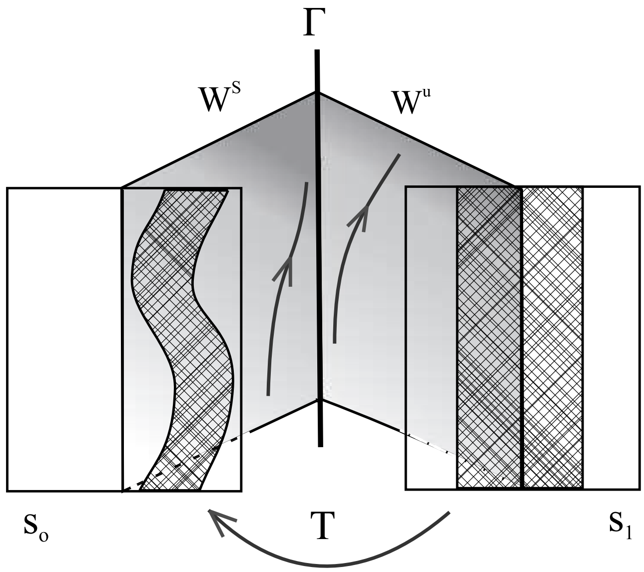


Fig. 3. Poincaré map taking a cross-section S_1 in the plane $x = \delta$ transverse to the unstable manifold W^u of the saddle-periodic orbit Γ onto a cross-section S_0 (transverse) to the stable manifold W^S.

both cases the period of the bifurcating limit cycle grows with no bound as the small parameter tends to zero.

As far as the perturbed system

$$\begin{aligned} \overline{x} &= \lambda x + f(x, y, \mu) + \mu p(x, y, t, \mu), \\ \overline{y} &= \gamma y + g(x, y, \mu) + \mu q(x, y, t, \mu) \end{aligned} \tag{5}$$

is concerned, where p and q are 2π-periodic functions in t, we shall also suppose that the Melnikov function is positive, which means that the stable W^s and unstable W^u manifolds of the saddle periodic orbit (passing near the origin ($x = y = 0$)) do not cross, see Fig. 3.

Now, one can easily see that the plane $x = \delta$ in the space $\{x, y, t\}$ is a cross-section for system (5) at small δ. The corresponding Poincaré map T_μ is in the form close to the following modeling map:

$$\begin{aligned} \overline{y} &= [y + \mu(1 + f(\theta))]^\nu \\ \overline{\theta} &= \theta + \omega - \frac{1}{\gamma} \ln[y + \mu(1 + f(\theta))], \end{aligned} \tag{6}$$

where $\nu = -(\lambda/\gamma) > 1$, ω is a constant, and $\mu(1+f(\theta))$ is the Melnikov function with $\langle f(\theta)\rangle = 0$. The right-hand side of the second equation is to be evaluated in modulo 2π since θ is an angular variable. The last can be interpreted as the phase difference between the external force and the response of the system. Thus, attracting fixed points (for which $\overline{\theta} = \theta \bmod 2\pi$) of the above map correspond to the regime of synchronization.

The limit set of the map T_μ at sufficiently small μ lies within an annulus $K_\mu = \{0 < x < C\mu^\nu, 0 \le \theta < 2\pi\}$ with some $C > 0$. After rescaling $y \to \mu^\nu y$ the map assumes the form

$$\begin{aligned} \overline{y} &= [1 + f(\theta)]^\nu + \cdots, \\ \overline{\theta} &= \theta + \tilde{\omega} - \frac{1}{\gamma} \ln[1 + f(\theta)] + \cdots, \end{aligned} \tag{7}$$

where the ellipsis stand for the terms converging to zero along with their derivatives, while $\tilde{\omega} = (\omega - (1/\gamma) \ln \mu)$ tends to infinity as $\mu \to +0$, i.e. $\tilde{\omega} \bmod 2\pi$ assumes arbitrary values in the interval $[0, 2\pi)$ countably many times. Hence, the dynamics of the Poincaré map is dominated largely by the properties of the family of circle maps:

$$\overline{\theta} = \theta + \tilde{\omega} + \mathcal{F}(\theta) \bmod 2\pi, \tag{8}$$

where $\mathcal{F}(\theta) = -(1/\gamma) \ln[1 + f(\theta)]$.

Assertion 1. [Afraimovich & Shilnikov, 1991] 1. In the case where

$$\frac{1}{\gamma} \frac{f'(\theta)}{1 + f(\theta)} < 1, \tag{9}$$

the map T_μ has an attracting smooth invariant closed curve of the form $y = h(\theta, \mu)$ that contains ω-limit set of any trajectory in K_μ.

Assertion 2. Let an interval $I = [\theta_1, \theta_2]$ exist such that either

$$f'(\theta) < 0 \quad \text{everywhere on } I \tag{10}$$

and

$$\frac{1}{\gamma} \ln \frac{1 + f(\theta_1)}{1 + f(\theta_2)} > 2\pi(m + 1), \quad m \ge 2, \tag{11}$$

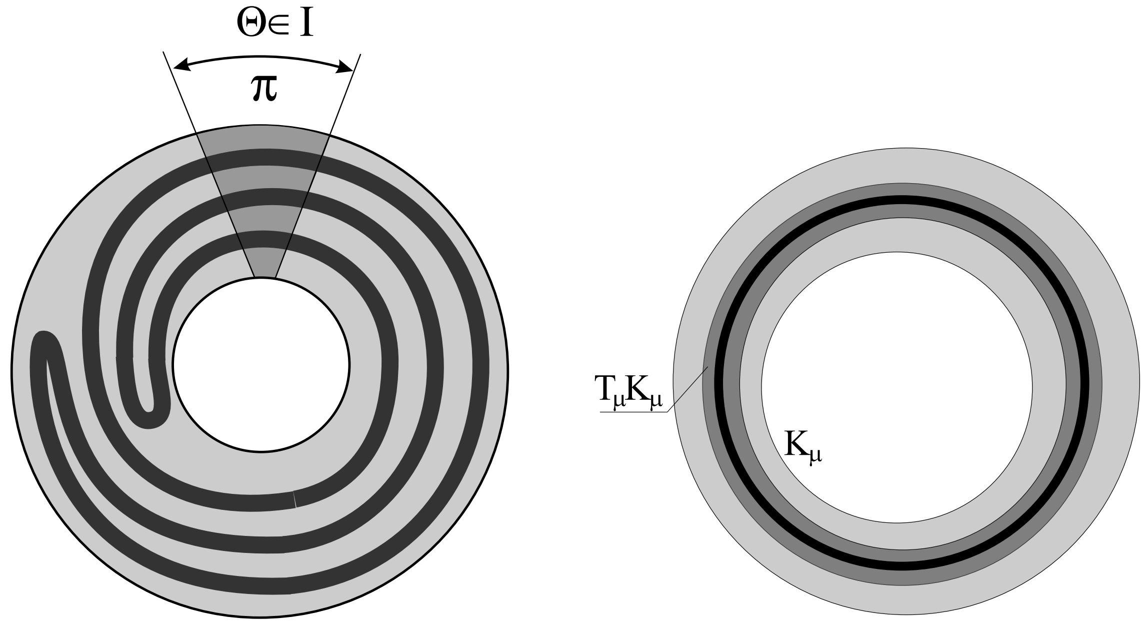
or

$$\frac{1}{\gamma} \frac{f'(\theta)}{1 + f(\theta)} > 2 \quad \text{everywhere on } I \tag{12}$$

and

$$\frac{1}{\gamma} \ln \frac{1 + f(\theta_2)}{1 + f(\theta_1)} > 2(\theta_2 - \theta_1) + 2\pi(m + 1), \quad m \ge 2. \tag{13}$$

Then, for all sufficiently small $\mu > 0$ the map T_μ will have a hyperbolic set Σ_μ conjugated with the Bernoulli subshift on m symbols.



(a) $m = 2$ (b)

Fig. 4. (a) The image of the region Π under T_μ overlaps with Π at least m times. (b) The image of the annulus K_μ under T_μ has no folds.

For example, in case $f(\theta) = A \sin\theta$ we have that if $A < (\gamma/\sqrt{1+\gamma^2})$, then the invariant closed curve is an attractor of the system for all small μ, while we have complex dynamics for $A > \tanh 3\pi\gamma$.

The meaning of conditions (10) and (12) is that they provide expansion in the θ-variable within the region $\Pi : \theta \in I$ and, therefore, hyperbolicity of the map (6) in the same region (contraction in y is always achieved at y sufficiently small since $\nu > 1$). Furthermore, if the conditions (11) and (13) are fulfilled, then the image of the region Π overlaps with Π at least m times [see Fig. 4(a)]. Hence, we obtain a construction analogous to the Smale horseshoe; then the second assertion above becomes evident by say, referring to the lemma on a saddle fixed point in a countable product of Banach spaces [Shilnikov, 1967].

In the first assertion, the condition (9) leads to the image of the annulus K_μ under T_μ having no folds for small μ [see Fig. 4(b)], in other words it is also an annulus bounded by two curves of the form $y = h^{\pm}(\theta)$. The subsequent image of this annulus is self-alike too, and so on. As a result, we obtain a sequence of embedded annuli; moreover, the contraction in the y-variable guarantees that they intersect in a single and smooth closed curve. This curve is invariant and attracting as follows, say, from the annulus principle introduced in [Afraimovich & Shilnikov, 1974a, 1974b, 1977], (see also [Shilnikov *et al.*, 1998]).

In an attempt for a comprehensive investigation of the synchronization zones we restrict ourself to the case $f(\theta) = A \sin\theta$ (or $f(\theta) = Ag(\theta)$, where $g(\theta)$ is a function with preset properties). This choice lets us build a quite reasonable bifurcation diagram (Figs. 5 and 6) in the plane of the parameters $(A, -\ln\mu)$ in the domain $\{0 \le A < 1, 0 < \mu < \mu_0\}$, where μ_0 is sufficiently small.

Each such region can be shown to adjoin to the axis $-\ln\mu_0$ at a point with the coordinates $(2\pi k; 0)$, where k is a large enough integer. Inside, there coexists a pair of fixed points of the Poincaré map such that $\overline{\theta} = \theta + 2\pi k$. Their images in the system (5) are the periodic orbits of period $2\pi k$. The borders of a resonant zone D_k are the bifurcation curves B_k^1 and B_k^2 on which the fixed points merge into a single saddle-node. The curves B_k^1 continue up to the line $A = 1$, while the curves B_k^2 bend to the left (as μ increases) staying below $A = 1$. Therefore, eventually these curves B_k^2 will cross the curves B_m^1 and B_m^2 with $m < k$.

Inside the region D_k one of the fixed points of the map, namely Q_k is always of saddle type. The other point P_k is stable in the region S_k right between the curves B_k^1, B_k^2 and B_k^-. Above B_k^- the point P_k loses stability that goes to a cycle of period

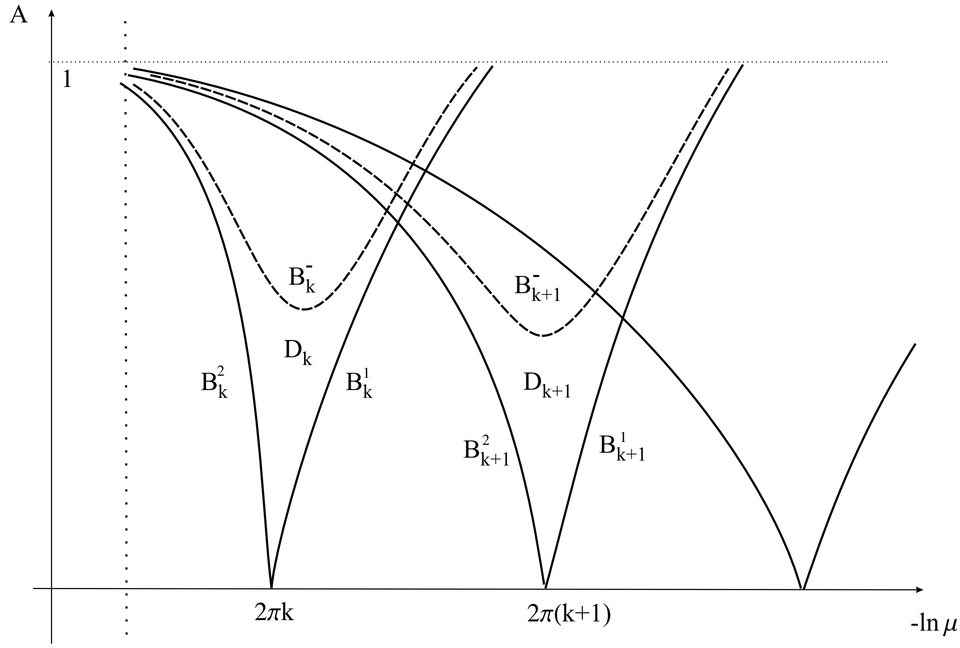


Fig. 5. Overlapping resonant zones.

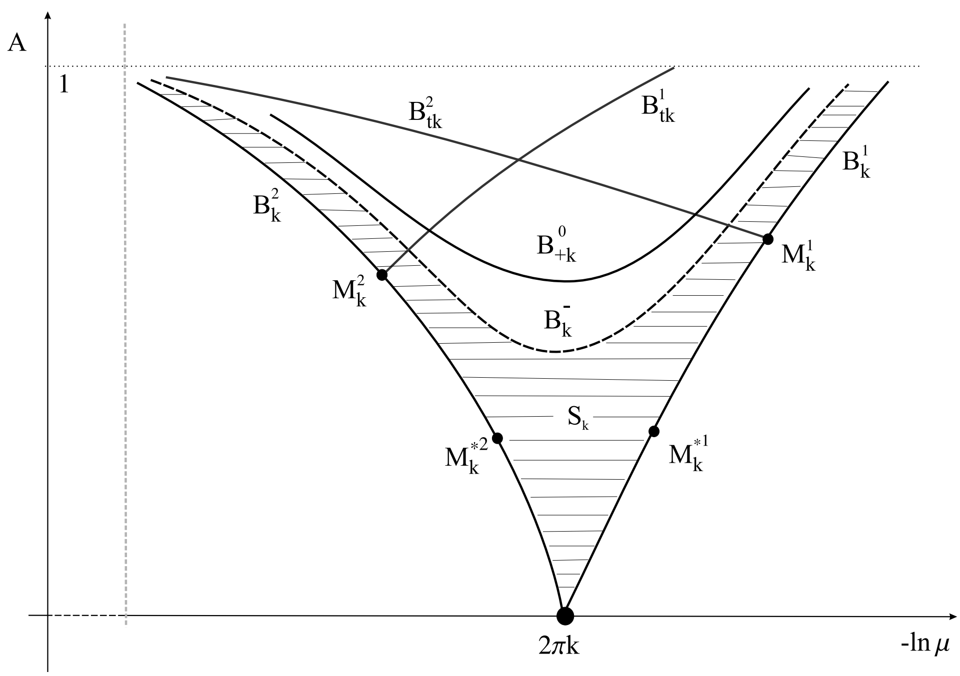


Fig. 6. Inner constitution of a resonant zone.

2 bifurcating from it. The region S_k is the synchronization zone as it corresponds to the existence of a stable periodic orbit of period $2\pi k$. Note that for any large enough integers k and m the intersection of the regions S_k and S_m is non-empty — in it, the periodic points of periods $2\pi k$ and $2\pi m$ coexist.

Within the region D_k the closed invariant curve, existing at small A (at least for $A < \gamma/\sqrt{1+\gamma^2}$, see Assertion 1) is the unstable manifold W^u of the saddle fixed point Q_k which closes on the stable point P_k, as sketched in Fig. 7.

After crossing B^-_k the invariant curve no longer exists, see Fig. 8.

Another mechanism of breakdown of the invariant circle is due to the onset of homoclinic tangencies produced by the stable and unstable

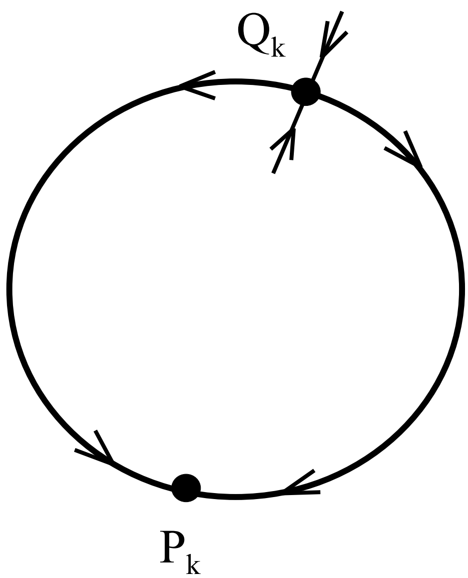


Fig. 7. The closure of the unstable manifold of the saddle fixed point Q_k is a closed invariant curve.

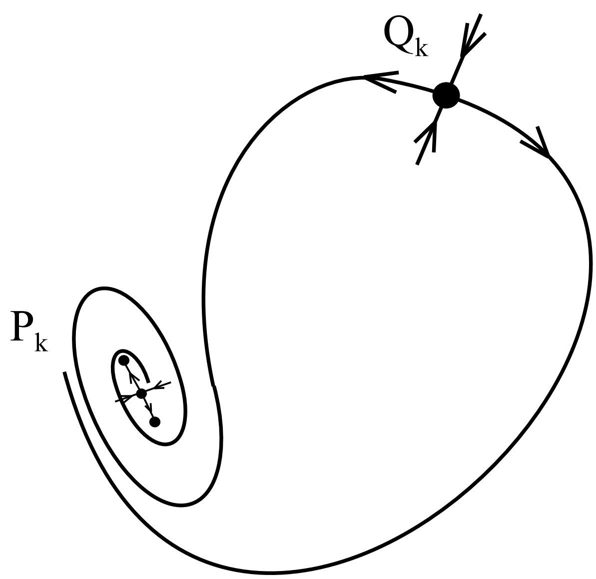


Fig. 8. After a period-doubling bifurcation on B_k^-, the closure of the unstable manifold of the saddle-fixed point is no longer homeomorphic to circle.

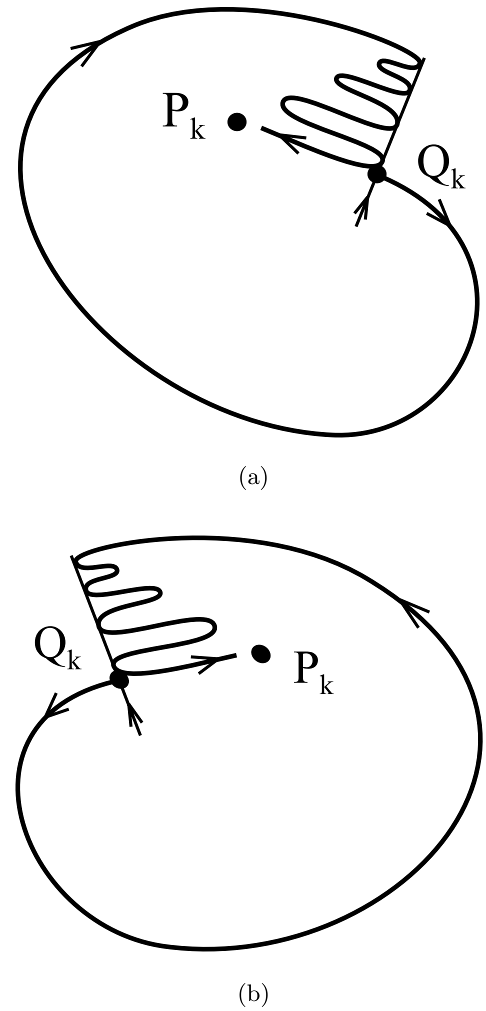


Fig. 9. End of the closed invariant curve: the very first homoclinic touches of the stable manifold of saddle fixed point with the unstable one that occur on the curves B_{tk}^1 and B_{tk}^2, respectively.

manifolds of the saddle point Q_k. The tangencies occur on the bifurcation curves B_{tk}^1 and B_{tk}^2 where each corresponds to a homoclinic contact of one of the components of the set $W^u \backslash Q$ with $W^S(Q_k)$, (see Fig. 9).

The curves B_{tk}^1 and B_{tk}^2 are noteworthy because they break each sector S_k into regions with simple and complex dynamics. Below the curves B_{tk}^1 and B_{tk}^2 in the zone S_k the stable point P_k is a single attractor grabbing all the trajectories other than the saddle fixed point Q_k. In the region above the curves B_{tk}^1 and B_{tk}^2, the point Q_k has a transverse homoclinic trajectory, and, consequently, the map must possess a nontrivial hyperbolic set [Shilnikov, 1967]. Note that the zone S_m $(m < k)$ should overlap with the zone S_k always above the curve B_{tk}^2. Hence, in the region $S_k \cap S_m$ where a pair of stable periodic orbits of periods $2\pi k$ and $2\pi m$ coexists, the dynamics is always complex like the corresponding case of the van der Pol equation.

In fact, other stable points of other periods may also exist in the region above the curves B_{tk}^1 and B_{tk}^2: since the homoclinic tangencies arising on the curves B_{tk}^1 and B_{tk}^2 are not degenerate (quadratic), it follows from [Newhouse, 1979] that above these curves in the parameter space there exist the

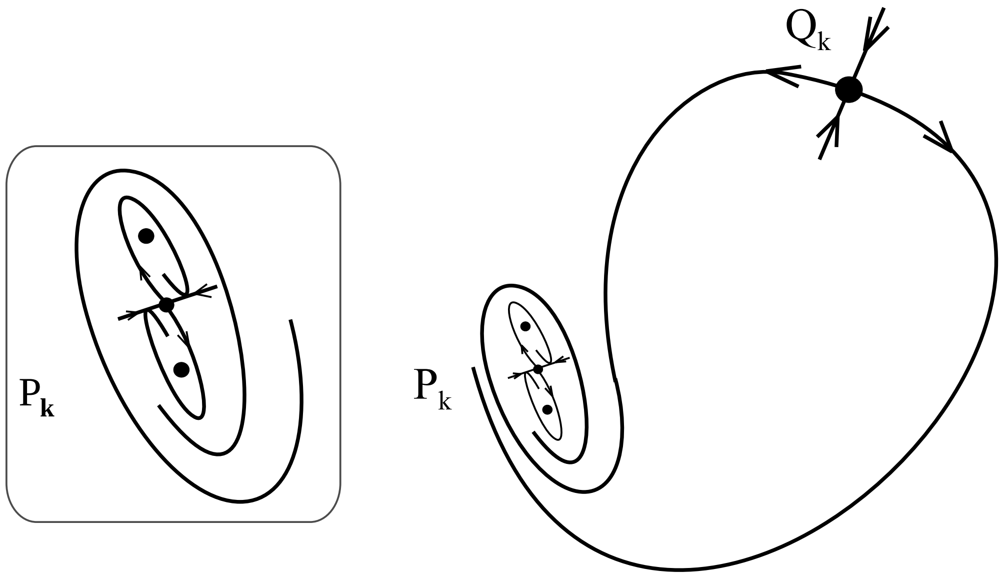


Fig. 10. Homoclinic contacts between the invariant manifolds of the saddle point P_k, see the left inset.

so-called Newhouse regions where the system has simultaneously an infinite set of stable periodic orbits for dense set of parameter values. On the other hand, it follows from [Newhouse & Palis, 1976] (see [Sten'kin & Shilnikov, 1998] for a higher dimensional case) that in addition there will be regions of hyperbolicity as well; moreover, the stable point P_k may be shown to be the only attractor for parameter values from these regions.

It should be remarked that the synchronization is always *incomplete* in the synchronization zone S_k above the curves B^1_{tk} and B^2_{tk}. This is due to the likelihood of the presence of other stable periodic orbits of different periods that coexist along with the orbit L_k corresponding to the stable fixed point P_k of the Poincaré map. However, even if this is not the case and L_k is still the only attractor, the phase difference between L_k and the trajectories from the hyperbolic set nearby the transverse homoclinics to the saddle point Q_k will grow at the asymptotically linear rate, i.e. phase locking may be broken at least within the transient process.

We should remark too that chaos itself is less important for desynchronization than the presence of homoclinics to the saddle point Q_k. So, for example, in the region $D_k \backslash S_k$ beneath the curves B^1_{tk} and B^2_{tk} where Q_k has no homoclinics, the difference in the phase stays always bounded, which means a *relative* synchronization, so to speak. Meanwhile the dynamics can be nonetheless chaotic: for instance, in the region above the curve B^0_{tk}, the fixed point P_k is no longer stable but a saddle with a transverse homoclinic orbit. On the curve B^0_{tk}, the stable and unstable manifolds of P_k have a homoclinic tangency of the third class in terminology introduced in [Gavrilov & Shilnikov, 1972] (illustrated in Fig. 10) which implies particularly the complex dynamics persisting below the curve B^0_{tk} as well.

Thus, the region D_k corresponding to the existence of the $2\pi k$-periodic orbit, may be decomposed into the zones of complete, incomplete and relative synchronization. The regime of incomplete synchronization, where there are periodic orbits with different rotation numbers, always yields complex dynamics. Further, in the zone of relative synchronization there is another "non-rotating" type of chaotic behavior. It can be shown that such a tableau of the behavior in the resonance zone D_k is drawn not only for $f(\theta) = A \sin \theta$, but in generic case too for an arbitrary function f.

The following question gets raised: what will happen upon leaving the synchronization zone S_k through its boundary B^2_k or B^1_k, i.e. as the saddle-node orbit vanishes? The answer to this question relies essentially on the global behavior of the unstable manifold W^u of the saddle-node. Above the points M^1_k and M^2_k ending up, respectively, the curves B^1_{tk} and B^2_{tk} corresponding to the beginning of the homoclinic trajectories, the unstable manifold W^u of the saddle-node has points of transverse crossings with its strongly stable manifold W^{ss} (see Fig. 11).

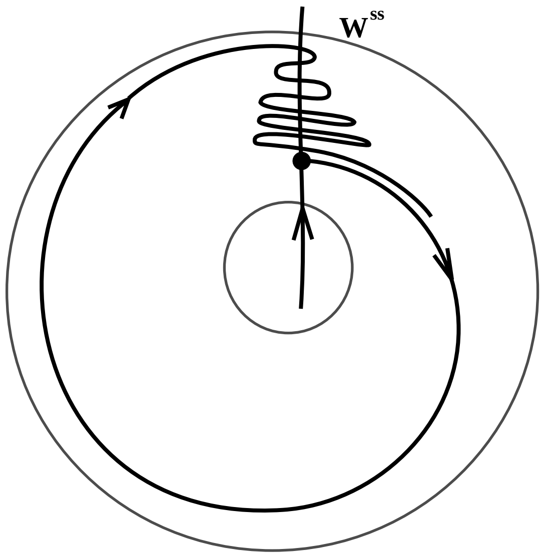
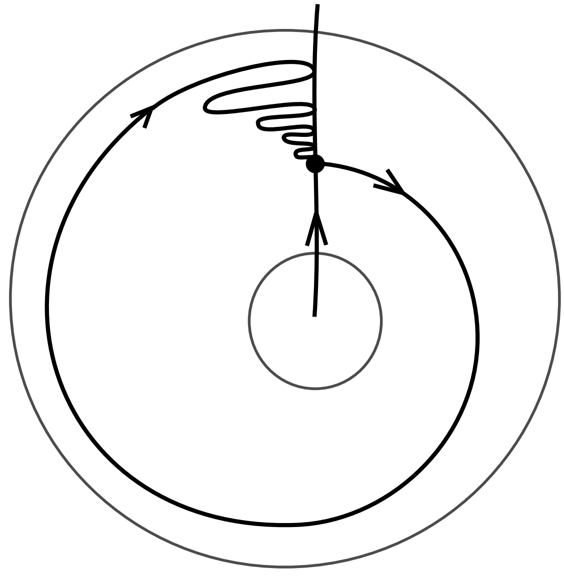
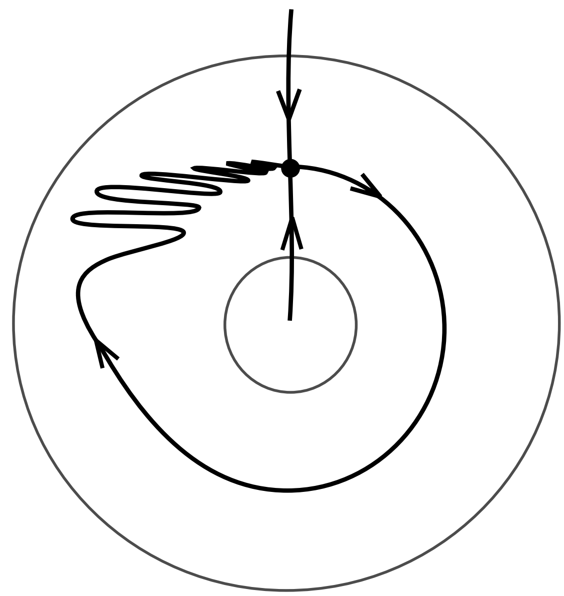


Fig. 11. Homoclinic crossings between the unstable manifold W^u with the strongly stable one W^{ss} of a saddle-node.



(a)



(b)

Fig. 12. (a) Homoclinic tangencies involving the unstable W^u and the strongly stable W^{ss} manifolds of a saddle-node. (b) Pre-wiggles of the unstable manifold of the saddle-node.

It is shown in [Lukyanov & Shilnikov, 1978] that this homoclinic structure generates the nontrivial hyperbolic set similar to that existing nearby a transverse homoclinic trajectory to a saddle. Upon getting into the region D_k the saddle-node disintegrates becoming a stable node and a saddle, and the latter inherits the homoclinic structure, and hence the hyperbolic set persists. Upon exiting D_k the saddle-node dissolves, however a great portion of the hyperbolic set survives, i.e. as soon as the boundaries B_k^2 and B_k^1 are crossed above the points M_k^1 and M_k^2 we enter the land of desynchronization ("rotational chaos").

The points M_k^1 and M_k^2 correspond to the homoclinic contacts between the manifolds W^u and W^{ss} (see Fig. 12). The limit set for all trajectories in W^u is the saddle-node itself, and hence W^u is homeomorphic to a circle. Below these points the manifold W^u returns always to the saddle-node from the node region. It is yet homeomorphic to a circle. Note that for the parameter values near M_k^1 and M_k^2 the folds on W^u persist; this implies that W^u cannot be a smooth manifold (a vector tangent to W^u wiggles as the saddle-node is being approached from the side of the node region and has no limit). This is always so until W^u touches a leaf of the strongly stable invariant foliation F^{ss} at some point in the node region (see [Newhouse *et al.*, 1983] and [Shilnikov *et al.*, 2001]). On the other hand, if W^u crosses the foliation F^{ss} transversely, then the former adjoins to the saddle-node smoothly. This is the case when A is small enough: here the smooth invariant closed curve services as the only attractor; it coincides with W^u on the curves $B_k^{1,2}$. We denote by M_k^{*1} and M_k^{*2} the points on the curves B_k^1 and B_k^2, respectively, such that below them the manifold W^u adjoins to the saddle-node smoothly, and nonsmoothly above them.

We will show in the next section that if either boundary B_k^1 or B_k^2 of the complete synchronization zone is crossed outbound below the points

M_k^{*1} or M_k^{*2}, respectively, the stable smooth invariant closed curve persists. It contains either a dense quasi-periodic trajectory (with an irrational rotation number) or an even number of periodic orbits of rather long periods at rational rotation numbers. In application, these double-frequency regimes are practically indistinguishable. When the synchronization zone is deserted through its boundaries B_k^1 and B_k^2 above the points M_k^{*1} and M_k^{*2}, we flow either into chaos right away, or we enter the land where the intervals of the parameter values corresponding to chaotic and simple dynamics may alter. The former situation always takes place by the points $M_k^{1,2}$ (as above as below), while the alternation occurs near and above the points M_k^{*1} and M_k^{*2}.

2. Disappearance of the Saddle-Node

In this section we will analyze a few versions of global saddle-node bifurcations. The analysis will be carried out with concentration on continuous time systems because they provide a variety unseen in maps.

Let us consider a one-parameter family of $\mathcal{C}^2$-smooth $(n+2)$-dimensional dynamical systems depending smoothly on $\mu \in \mu(-\mu_0; \mu_0)$. Suppose the following conditions hold:

(1) At $\mu = 0$ the system has a periodic orbit L_0 of the simple saddle-node type. This means that all multipliers besides a single one equal to $+1$, lie in the unit circle, and the first Lyapunov coefficient is not zero.
(2) All the trajectories in the unstable manifold W^u of L_0 tend to L_0 as $t \to \infty$ and $W^u \cap W^{ss} = \emptyset$, i.e. the returning manifold W^u approaches L_0 from the node region.
(3) The family under consideration is transverse to the bifurcational set of systems with a periodic orbit of the saddle-node type. This implies that as μ changes the saddle-node bifurcates: it decouples into a saddle and a node when, say, $\mu < 0$, and does not exist when $\mu > 0$.

According to [Shilnikov *et al.*, 2000], one may introduce coordinates in a small neighborhood of the orbit L_0 so that the system will assume the following form

$$\begin{aligned} \dot{x} &= \mu + x^2[1 + p(x, \theta, \mu)]\,, \\ \dot{y} &= [A(\mu) + q(x, \theta, y, \mu)]y\,, \\ \dot{\theta} &= 1\,, \end{aligned} \tag{14}$$

where the eigenvalues of the matrix A lie in the left open half-plane. Here, θ is an angular variable defined modulo of 1, i.e. the points $(x, y, \theta = 0)$ and $(x, \sigma y, \theta = 1)$ are identified, where σ is some involution in $\mathbb{R}^n$ (see [Shilnikov *et al.*, 1998]). Thus p is a 1-periodic function in θ, whereas q is of period 2. Moreover, we have $p(0, \theta, 0) = 0$ and $q(0, \theta, 0) = 0$. In addition, the mentioned coordinates are introduced so that p becomes independent of θ at $\mu = 0$ (the Poincaré map on the center manifold is imbedded into an autonomous flow; see [Takens, 1974]).

The saddle-node periodic orbit L_0 is given by equation $(x = 0, y = 0)$ at $\mu = 0$. Its strongly stable manifold W^{ss} is locally given by equation $x = 0$. The manifold W^{ss} separates the saddle region (where $x > 0$) of L_0 from the node one where $x < 0$. The manifold $y = 0$ is invariant, this is a center manifold. When $\mu < 0$, it contains two periodic orbits: stable L_1 and saddle L_2, both coalesce in one L_0 at $\mu = 0$. When $\mu > 0$ there are no periodic orbits and a trajectory leaves a small neighborhood of the phantom of the saddle-node.

At $\mu = 0$ the x-coordinate increases monotonically. In the region $x < 0$ it tends slowly to zero, at the rate $\sim 1/t$. Since the y-component decreases exponentially, it follows that all trajectories in the node region tend to L_0 as $t \to +\infty$ tangentially to the cylinder given by $y = 0$. In the saddle region $x(t) \to 0$ now as $t \to -\infty$, and since y increases exponentially as t decreases, the set of trajectories converging to the saddle-node L_0 as $t \to -\infty$, i.e. its unstable manifold W^u, is the cylinder $\{y = 0, x \geq 0\}$.

As time t increases, a trajectory starting in $W^u \backslash L_0$ leaves a small neighborhood of the saddle-node. However, in virtue of Assumption 2, it is to return to the node region as $t \to +\infty$, i.e. it converges to L_0 tangentially to the cylinder $y = 0$. Hence, a small $d > 0$ can be chosen so that W^u will cross the section $S_0 : \{x = -d\}$. Obviously, $\bar{l} = W^u \cap S_0$ will be a closed curve. It can be imbedded in S_0 variously. We will assume that the median line $l_0 : \{y = 0\}$ in the cross-section S_0 is oriented in the direction of increase of θ, so is the median line $l_1 : \{y = 0\}$ of the section $S_1 : \{x = +d\}$. Because $l_1 = W^u \cap S_1$, it follows that the curve $\bar{l}$ is an image of the curve l_1 under the map defined by the trajectories of the system, and therefore the orientation on l_1 determines the orientation on $\bar{l}$ too. Thus, taking the orientation into account the curve $\bar{l}$ becomes homotopic to ml_0, where $m \in Z$. In the case $n = 1$,

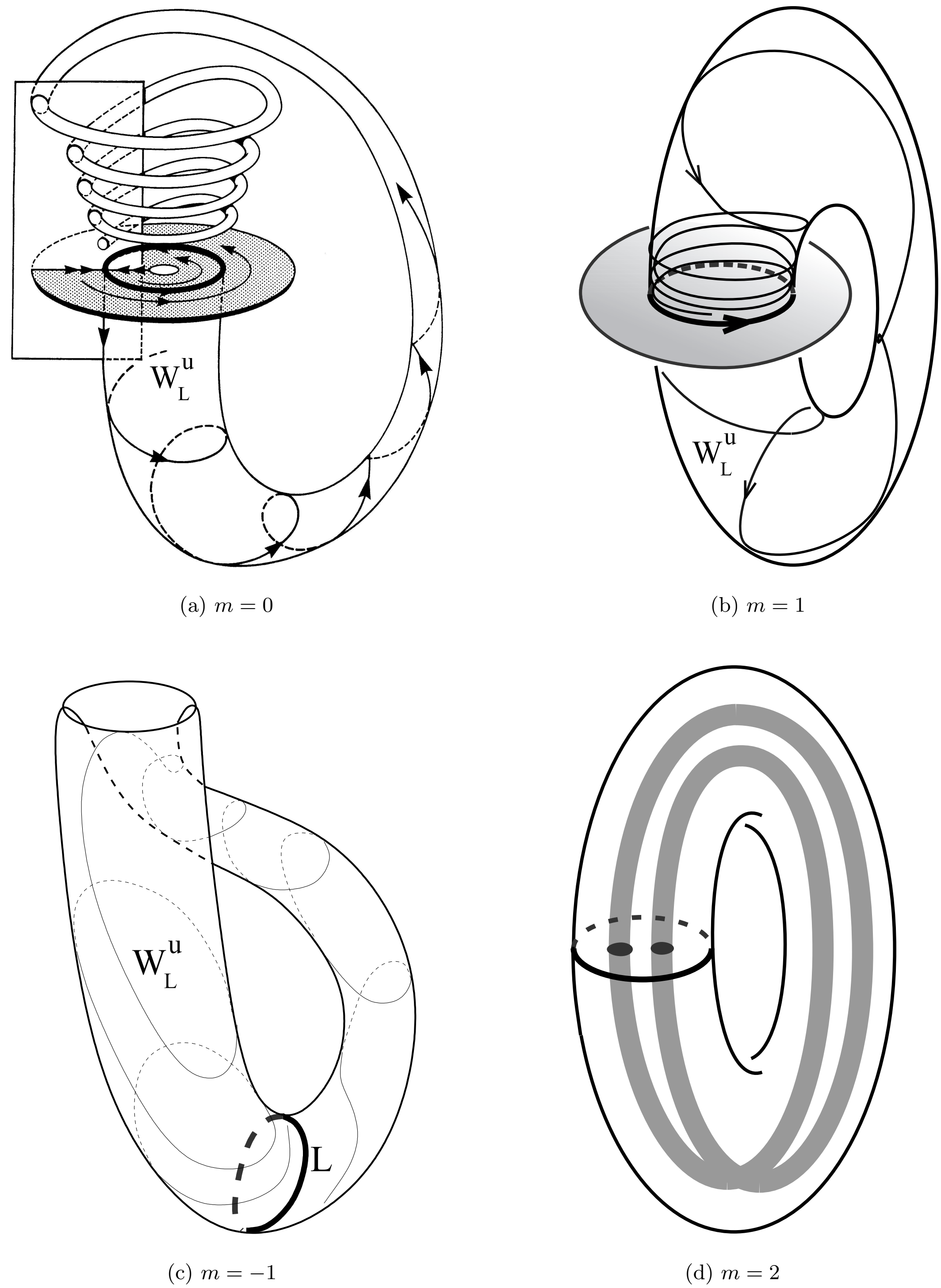


Fig. 13. Case $m = 0$ — the blue sky catastrophe. Cases $m = 1$ and -1: closure of the unstable manifold of the saddle-node periodic orbit is a smooth 2D torus and a Klein bottle, respectively. Case $m = 2$ — the solid-torus is squeezed, doubly stretched and twisted, and inserted back into the original and so on, producing the Wietorius–van Danzig solenoid in the limit.

i.e. when the system defined in R^3 and S_0 is a 2D ring, the only possible cases are where $m = 0$, or $m = +1$. However, if $n \geq 2$ all integers m become admissible. The behavior of W^u in case $m = 0$ is depicted in Fig. 13(a). When $m = 1$ the manifold W^u is homeomorphic to a 2D torus, and to a Klein bottle in case $m = -1$, see Figs. 13(b) and 13(c). When $|m| \geq 2$ the set W^u is a $|m|$-branched manifold (exactly $|m|$ pieces of the set W^u adhere to any point of the orbit L_0 from the node region). Note also that in the above discussed problem on a periodically forced oscillatory system, as for example (5), the first case $m = 1$ can only occur regardless of the dimension of the phase space.

The analysis of the trajectories near W^u presents interest only when $\mu > 0$ (it is trivial when $\mu \leq 0$). When $\mu > 0$ the Poincaré map $T : S_1 \to S_1$ is defined as the superposition of two maps by the orbits of the system: $T_1 : S_1 \to S_0$ followed by $T_0 : S_0 \to S_1$.

As shown in [Shilnikov *et al.*, 2000] if the system was brought to form (14) and the function p is independent of θ at $\mu = 0$, the map $T_0 : (y_0, \theta_0) \in S_0 \mapsto (y_1, \theta_1) \in S_1$ can be written as

$$\begin{aligned} y_1 &= \alpha(y_0, \theta_0, \nu)\,, \\ \theta_1 &= \theta_0 + \nu + \beta(\theta_0, \nu)\,, \end{aligned} \tag{15}$$

where $\nu(\mu)$ is the flight time from S_0 to S_1. As $\mu \to +0$, this time ν tends monotonically to infinity: $\nu \sim 1/\sqrt{\mu}$; meanwhile the functions α and β converge uniformly to zero, along with all derivatives. Thus, the image of the cross-section S_0 under action of the map T_0 shrinks to the median line l_1 as $\mu \to +0$.

It takes a finite time for trajectories of the system to travel from the cross-section S_1 to S_0. Hence, the map $T_1 : S_1 \to S_0$ is smooth and well-defined for all small μ. It assumes the form

$$\begin{aligned} y_0 &= G(y_1, \theta_1, \mu)\,, \\ \theta_0 &= F(y_1, \theta_1, \mu)\,. \end{aligned} \tag{16}$$

The image of l_1 will, consequently, be given by

$$y_0 = G(0, \theta_1, 0)\,, \quad \theta_0 = F(0, \theta_1, 0)\,. \tag{17}$$

The second equation in (16) is a map taking a cycle into another one. Hence this map may be written as

$$\theta_0 = m\theta_1 + f(\theta_1)\,, \tag{18}$$

where $f(\theta)$ is a 1-periodic function, and the degree m of the map is the homotopy integer index discussed above.

By virtue of (15), (16) and (18) the Poincaré superposition map $T = T_0 T_1 : S_1 \to S_1$ can be recast as

$$\begin{aligned} \overline{y} &= g_1(y, \theta, \nu)\,, \\ \overline{\theta} &= m\theta + \nu + f(\theta) + f_1(y, \theta, \nu)\,, \end{aligned} \tag{19}$$

where the functions f_1 and g_1 tend to zero as $\nu \to +\infty$, so do all their derivatives. Thus, we may see that if the fractional part of ν is fixed, then as its integral part ν tends to infinity, the map T degenerates into the circle map $\tilde{T}$:

$$\overline{\theta} = m\theta + f(\theta) + \nu \mod 1\,. \tag{20}$$

It becomes evident that the dynamics of the map (19) is dominated by the properties of the map (20). The values of ν with equal fractional parts give the same map $\tilde{T}$. Hence, the range of the small parameter $\mu > 0$ is represented as a union of the countable sequence of intervals $J_k = [\mu_{k+1}; \mu_k)$ (where $\nu(\mu_k) = k$) such that the behavior of the map T for each such segment J_k is likewise in main.

Let us next outline the following two remarkable cases $m = 0$ and $|m| \geq 2$ considered in [Turaev & Shilnikov, 1995, 1997, 2000; Shilnikov *et al.*, 2001].

Theorem [Turaev & Shilnikov, 1997, 2000]. *At $m = 0$ the map T has, for all sufficiently small μ, a single stable fixed point if $|f'(\theta)| < 1$ for all θ.*

After the map T was reduced to the form (19), the claim of the theorem follows directly from the principle of contracting mappings. It follows from the theorem that as the orbit L_0 vanishes, the stability goes to a new born, single periodic orbit whose length and period both tend to infinity as $\mu \to +0$. Such bifurcation is called a blue sky catastrophe. The question of a possibility of infinite increase of the length of a stable periodic orbit flowing into a bifurcation was set first in [Palis & Pugh, 1975]; the first such example (of infinite codimension) was built in [Medvedev, 1980]. Our construction produces the blue sky catastrophe through a codimension-1 bifurcation. We may refer the reader to the example of a system with the explicitly given right-hand side where such catastrophe is constructed [Gavrilov & Shilnikov, 1999]. Point out also [Shilnikov *et al.*, 2001; Shilnikov, 2003] that show the blue-sky catastrophe in our setting is typical for singularly perturbed systems with at least two fast variables.

Theorem [Turaev & Shilnikov, 1995, 1997]. *Let $|m| \geq 2$ and $|m + f'(\theta)| > 1$ for all θ. Then, the map T will have the hyperbolic Smale–William attractor for all small $\mu > 0$.*

In these conditions, the map T acts similarly to the construction proposed by Smale and Williams.

Namely, a solid torus S_0 is mapped into itself in such a way that the limit $\Sigma = \cap_{k \geq 0} T^k S_0$ is a Wietorius–van Danzig solenoid which is locally homeomorphic to the direct product of a Cantor set by an interval. Furthermore, the conditions of the theorem guarantee a uniform expansion in θ and a contraction in y, i.e. the attractor Σ is a uniformly hyperbolic set. Besides, since the map $T|_\Sigma$ is topologically conjugated to the inverse spectrum limit for the expanding circle map (of degree $|m|$), it follows that all the points of Σ are nonwandering. In other words, we do have here a genuine hyperbolic attractor (see more in [Turaev & Shilnikov, 1995, 1997]).

As mentioned above, in the case $m = \pm 1$ the surface W^u at $\mu = 0$ may adjoin to the saddle-node L_0 smoothly as well as nonsmoothly, depending upon how W^u crosses the strongly stable invariant foliation F^{ss} in the node region. When the system is reduced to the form of (14), the leaves of the foliation are given by $\{x = \text{const}, \theta = \text{const}\}$, i.e. on the cross-section S_0 the leaves of F^{ss} are the planes $\{\theta_0 = \text{const}\}$. The intersection $W^u \cap S_0$ is the curve (17). Therefore [see (18)], W^u adjoins to L_0 smoothly if and only if

$$m + f'(\theta) \neq 0 \qquad (21)$$

for all θ. This condition is equivalent to the limiting map $\tilde{T}$ [see(20)] being a diffeomorphism of the circle for all ν.

Theorem [Afraimovich & Shilnikov, 1974a, 1974b]. *If the limit map $\tilde{T}$ is a diffeomorphism, then for all $\mu > 0$ sufficiently small the map (19) has a closed stable invariant curve attracting all the trajectories of the map.*

The proof of the theorem follows directly from the annulus principle [Afraimovich & Shilnikov, 1977; Shilnikov *et al.*, 1998]. Remark that this smooth stable invariant circle of the Poincaré map T corresponds to a smooth attractive 2D torus in the original system in the case $m = 1$, and to a smooth invariant Klein bottle in the case $m = -1$.

The case where the map $\tilde{T}$ is no diffeomorphism is more complex. We put the case $m = -1$ aside and focus on $m = 1$ because the latter is characteristic for synchronization problems.

Thus we have $m = 1$ and the limiting map $\tilde{T}$ has critical points. Introduce a quantity δ defined as

$$\delta = \sup_{\theta_1 < \theta_2} (\theta_1 + f(\theta_1) - \theta_2 - f(\theta_2)) \,.$$

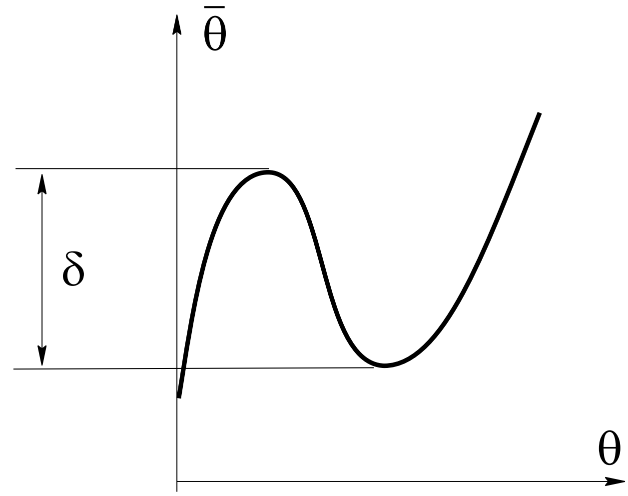


Fig. 14. δ is the absolute value of the difference between certain minimal value of the right-hand side of the map and the preceding maximal one.

It becomes evident [see (20)] that $\delta = 0$ if and only if the map $\tilde{T}$ is a homeomorphism for all ν, i.e. when its graph is an increasing function. If $\delta > 0$, this map is to have at least one point of a maximum as well as one point of a minimum; in essence, δ determines the magnitude between the given minimal value of the right-hand side of the map (18) and the preceding maximal one (see Fig. 14). One can easily evaluate $\delta(A) = (1/\pi)(\sqrt{4\pi^2 A^2 - 1} - \arctan\sqrt{4\pi^2 A^2 - 1})$ for the case $f = A \sin 2\pi\theta$, for example.

When $\delta \geq 1$ each θ has at least three preimages with respect to map (18). In terms of the original system the condition $\delta > 0$ holds true if and only if some leaf of the foliation F^{ss} has more than one (three indeed) intersection with the unstable manifold W^u of the saddle-node orbit at $\mu = 0$, and that $\delta \geq 1$ when and only when W^u crosses each leaf of the foliation F^{ss} at least three times.

Borrowing the terminology introduced in [Afraimovich & Shilnikov, 1974a, 1974b] we will refer to the case of $\delta > 1$ as the case of the big lobe.

Theorem [Turaev & Shilnikov, 1986]. *In case of the big lobe the map T has complex dynamics for all $\mu > 0$ sufficiently small.*

For its proof we should first show that the map $\tilde{T}$ for each ν has a homoclinic orbit of some fixed point of the map. Recall that a homoclinic orbit in a noninvertible map reaches the fixed

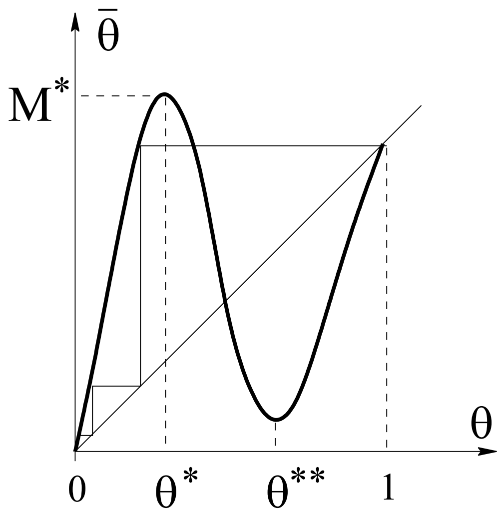


Fig. 15. Homoclinic orbit to $\theta = 0$ in the case $M^* > 1$.

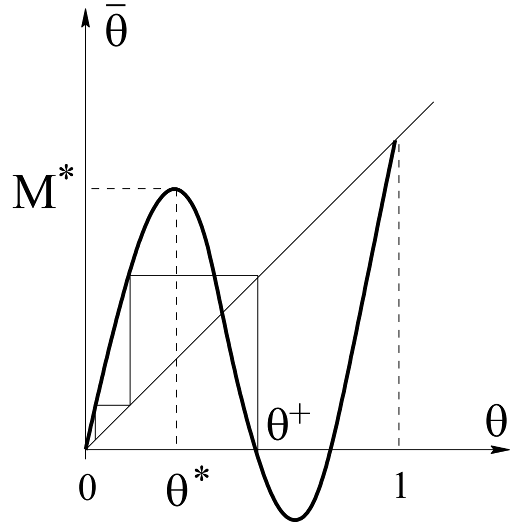


Fig. 16. Homoclinic orbit to $\theta = 0$ in the case $\theta^+ < M^* \leq 1$.

point after a finite number of iterations in forward direction and after infinitely many backward iterations.

Let $\theta^* < \theta^{**}$ be the maximum and minimum points of the function in the right-hand side of the map $\tilde{T}$ in (20), such that

$$M^* \equiv \nu + \theta^* + f(\theta^*) = \nu + \theta^{**} + f(\theta^{**}) + \delta \equiv M^{**} + \delta$$

It follows from the condition $\delta > 1$ that the difference between $f(\theta^*)$ and $f(\theta^{**})$ exceeds 1. Therefore, by adding, if needed, a suitable integer to ν, one can always achieve that $\nu + f(\theta)$ has some zeros. They are the fixed points of the map $\tilde{T}$. Let θ_0 be a fixed point next to θ^* from the left. By translating the origin, one can achieve $\theta_0 = 0$. Thus, we let $\nu + f(0) = \nu + f(1) = 0$, and hence $M^* \geq 0$.

For the beginning, let $M^* > \theta^*$. Then the fixed point at the origin is unstable (at least where $\theta > 0$) and each $\theta \in (0, M^*]$ has a preimage with respect to the map $\tilde{T}$ that is less than θ and positive. Consequently, for each point $\theta \in (0, M^*]$ there exists a negative semi-trajectory converging to the fixed point at the origin. Thus, in the case $M^* > 1$ (see Fig. 15) we obtain the sought homoclinic orbit (in backward time it converges to $\theta = 0$, while in forward time it jumps at the point $\theta = 1$ equivalent to $\theta = 0$ in modulo 1).

Whenever $M^* \leq 1$, it follows that $M^{**} < 0$. Since $M^* > 0$, the segment (θ^*, θ^{**}) contains a preimage of zero which we denote by θ^+. If $\theta^+ < M^*$, then this point has a negative semi-trajectory tending to zero, i.e. there is a desired homoclinics, see Fig. 16.

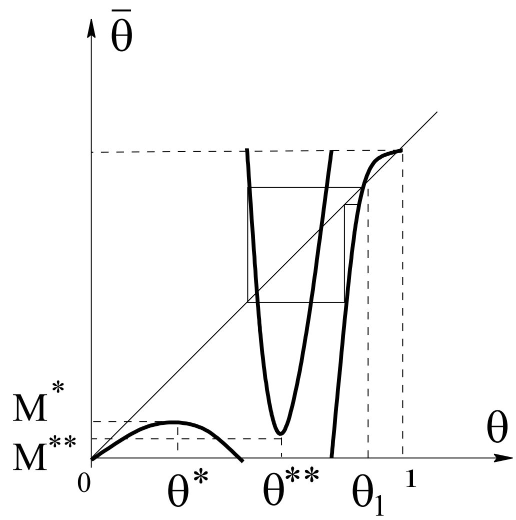


Fig. 17. Homoclinic orbit to θ_1.

In all remaining cases — $M^* \leq \theta^+$ or even $M^* \leq \theta^*$ we have that $M^{**} < \theta^{**} - 1$. Let $\theta_1 \leq 1$ be a fixed point closest to θ^{**}. Because $M^{**} < \theta^{**}$, it follows that θ_1 is an unstable point and that each $\theta \in [M^{**}, \theta_1)$ has a preimage greater than θ but less than θ_1, i.e. there exists a negative semi-trajectory tending to θ_1. Now, since $M^{**} < \theta^{**} - 1 < \theta_1 - 1$, we have that the point $\theta = \theta_1 - 1$ begins a negative semi-trajectory tending to θ_1, which means that there is homoclinics in the given case too, see Fig. 17.

The obtained homoclinic trajectory is structurally stable when it does not pass through a

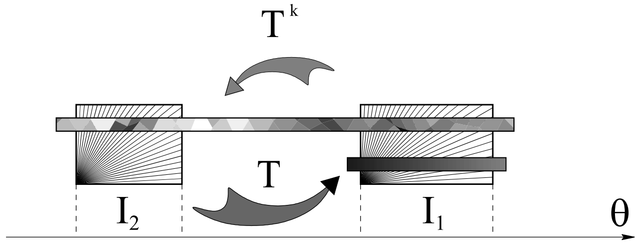


Fig. 18. The image of I_1 covers both I_1 and I_2. The image of I_2 covers I_1.

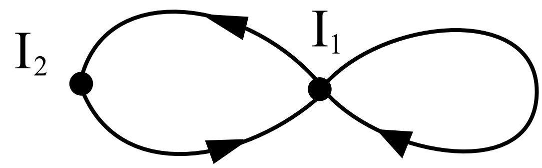


Fig. 19. On the set of points whose trajectories never leave the region $\theta \in I_1 \cup I_2$ the map T' is semi-conjugate to a subshift with positive entropy.

critical point of the map and when the absolute value of derivative of the map at the corresponding fixed point of the map does not equal 1. Then, the original high-dimensional map T has, for all μ sufficiently small, a saddle fixed point with a transverse homoclinic orbit. This implies automatically a complex dynamics. One, nonetheless, cannot guarantee the structural stability of the obtained homoclinic orbits in the map (20) for all ν and for all functions f. Therefore to complete the proof of the theorem we need additional arguments.

Observe first that the constructed homoclinic trajectory of the map $\tilde{T}$ has the following feature. Let θ^0 be a fixed point, and let θ^1 be some point on the homoclinic trajectory picked near θ^0. As the map is iterated backward, the preimages θ^2, $\theta^3, \ldots$ of the point θ^1 converge to θ^0, at least from one side. By definition, the image $\tilde{T}^{k_0}\theta^1$ is the point θ^0 itself at some k_0. The property we are speaking about is that the forward images of an open interval I_1 containing θ^1 covers a half-neighborhood of the point θ^0, which hosts preimages of the point θ^1. It follows that the image of the interval I_1 after a large number of iterations of the map $\tilde{T}$ will contain the point θ^1 and its preimage θ^2 along with the interval I_1 itself and some small interval I_2 around the point θ^2 such that the image of I_2 covers I_1, as shown in Fig. 18.

Thus we have shown that if $\delta > 1$, then for each ν there is a pair of intervals I_1 and I_2, and the integer k so the image $\tilde{T}(I_2)$ covers I_1, while the image $\tilde{T}^k(I_1)$ covers both I_1 and I_2. In virtue of the closeness of the map (19) to (20) we obtain that for all μ sufficiently small there exists a pair of intervals I_1 and I_2 such that the images of the back sides $\{\theta = \text{const}\}$ of the cylinder $\theta \in I_2$ mapped by T will be on the opposite side of the cylinder $\theta \in I_1$, while the images of the back sides of the cylinder $\theta \in I_1$ due to the action of the map T^k will be on the opposite sides of the union of the cylinders $\theta \in I_1$ and $\theta \in I_2$, see Fig. 18. This picture is quite similar to the Smale horseshoe with a difference that we do not require hyperbolicity (uniform expansion in θ). It is not hard to show that the map T' which is equal to T at $\theta \in I_2$ and to T^k at $\theta \in I_1$, in restriction to the set of trajectories remaining in the region $\theta \in I_1 \cup I_2$ is semi-conjugate to the topological Markov chain shown in Fig. 19.

It follows that the topological entropy of the map T is positive for all small $\mu > 0$. Note also that here there is always a hyperbolic periodic orbit with a transverse homoclinic trajectory. Indeed, the positiveness of the topological entropy implies (see [Katok, 1980]) the existence of an ergodic invariant measure with a positive Lyapunov exponent. Moreover, since the map (19) contracts two-dimensional areas for small μ, the remaining Lyapunov exponents are strictly negative so that the existence of a nontrivial uniformly hyperbolic set follows in the given case right away from [Katok, 1980]. This completes the proof of the theorem.

We see that if δ is large enough, the complex dynamics exists always as the saddle-node disappears. Two theorems below show that when δ is small the intervals of simple dynamics alter with those of complex dynamics as $\mu \to 0$.

Theorem [Turaev & Shilnikov, 1986]. *If $\delta > 0$ in the map (20) and all its critical points are of a finite order, then the map T has complex dynamics in the intervals of μ values which are located arbitrarily close to $\mu = 0$.*

After the Poincaré map T is brought to the form (19), this theorem follows almost immediately from the Newhouse–Palis–Takens theory of rotation numbers for noninvertible maps of a circle. Indeed, according to [Newhouse *et al.*, 1983], when all critical points of the circle map (20) are of finite order, a periodic orbit must have a homoclinic at some $\nu = \nu_0$, provided $\delta > 0$. Therefore, arguing the same as above, when $\delta > 0$ there is a value of ν_0

such that a pair of intervals I_1 and I_2 can be chosen so that the image $\tilde{T}^{k_1}(I_2)$ covers I_2, and the image $\tilde{T}^{k_2}(I_1)$ covers both I_1 and I_2, for certain k_1 and k_2. The rest follows exactly as in the previous theorem: due to the closeness of the maps T and $\tilde{T}$ we obtain the existence of an invariant subset of the map T on which the latter is semi-conjugate with a nontrivial topological Markov chain for all μ small such that $\mu \bmod 1$ is close to ν_0.

Theorem [Turaev & Shilnikov, 1986]. *If* $2\delta \max_\theta f''(\theta) < 1$, *then arbitrarily close to* $\mu = 0$ *there are intervals of values of* μ *where the map* T *has trivial dynamics: all trajectories tend to a continuous invariant curve, homeomorphic to a circle, with a finite number of fixed points.*

Proof. Let θ_0 be a minimum of f, i.e. $f(\theta) \geq f(\theta_0)$ for all θ. Choose $\nu = \nu^*$ so that this point becomes a fixed one for the map $\tilde{T}$, i.e. $\nu^* = -f(\theta_0)$ [see (20)]. By construction, the graph of the map $\tilde{T}$ is nowhere below the bisectrix $\overline{\theta} = \theta$ and only touches it at the point θ_0. Let θ_1 be a critical point of $\tilde{T}$ closest to θ_0 on the left. The derivative of the map vanishes at this point, equals 1 at the point θ_0 and is positive everywhere between θ_1 and θ_0. One can then derive that

$$\theta_1 + f(\theta_1) \leq \theta_0 + f(\theta_0) - \frac{1}{2 \max_\theta f''(\theta)}\,. \tag{22}$$

Indeed,

$$\theta_0 + f(\theta_0) - (\theta_1 + f(\theta_1)) = \int_{\theta_1}^{\theta_0} (1 + f'(\theta))d\theta\,.$$

Since $1 + f'(\theta) \geq 0$ in the integration interval, we have

$$\theta_0 + f(\theta_0) - (\theta_1 + f(\theta_1)) \geq \frac{1}{\max_\theta f''(\theta)} \int_{\theta_1}^{\theta_0} (1 + f'(\theta))f''(\theta)d\theta\,,$$

i.e.

$$\theta_0 + f(\theta_0) - (\theta_1 + f(\theta_1)) \geq \frac{1}{2 \max_\theta f''(\theta)} [(1 + f'(\theta_0))^2 - (1 + f'(\theta_1))^2]\,.$$

Now, since $f'(\theta_0) = 0$ and $f'(\theta_1) = -1$, we obtain the inequality (22).

Now, by the condition of the theorem, it follows from (22) that the value $\overline{\theta}$ which the map takes on at the point θ_1 is less than the corresponding value at the fixed point θ_0 and the difference does exceed

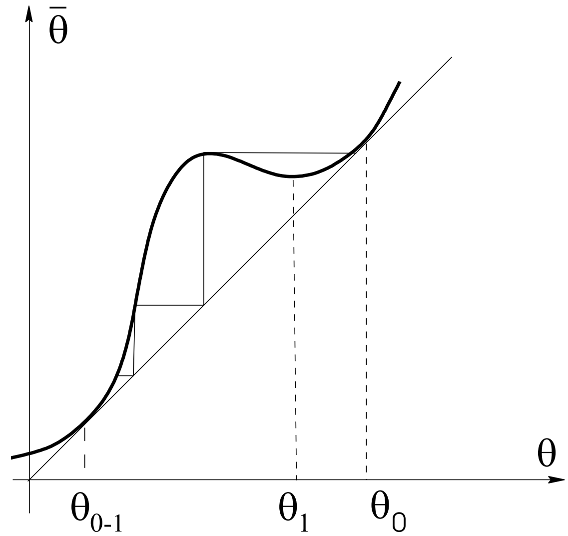


Fig. 20. The map has only semi-stable fixed points and trajectories tending to them.

δ. It follows then from the definition of δ that when $\theta < \theta_1$ the value of $\overline{\theta}$ is strictly less than θ_0. By construction, the same is also true for $\theta \in [\theta_1, \theta_0)$.

Thus, at the given ν, the graph of the map on $(\theta_0 - 1, \theta_0)$ belongs entirely to the region $\theta \leq \overline{\theta} < \theta_0$, see Fig. 20.

This means that there are only semi-stable fixed points and trajectories tending to them. Now, we can pick ν a bit less than ν^* so that these fixed points disintegrate in a finite number of stable and unstable ones; the remaining trajectories will go to the stable fixed point in forward time and to the unstable ones in backward time.

This is a structurally stable situation. It persists within a small interval Δ of values of ν. It persist too for all closed maps, i.e. for the map T at sufficiently small μ such that $\nu(\mu) \pmod 1 \in \Delta$. Here, the stable fixed point of the map $\tilde{T}$ correspond to those of map T, while unstable ones to saddles. The one-dimensional unstable separatrices of the saddles tend to the stable fixed points; their closures form an invariant circle. ■

3. Conclusion

We can resume now that the common feature of the problems in classical synchronization theory is that they are often reducible to the study of an annulus map, i.e. a map in the characteristic form

$$\begin{aligned} \overline{x} &= G(x, \theta)\,, \\ \overline{\theta} &= \theta + \omega + F(x, \theta) \mod 1\,, \end{aligned} \tag{23}$$

where the functions F and G are 1-periodic in θ. We focused on the simplest case where the map (23) contracts areas. We described the structure of the bifurcation diagrams typical for this case, possible synchronization regimes and the connection between desynchronization and chaos.

We should mention that when the area-contraction property does not hold and higher dimensions become involved, the situation may be more complicated. So for example, in addition to saddle-node and period-doubling bifurcations, the system may possess a periodic orbit with a pair of complex-conjugate multipliers on the unit circle. Furthermore, when the corresponding bifurcation curve meets the boundary of the synchronization zone we will already have a periodic orbit with the triplet of multipliers $(1, e^{\pm i\varphi})$. An appropriate local normal form for this bifurcation will be close to that of an equilibrium state with characteristic exponents equal to $(0, \pm i\omega)$ (the so-called Gavrilov–Guckenheimer point), and its local bifurcations are quite nontrivial [Gavrilov, 1978; Guckenheimer, 1981]. If, additionally, on the boundary of the synchronization zone this periodic orbit has a homoclinic trajectory, then one arrives at the necessity of studying global bifurcations of Gavrilov–Guckenheimer points with homoclinic orbits. Such a bifurcation has already been seen in the example of the blue sky catastrophe [Gavrilov & Shilnikov, 1999]. It has also been noticed recently in a synchronization problem [Krauskopf & Wieczorek, 2002]. It is becoming evident that other codimension-two cases like $(\pm 1, \pm 1)$ and $(-1, e^{\pm i\varphi})$ with homoclinic orbits are worth a detailed analysis and will be called for soon too.

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