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Theme Issue (Part 2)

COALESCENCE OF REVERSIBLE HOMOCLINIC ORBITS CAUSES ELLIPTIC RESONANCE

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Reversible flows can possess a robust homoclinic orbit to a saddle equilibrium: the orbit is preserved under small perturbations that do not destroy the reversibility of the system. Such a homoclinic orbit is a limit of a unique one-parameter family of periodic orbits. All these orbits are saddles if the equilibrium state is a saddle. There are both saddle and elliptic periodic orbits in this family if the equilibrium state is a saddle-focus. In the present paper, we study the coalescence of two such homoclinic orbits in a one-parameter family of reversible flows. We show that, even in the case where all eigenvalues of the corresponding equilibrium are real, a family of elliptic periodic orbits arises at this bifurcation.

1. Introduction and Main Results

Time reversible flows $\varphi_t(z)$ are characterized by a time reversal operator R. Here, $z \in \mathbb{R}^n$ and R is an involution, $R^2 = R \circ R = id$, such that the diffeomorphisms $\varphi_t(\cdot)$ of $\mathbb{R}^n$ again provide an involution when composed with R:

$$(R\varphi_t)^2 = id\,, \tag{1.1}$$

or, in other words,

$$R^{-1}\varphi_t R = \varphi_{-t}\,. \tag{1.2}$$

This identity means that the involution R maps orbits of the flow φ_t into orbits of the same flow, reversing the direction of time.

Let $\mathrm{Fix}(R) := \{Rz = z\}$ be the fix space of R. We assume $\mathrm{Fix}(R) \neq \emptyset$, or else (1.2) does not impose significant restrictions on the orbits $z(t) = \varphi_t(z_0)$. Moreover, we can linearize the action of R, locally near any point $z_0 \in \mathrm{Fix}(R)$, by the explicit transformation

$$h(z) := \frac{1}{2}(z_0 + R'(z_0)(z - z_0) + R(z))\,. \tag{1.3}$$

Since this will not restrict generality below, we will henceforth assume R to be linear, globally.

For general reference on time reversible systems see Devaney [1976], Vanderbauwhede [1982], Arnol'd & Sevryuk [1986], Sevryuk [1986], Fiedler & Heinze [1996a, 1996b], and references therein. For the convenience of our readers we now summarize some relevant aspects before stating our main results in Theorems 1 and 2 below.

In terms of the generating vector field $f(z) := \frac{\partial}{\partial t}|_{t=0}\, \varphi_t(z)$ of the flow φ_t, the ordinary differential equation

$$\dot{z} = f(z) \tag{1.4}$$

is also reversible under the matrix R, that is,

$$Rf(z) = -f(Rz) \tag{1.5}$$

for all $z \in \mathbb{R}^n$.

Because the structure of time reversibility does not impose significant restrictions on orbits $z(t)$ which stay away from $\mathrm{Fix}(R)$, uniformly, we focus on the following objects:

- *reversible equilibria* $z_0 \in \mathrm{Fix}(R)$, with $f(z_0) = 0$;
- *reversible periodic orbits* $z(t+T) = z(t)$ with period $T > 0$ and $z_0 = z(0) \in \mathrm{Fix}(R)$;
- *reversible homoclinic loops* $\Gamma = \{z(t)\}$ with $z_0 = z(0) \in \mathrm{Fix}(R)$ such that the α- and ω-limit sets are the same single equilibrium state O:

$$O = \lim_{t\to\pm\infty} z(t). \tag{1.6}$$

In fact, O is a reversible equilibrium, because

$$\begin{aligned} O &= \lim_{t\to\pm\infty} \varphi_{-t}(z_0) \\ &= \lim_{t\to\pm\infty} R\varphi_t(Rz_0) \\ &= R \lim_{t\to\pm\infty} \varphi_t(z_0) \\ &= RO. \end{aligned}$$

At first glance, the condition

$$f(z_0) = 0 \in \mathbb{R}^n \tag{1.7}$$

seems like a few conditions too many to be solved for a reversible equilibrium z_0 in the subspace $\mathrm{Fix}(R)$ of $\mathbb{R}^n$. Note, however, that reversibility (1.5) implies

$$f : \mathrm{Fix}(R) \to \mathrm{Fix}(-R), \tag{1.8}$$

where $\mathrm{Fix}(-R)$ denotes the (representation) subspace where R acts as $-id$. In particular, reversible equilibria will generically be isolated if f in (1.8) is a mapping between spaces of equal dimension. Therefore, we assume n to be even and

$$\dim \mathrm{Fix}(R) = \dim \mathrm{Fix}(-R) = n/2, \tag{1.9}$$

from now on.

An example for reversibility is given by systems of second order differential equations

$$\ddot{u} + g(u, \dot{u}) = 0 \in \mathbb{R}^{n/2} \tag{1.10}$$

with $g = g(u, p)$, even in p. Indeed, rewriting (1.10) as a system of first order for $z = (u, v) \in \mathbb{R}^n$ with $f(u, v) := (v, -g(u, v))$, the system becomes reversible under $R(u, v) := (u, -v)$. In particular, the u-"axis" is $\mathrm{Fix}(R)$ and the v-"axis", alias the "$\dot{u}$-axis", is $\mathrm{Fix}(-R)$.

A reversible periodic orbit intersects $\mathrm{Fix}(R)$ in precisely two points, in general, half a period apart. Indeed, for $z_0 \in \mathrm{Fix}(R)$ on a reversible periodic orbit, we have

$$\varphi_{-t}(z_0) = R\varphi_t(z_0). \tag{1.11}$$

Picking $t = T/2$ with T the minimal period of z_0, this implies $z_1 := \varphi_{T/2}(z_0) \in \mathrm{Fix}(R)$ is another intersection of the periodic orbit with $\mathrm{Fix}(R)$. Conversely, for any such intersection $\varphi_t(z_0)$, by (1.11), the time t must be a multiple of $T/2$. This proves the claim.

Specifically, for Eq. (1.10), the half-period arcs between $z_0 \in \mathrm{Fix}(R)$ and $\varphi_{T/2}(z_0) \in \mathrm{Fix}(R)$ of reversible periodic orbits are solutions of the associated Neumann boundary value problem on the interval $0 \le t \le T/2$.

In general, reversible periodic orbits correspond to intersection points

$$z_0 \in \mathrm{Fix}(R) \cap \varphi_{T/2}(\mathrm{Fix}(R)), \tag{1.12}$$

by (1.11), for some $T > 0$. Generically, again, the intersection of the two surfaces of dimension $n/2$ in (1.12) will be transverse for fixed T. In particular, intersection points will persist under small perturbations of T. Therefore, reversible periodic orbits typically appear in one-parameter families.

Reversible homoclinics Γ are a limiting case of reversible periodics, if we let the "other" intersection point $z_1 := \varphi_{T/2}(z_0) \in \mathrm{Fix}(R)$ tend to a reversible equilibrium $O \in \mathrm{Fix}(R)$, along the one-parameter reversible periodic family. Of course, $T \to +\infty$ in this limit. Suppose now that O is hyperbolic, with associated stable manifold W^s and unstable manifold W^u. The time reversibility implies [see (1.2)] that the involution R maps orbits which are asymptotic to O as $t \to +\infty$ into orbits asymptotic to O as $t \to -\infty$, that is,

$$W^u = RW^s. \tag{1.13}$$

In particular, the manifolds W^u and W^s have equal dimension: half the dimension of the phase space.

Homoclinic orbits $\Gamma = \{z(t)\}$ to O are produced by intersections of W^u with W^s, other than the equilibrium O itself. By (1.13), such homoclinic loops either occur in pairs, related by R, or intersect $\mathrm{Fix}(R)$ in, say, $z_0 = z(0)$. In this latter case,

$\Gamma = \{z(t)\}$ is a reversible homoclinic. It arises by intersections

$$z_0 \in \mathrm{Fix}(R) \cap W^u, \tag{1.14}$$

other than the trivial intersection O. Indeed, (1.14) implies

$$z_0 = Rz_0 \in \mathrm{Fix}(R) \cap RW^u = \mathrm{Fix}(R) \cap W^s, \tag{1.15}$$

so that $z_0 \in W^u \cap W^s$ is reversibly homoclinic to O. Generically, again, we can assume the intersection in (1.14) to be transverse: generic reversible homoclinic loops are robust under perturbations.

Moreover, as was observed by Devaney [1976], any such *transverse reversible homoclinic* is indeed accompanied by a one-parameter family of reversible periodic orbits, as indicated above; see also Vanderbauwhede & Fiedler [1992]. This is a consequence of the λ-Lemma: $\varphi_{T/2}(\mathrm{Fix}(R))$ spreads out along W^u, as T increases, and (1.14) induces reversible periodic orbits by (1.12).

In contrast, our *main goal* in the present paper is to investigate a simple nontransverse intersection in (1.14), that is: a quadratic tangency between W^u and $\mathrm{Fix}(R)$. Varying a real parameter μ, this will correspond to a "saddle-node" bifurcation of the associated reversible homoclinic orbits: they coalesce and disappear.

The dynamical effects associated to this simple geometric bifurcation crucially depend on the detailed dynamics near the reversible equilibrium O. Specifically, reversibility restricts the possible spectra at reversible equilibria like O, and of reversible periodics. Moreover, these spectra are related near homoclinics through O, by passage near O. Differentiating (1.5) at $z = O \in \mathrm{Fix}(R)$, we obtain

$$Rf'(O)R^{-1} = -f'(O). \tag{1.16}$$

Therefore, the spectrum spec $\subseteq \mathbb{C}$ of $f'(O)$ is symmetric to the origin:

$$\mathrm{spec} = -\mathrm{spec}. \tag{1.17}$$

Assuming spec to consist of simple eigenvalues, four cases arise in two degrees-of-freedom, that is, for

$$z \in \mathbb{R}^4.$$

Firstly, if all eigenvalues are real, then

$$\mathrm{spec} = \{\pm 1, \pm\gamma\}, \tag{1.18}$$

for some $\gamma > 1$. Here, we have rescaled time so that the leading smaller positive eigenvalue is normalized to 1. Note that zero cannot be an eigenvalue, because it necessarily possesses even algebraic multiplicity in our even-dimensional phase space $\mathbb{R}^n = \mathbb{R}^4$.

A *second* possibility in $\mathbb{R}^4$ are all complex eigenvalues, spec $= \{\pm\alpha \pm i\beta\}$ with α, β nonzero. It was noticed by Härterich [1993] that reversible k-homoclinic orbits then arise near a primary reversible transverse homoclinic Γ, for any $k \geq 2$. See also Champneys [1994]. Here, *k-homoclinic* means that these secondary homoclinic loops complete k revolutions, in a small tubular neighborhood of Γ, before closing up homoclinically at O. This effect is produced by the $\beta \neq 0$ spinning of the vector field near O.

As Devaney [1977] has observed, the spinning at O also produces elliptic reversible periodics in the periodic family generated by Γ, with accompanying subharmonic bifurcations [Vanderbauwhede, 1990]. *Elliptic reversible periodics* are characterized by nonreal Floquet multipliers on the unit circle. Note that the Floquet spectrum floq of the linearized flow $\varphi'_T(z_0)$ along a reversible periodic through $z_0 \in \mathrm{Fix}(R)$ of minimal period T satisfies

$$\mathrm{floq}^{-1} = \mathrm{floq}. \tag{1.19}$$

Indeed, linearization of (1.2) at $z_0 \in \mathrm{Fix}(R)$ implies

$$R^{-1}\varphi'_T(z_0)R = (\varphi'_T(z_0))^{-1}. \tag{1.20}$$

Along the one-parameter reversible family of periodics, we note that floq always contains a trivial Floquet multiplier $+1$ of algebraic multiplicity two. Therefore, the only possible combinations in $\mathbb{R}^4$ of Floquet multipliers $s \in \mathbb{C}$ are

$$\mathrm{floq} = \{s, s^{-1}, 1, 1\}. \tag{1.21}$$

The elliptic case corresponds to nonreal $s^{-1} = \bar{s}$, that is, $|s| = 1$. In particular, ellipticity arises in open intervals along our one-parameter families of reversible periodics, inducing subharmonics accordingly when floq is crossing roots of unity along the way.

If $s \neq \pm 1$ is real, in contrast, we call such reversible periodics *hyperbolic* or *saddle*. There are two subcases: *Möbius* orbits have $s < 0$, whereas *non-Möbius* orbits have $s > 0$. Note that the two-dimensional local strong stable manifold of a Möbius orbit is indeed a Möbius band, and hence is nonorientable. By symmetry R, the same holds true for the local strong unstable manifold.

As a *third* case of generic eigenvalue configurations at the reversible equilibrium $O \in \mathbb{R}^4$, we mention spec $= \{\pm\alpha, \pm i\}$, $\alpha \neq 0$. As was shown in Holmes *et al.* [1992] specifically for the Hamiltonian case (see also Lerman [1991]), the spinning at O due to the pure imaginary pair $\pm i$ may induce shift dynamics near the reversible homoclinic loop Γ.

The final *fourth* generic case of spec $= \{\pm i, \pm i\omega\}$ with nonresonant $\omega > 1$ does not produce reversible homoclinic orbits easily — at least not when the flow near O is in normal form.

The two former cases correspond to hyperbolic equilibria: the equilibrium state is called a *saddle* in the first case (all eigenvalues are real) and a *saddle-focus* in the second case (all complex eigenvalues).

Out of the two cases, only the saddle hyperbolic equilibrium O with spec $= \{\pm 1, \pm\gamma\}$ does not generate elliptic periodic orbits out of a transverse reversible homoclinic to O. Surprisingly, however, even a real saddle O with reversible homoclinic Γ is able to generate elliptic reversible periodics as soon as the homoclinic orbit Γ loses its transversality.

More precisely, consider reversible vector fields

$$\dot{z} = f(z, \mu) \tag{1.22}$$

such that $f(Rz, \mu) = -Rf(z, \mu)$ for real parameters $\mu \in \mathbb{R}$, and $z \in \mathbb{R}^4$. Then, at $\mu = 0$, the reversible homoclinic Γ may arise by a quadratic tangency of W^u with $\mathrm{Fix}(R)$, rather than a transverse intersection. Under suitable nondegeneracy conditions, which are generic in one-parameter families, this corresponds to a coalescence and subsequent annihilation in Γ of two transverse reversible homoclinics, as μ increases through zero. It turns out that, again generically, a wedge of elliptic reversible periodics is then generated from Γ, for μ on one side of zero; see Theorem 2 below. This is surprising because the necessary spinning effects are produced by the tangency of W^u with $\mathrm{Fix}(R)$ alone, rather than by any local spinning near the reversible equilibrium O.

We now fix all technical assumptions for our main results, Theorems 1 and 2, below. These will be valid throughout the remainder of this paper.

We assume that the equilibrium $O \in \mathbb{R}^4$ is hyperbolic with (nonzero) real and simple eigenvalues spec $= \{\pm 1, \pm\gamma\}$, $\gamma > 1$, as in (1.18). As we mentioned above,

$$RW^u = W^s\,, \tag{1.23}$$

where W^u and W^s are, respectively, the unstable and stable manifolds of O. By (1.23), we have $\dim W^u = \dim W^s = 2$.

We introduce coordinates $z = (x, y, u, v)$ in a neighborhood of O such that the vector field of the system in that neighborhood takes the form

$$\begin{aligned} \dot{x} &= -x + \cdots\,, \quad & \dot{u} &= -\gamma u + \cdots\,, \\ \dot{y} &= y + \cdots\,, \quad & \dot{v} &= \gamma v + \cdots\,, \end{aligned} \tag{1.24}$$

where the dots indicate nonlinear terms of higher order. Here, the equilibrium state O is in the origin; the coordinate axes are the eigendirections of the linearization matrix of the system at O. The unstable manifold is tangent to the plane $\{x = 0, u = 0\}$ and the stable manifold is tangent to the plane $\{y = 0, v = 0\}$. Since R maps coordinate axes onto the coordinate axes corresponding to the eigenvalues of the opposite sign [see (1.16)], it follows that R is given by

$$R(x, y, u, v) = (y, x, v, u) \tag{1.25}$$

in these coordinates. Obviously, the plane $\mathrm{Fix}(R)$ is given by

$$\mathrm{Fix}(R) = \{x = y,\ u = v\}\,. \tag{1.26}$$

It is also obvious that the affine planes $\{\frac{x+y}{2} = x_0, \frac{u+v}{2} = u_0\}$ are invariant with respect to R, and R acts as a center symmetry on each such plane. We use the notation R_M^- for these planes where $M = (x_0, y_0, u_0, v_0)$ is the point of intersection of the plane with $\mathrm{Fix}(R)$. Clearly, $R_M^- = M + \mathrm{Fix}(-R)$.

We suppose that the unstable manifold W^u of O intersects $\mathrm{Fix}(R)$ at some point M. Due to reversibility, the stable manifold W^s of O intersects $\mathrm{Fix}(R)$ at the same point. Therefore, the orbit Γ passing through M is a reversible homoclinic to O. We investigate a tangency of W^u with $\mathrm{Fix}(R)$ at the point M (Fig. 1), in this paper.

Specifically, we assume that

(A) *W^u possesses a tangency with* $\mathrm{Fix}(R)$ *at the point M.*

The tangency is assumed to be as simple as possible in the sense that:

(B) *the intersection of* $\mathrm{Fix}(R)$ *with the tangent plane of W^u at M is one-dimensional (a straight line), and*

(C) *the tangency is quadratic.*

We include our system in a one-parameter family, $f = f(z, \mu)$, depending smoothly on a parameter μ ["smoothly" means that the vector field is

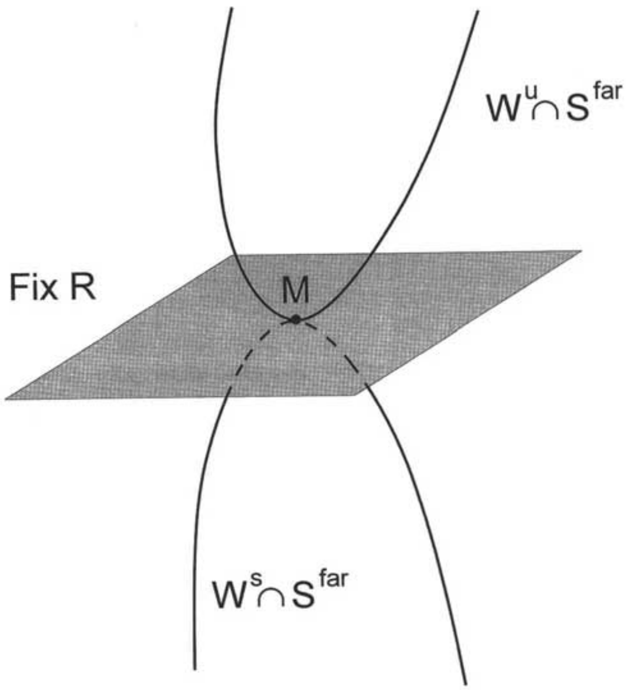

Fig. 1. The unstable manifold W^u of a reversible equilibrium O has a tangency with Fix(R) at some point M. Due to reversibility, the stable manifold W^s has a tangency with Fix(R) at the same point. Therefore, the orbit Γ passing through M is a reversible homoclinic to O.

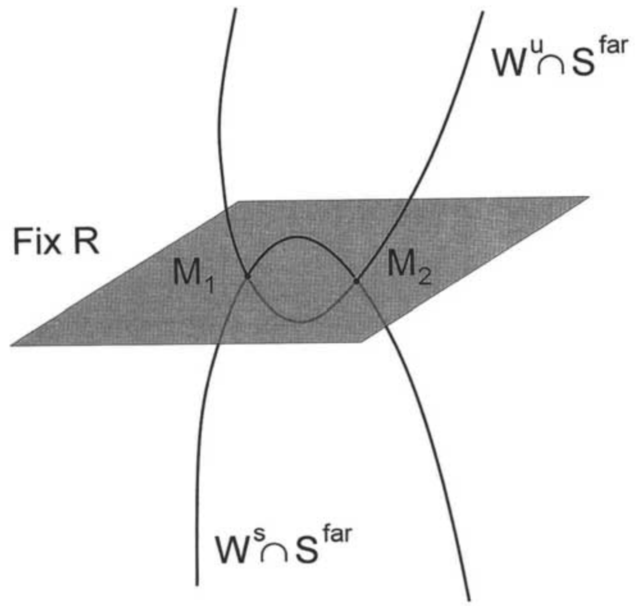

Fig. 2. The point of quadratic tangency disappears for $\mu > 0$: both W^u and its R-image W^s lift off from the Fix(R) plane simultaneously. For $\mu < 0$, on the other hand, two points M_1 and M_2 appear, at which W^u intersects Fix(R) transversely. The points M_1 and M_2 correspond to a pair of transverse reversible homoclinics Γ_1 and Γ_2.

C^r-smooth with respect to (x, y, u, v, μ)]. We also assume that:

(D) *W^u penetrates* Fix(R) *with a nonzero velocity as μ varies.*

Since the tangency is quadratic, the point of tangency disappears, say, for $\mu > 0$. In effect, both W^u and its R-image W^s lift off from the Fix(R) plane simultaneously. For $\mu < 0$, on the other hand, two points M_1 and M_2 appear, at which W^u intersects Fix(R) transversely. The points M_1 and M_2 correspond to a pair of transverse reversible homoclinics Γ_1 and Γ_2 (Fig. 2).

We need some additional nondegeneracy assumptions. First, we suppose that:

(E) Γ *does not belong to the strong unstable manifold W^{uu} of O.*

Recall that W^{uu} is the uniquely defined one-dimensional manifold which is tangent to $\{x = 0, y = 0, u = 0\}$ at O. Condition (E) means that Γ leaves the origin O in a leading direction, that is, tangent to the y-axis. Without loss of generality we may assume that Γ leaves the origin in the direction $y > 0$. Due to the reversibility of the system, Γ also returns to O in a leading direction: along the positive x-axis (Fig. 3).

For our remaining nondegeneracy assumptions, we consider a cross-section $S^{\text{out}} := \{y = \varepsilon\}$ to Γ, where $\varepsilon > 0$ is fixed small enough. Also, fix another cross-section S^{far} which contains the point M along with the local piece of Fix(R) near M (Fig. 4).

Note that in a small neighborhood of O there exists a (nonunique) three-dimensional invariant manifold W^{ue}_{loc} (so-called extended unstable manifold) which is C^1-smooth, tangent to the hyperplane $\{u = 0\}$ at O, and which contains W^u_{loc}; the general reference is Hirsch *et al.* [1977]; see Turaev [1996] for more detail. Let W^{ue} denote the forward continuation of W^{ue}_{loc} within a neighborhood of Γ and outside a small neighborhood of O.

We assume that:

(F) *W^{ue} is transverse to* Fix(R) *at the point* $M = \Gamma \cap \text{Fix}(R)$, *and*

(G) *W^{ue} is also transverse to the plane R^-_M (Fig. 5)*

Note that W^{ue} is not defined uniquely, but the tangent spaces of any two such manifolds coincide at all points of W^u. Thus, our transversality conditions (F) and (G) are well posed.

We have to distinguish two different geometric possibilities of how Fix(R) can adjoin to W^u.

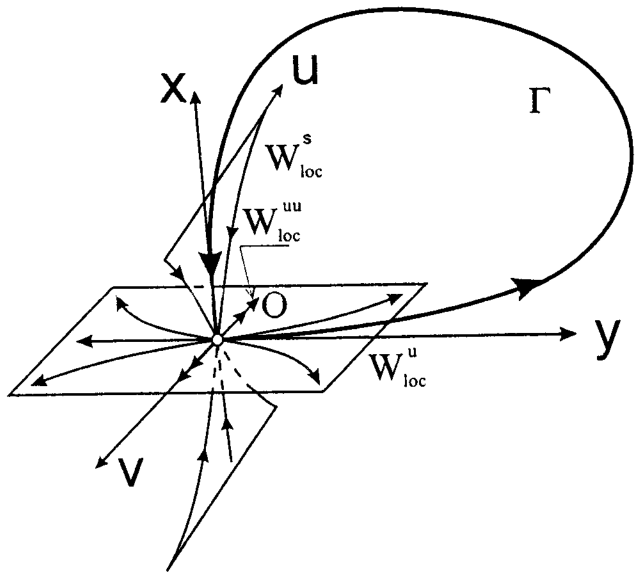

Fig. 3. By condition (E): $\Gamma \not\subset W^{uu}$, the homoclinic orbit Γ leaves the origin O in a leading direction, that is, tangent to the y-axis. Without loss of generality we assume that Γ leaves the origin in the direction $y > 0$. Due to the reversibility of the system, Γ also returns to O in leading direction: along the positive x-axis.

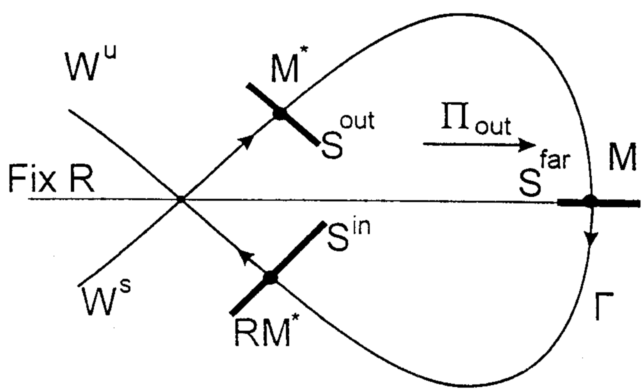

Fig. 4. Three cross-sections to Γ. The cross-sections $S^{\rm out}$ and $S^{\rm in} = RS^{\rm out}$ lie in a small neighborhood of O: $S^{\rm out}$ intersects $W^u_{\rm loc}$ and $S^{\rm in}$ intersects $W^s_{\rm loc}$. Another cross-section $S^{\rm far}$ contains the point $M = \Gamma \cap {\rm Fix}(R)$ along with the local piece of ${\rm Fix}(R)$ near M. The flow near Γ defines the map $\Pi^{\rm out} : S^{\rm out} \to S^{\rm far}$.

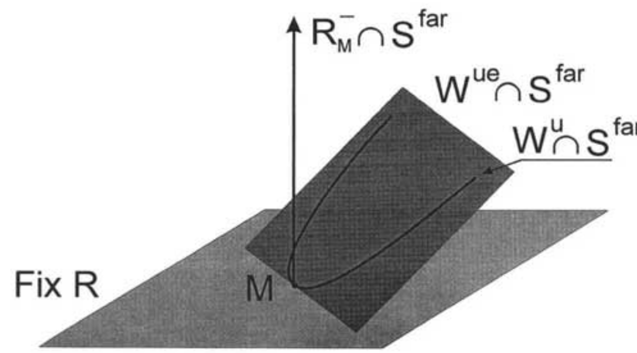

Fig. 5. The manifold W^{ue} *is assumed to be transverse to* ${\rm Fix}(R)$ *at the point* $M = \Gamma \cap {\rm Fix}(R)$ and *to the plane* $R^-_M \doteq M + {\rm Fix}(-R)$.

In a small neighborhood of the point $M^* := \Gamma \cap S^{\rm out}$ we have a flow defined Poincaré map $\Pi_{\rm out} : S^{\rm out} \to S^{\rm far}$. Due to our assumptions, the preimage $\Pi^{-1}_{\rm out}({\rm Fix}(R))$ possesses a quadratic tangency with the line $W^u_{\rm loc} \cap S^{\rm out} = \{x = 0, u = 0, y = \varepsilon\}$. Since $W^u_{\rm loc} \subset W^{ue}_{\rm loc}$, this line belongs to the surface $W^{ue}_{\rm loc} \cap S^{\rm out}$ which is close to the plane $\{u = 0, y = \varepsilon\}$. Below, we will show that this surface is tangent to the plane $\{u = 0, y = \varepsilon\}$ for a suitable choice of coordinates. By transversality of W^{ue} and ${\rm Fix}(R)$, the surface $W^{ue}_{\rm loc} \cap S^{\rm out}$ is transverse to $\Pi^{-1}_{\rm out}({\rm Fix}(R))$. Thus the intersection is a line l which possesses a quadratic tangency with the line $\{x = 0, u = 0, y = \varepsilon\}$. Moreover, the value x does not vanish everywhere on l (see Fig. 6). We will say that ${\rm Fix}(R)$ adjoins to W^u *from the positive side* if the line l belongs to the part of $W^{ue}_{\rm loc} \cap S^{\rm out}$ that corresponds to positive values of x, and *from the negative side* if $x < 0$ on l. Recall here that $x > 0$, within $W^u_{\rm loc}$, is the direction of return of our original homoclinic orbit Γ towards O.

Let U be a small neighborhood of $\Gamma \cup O$. We will study *one*-periodic orbits: orbits which are homotopic to $\Gamma \cup O$ in U. Specifically, we will study reversible one-periodic orbits, calling them *principal*. Similarly, Γ itself can be called *one*-homoclinic. Homoclinic orbits which complete k cycles in U and then close up at 0 are called *k-homoclinic.*

Theorem 1. *Let O be a reversible hyperbolic equilibrium, with real simple eigenvalues, of a four-dimensional reversible vector field of smoothness C^r, $r \geq 3$. Assume that O possesses a reversible homoclinic orbit along which the tangency and non-degeneracy conditions* (B)–(G) *specified above hold. Then, in a sufficiently small neighborhood U of the reversible 1-homoclinic Γ, and for small μ, there do not exist any k-homoclinic orbits, for any $k \geq 2$.*

For principal periodic orbits in U, and for small μ, the following holds.

If Fix(R) adjoins to W^u from the negative side, then there are no principal periodic orbits for $\mu \geq 0$. For fixed $\mu < 0$, there exists a one-parameter family

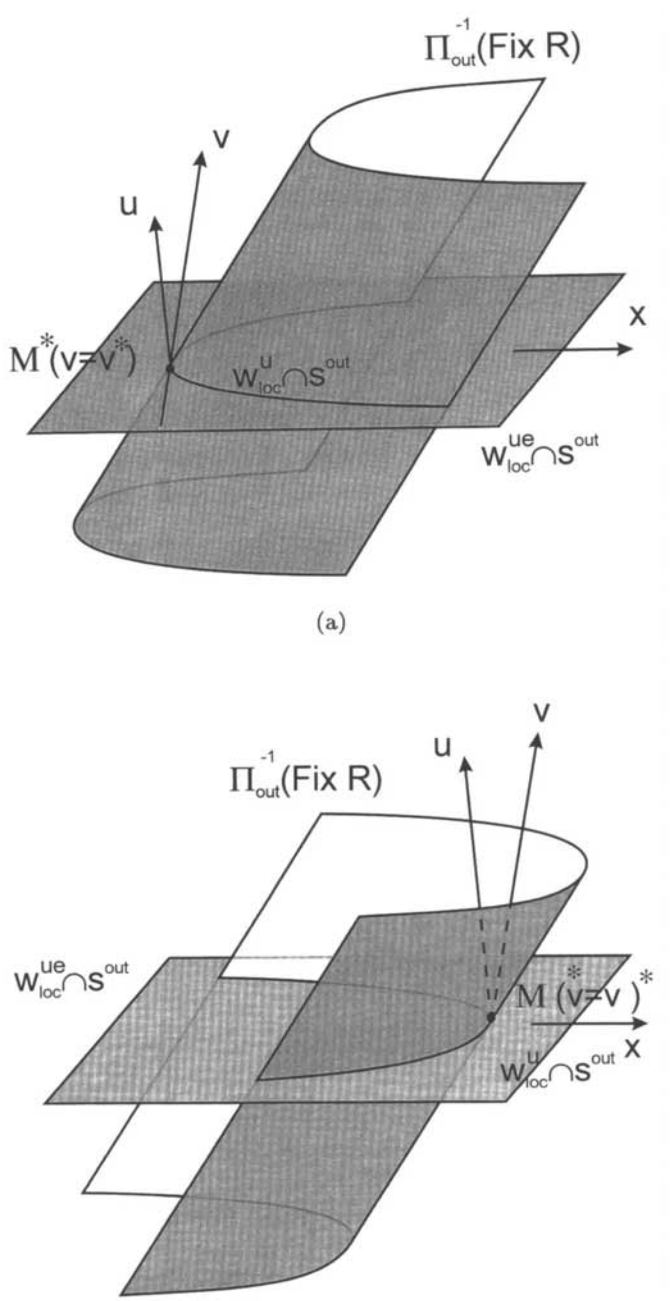

Fig. 6. Two different cases for quadratic tangency of W^u and Fix(R): (a) Fix(R) adjoins to W^u *from the positive side* and (b) Fix(R) adjoins to W^u *from the negative side.*

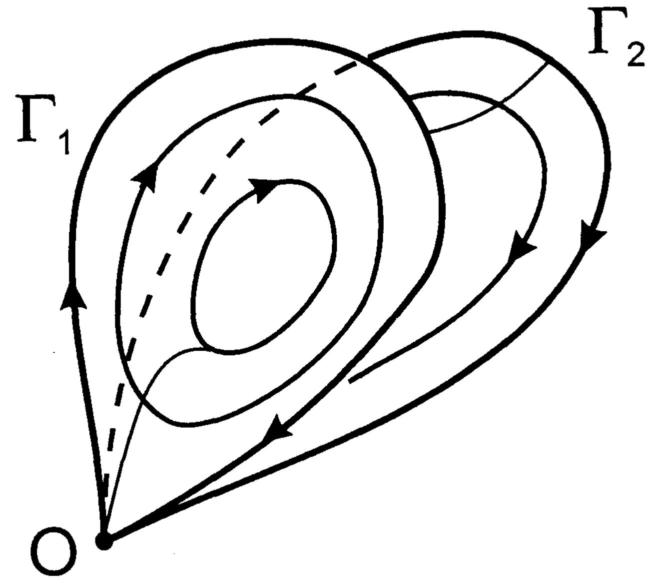

Fig. 7. The one-parameter family of principal periodic orbits joining the homoclinic orbits Γ_1 and Γ_2 for fixed $\mu < 0$ in the case where Fix(R) adjoins to W^u from the negative side.

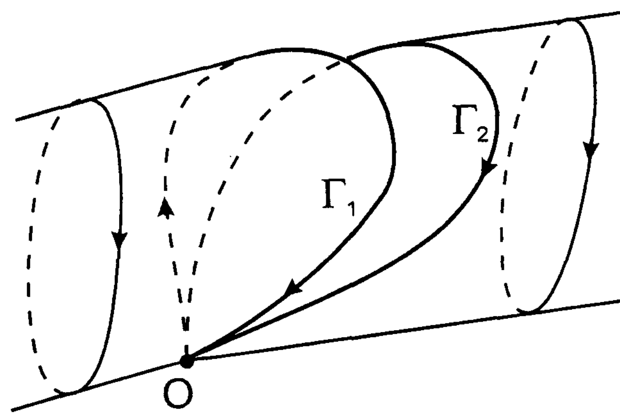

Fig. 8. The surface of principal periodic orbits in the case where Fix(R) adjoins to W^u from the positive side, for $\mu < 0$. It consists of two parts, one bounded by Γ_1 and the other by Γ_2.

of principal periodic orbits joining the homoclinic orbits Γ_1 and Γ_2 (Fig. 7).

If Fix(R) adjoins to W^u from the positive side, then there exists a one-parameter family of principal periodic orbits for $\mu > 0$. This family is split into two continuous paths by the homoclinic orbit Γ at $\mu = 0$. The two paths persist, separately, for $\mu < 0$, one bounded by Γ_1 and the other by Γ_2 (Fig. 8).

The families of principal periodic orbits form two-dimensional surfaces in $\mathbb{R}^4$. We will show that the set where these surfaces intersect the cross-section S^{out} lies on a smooth curve close to $W^u_{\text{loc}} \cap S^{\text{out}}$. Since the points on $W^u_{\text{loc}} \cap S^{\text{out}}$ are parametrized by the coordinate v, we can parametrize the principal periodic orbits by the v-coordinate of the point of intersection of the orbit with S^{out}. Since we study a small neighborhood of $\Gamma \cup O$, the value of v is close to v^* where v^* is the v-coordinate of the point $M^* = \Gamma \cap S^{\text{out}}$.

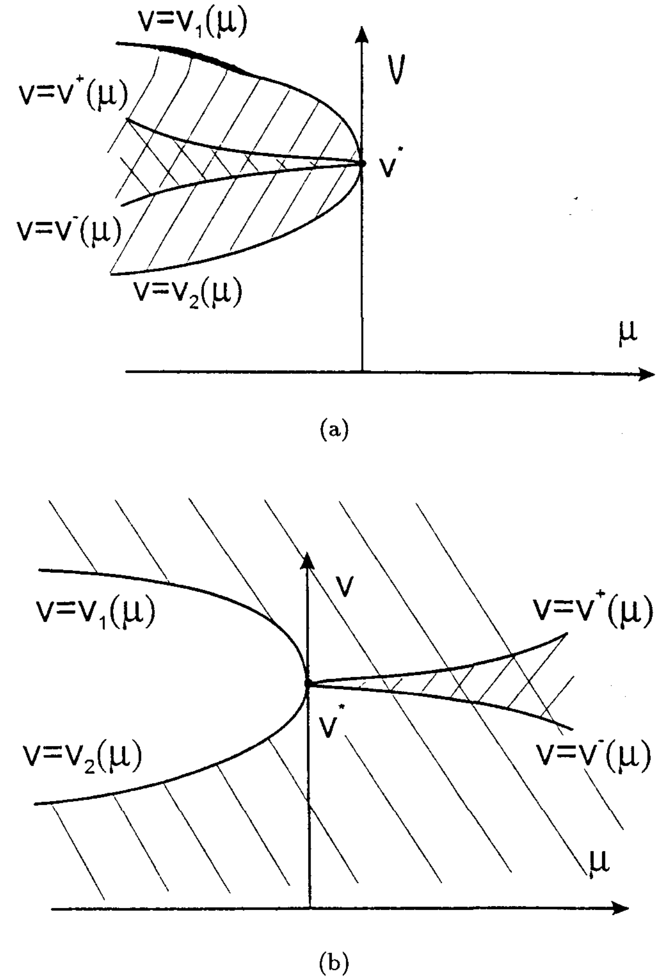

Fig. 9. An illustration to Theorems 1 and 2: (a) the negative case, (b) the positive case. The dashed regions correspond to the principal periodic orbits: the intersection of the dashed region with a line $\{\mu = \text{const}\}$ is the set of the v-values (the values of the v-coordinate of the point of intersection of the orbit with S^{out}) corresponding to the principal periodic orbits that exist for the given μ. The dashed regions are bounded by two curves $v = v_1(\mu)$ and $v = v_2(\mu)$ where v_1 and v_2 are the v-coordinates of the points M_1^* and M_2^*, respectively, where the transverse homoclinic orbits Γ_1 and Γ_2 emanating from Γ at $\mu < 0$ intersect S^{out}; the functions $v_{1,2}$ behave like $v^* \pm \sqrt{|\mu|}$, locally, for $\mu \le 0$. In the wedge bounded by the curves $v = v^+(\mu)$ and $v = v^-(\mu)$ the principal periodic orbits are elliptic; they are non-Möbius saddles for $v > v^+(\mu)$ and Möbius saddles for $v < v^-(\mu)$. Along the curves $v = v^+(\mu)$ and $v = v^-(\mu)$, respectively, algebraically double nontrivial Floquet multipliers $s = +1$ and $s = -1$ occur.

Theorem 1 can be illustrated by the two diagrams shown in Fig. 9: the dashed regions correspond to the principal periodic orbits; namely, the intersection of the dashed region with the line $\{\mu = \text{const}\}$ is the set of the v-values corresponding to the principal periodic orbits that exist for the given μ. The dashed regions are bounded by two curves $v = v_1(\mu)$ and $v = v_2(\mu)$. Here, v_1 and v_2 are the v-coordinates of the points M_1^* and M_2^*, respectively, where the transverse homoclinic orbits Γ_1 and Γ_2 emanating from Γ at $\mu < 0$ intersect S^{out}. Recall that Γ_1 and Γ_2 pass through points M_1 and M_2, the intersection points of W^u with $\text{Fix}(R)$. Therefore, the points M_1^* and M_2^* are the points of intersection of $W^u_{\text{loc}} \cap S^{\text{out}}$ with $\Pi^{-1}_{\text{out}}(\text{Fix}(R))$. Since the tangency of $W^u_{\text{loc}} \cap S^{\text{out}}$ with $\Pi^{-1}_{\text{out}}(\text{Fix}(R))$ at $\mu = 0$ is quadratic and the parameter μ is chosen generically, the functions $v_{1,2}$ behave like $v^* \pm \sqrt{|\mu|}$, locally, for $\mu \le 0$.

Our main result on reversible periodics near the tangent reversible homoclinic Γ is the appearance of a wedge of elliptic reversible orbits, separating Möbius from non-Möbius types.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then there exist two smooth functions $v^+(\mu)$ and $v^-(\mu)$, defined for $\mu > 0$ in the positive case, and for $\mu < 0$ in the negative case, $v^\pm(\mu) \to v^*$ and $\frac{d}{d\mu}(v^+(\mu) - v^-(\mu)) \to 0$ for $\mu \to 0$ (Fig. 9), such that the following holds.*

The principal periodic orbits are elliptic for $v^-(\mu) < v < v^+(\mu)$, non-Möbius saddles for $v > v^+(\mu)$, and Möbius saddles for $v < v^-(\mu)$. Along the curves $v = v^+(\mu)$ and $v = v^-(\mu)$, respectively, algebraically double nontrivial Floquet multipliers, $s = +1$ and $s = -1$, occur.

As of this writing we would like to mention the forthcoming work by Knobloch & Sandstede [1995], which also addresses the issue of higher singularities of the homoclinic set, in the multiparameter situation. In the present context, it appears possible to find a Smale horseshoe in an appropriate return map.

2. Proof of Theorem 1

We first prove the claims concerning reversible periodics near the primary reversible homoclinic Γ. The absence of k-homoclinics will be proved at the end of this section.

We recall from (1.12), that a reversible periodic orbit intersects the $\text{Fix}(R)$ plane in exactly two points. Conversely, any orbit that intersects $\text{Fix}(R)$ twice is a reversible periodic orbit. For a principal periodic orbit, in a small neighborhood U of $\Gamma \cup O$, one intersection point with $\text{Fix}(R)$ lies in a small neighborhood of O. The other intersection occurs in a small neighborhood of the point

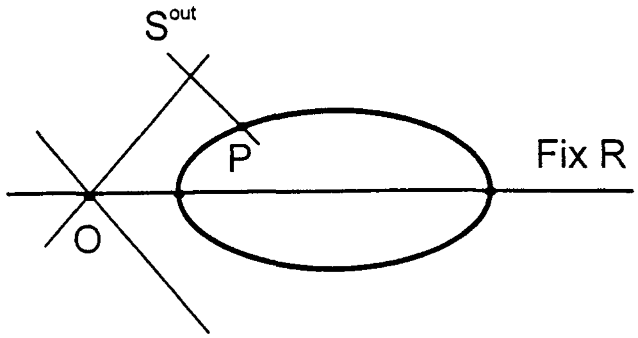

Fig. 10. A principal periodic orbit in a small neighborhood of $\Gamma \cup O$. It intersects the Fix(R) plane in exactly two points: one intersection point with Fix(R) lies in a small neighborhood of O and the other intersection occurs in a small neighborhood of the point $M = \Gamma \cap \text{Fix}(R)$. The point P, close to $M^* = \Gamma \cap S^{\text{out}}$, is the intersection point with S^{out}.

$M = \Gamma \cap \text{Fix}(R)$. Indeed, one-periodic orbits in U intersect cross-sections to U precisely once.

Let P, close to $M^* = \Gamma \cap S^{\text{out}}$, denote the intersection point with S^{out} (Fig. 10). The point P belongs to the line $\mathcal{L}$ which is the intersection of two surfaces in S^{out}. The first surface is the exit set on which the forward flow applied to the small piece of Fix(R) near O intersects S^{out}. The second surface is the set $\Pi_{\text{out}}^{-1}(\text{Fix}(R))$: the intersection with S^{out} of the backward flow applied to the piece of Fix(R) near M. We will use the notation $(\text{Fix}(R))^+$ and $(\text{Fix}(R))^-$ for these two surfaces: $\mathcal{L} = (\text{Fix}(R))^+ \cap (\text{Fix}(R))^-$. By definition, the orbit passing through any point of $\mathcal{L}$ intersects Fix(R) twice: near O and near M. Therefore, any such orbit is indeed a principal periodic orbit. Our proof of Theorem 1 consists of an approximate algebraic description of the surfaces $(\text{Fix}(R))^{\pm}$, near O, and hence of their intersection $\mathcal{L}$.

The surface $(\text{Fix}(R))^- = \Pi_{\text{out}}^{-1}(\text{Fix}(R))$ is easily described, algebraically. In fact, for $\mu = 0$ the surface $(\text{Fix}(R))^-$ possesses a quadratic tangency with the line $\{x = 0,\ u = 0\} = W_{\text{loc}}^u \cap S^{\text{out}}$ at the point $M^*(x = 0,\ u = 0,\ v = v^*) = \Gamma \cap S^{\text{out}}$ (see Fig. 6). This allows us to express $(\text{Fix}(R))^-$ in the form

$$a_1 x + a_2 u = c'(v - v^*)^2 + \cdots\,, \qquad (2.1)$$

for $\mu = 0$; here, the values c' and $|a_1| + |a_2|$ are nonzero and the dots indicate higher order terms. Moreover, $(\text{Fix}(R))^-$ is transverse to the surface $W_{\text{loc}}^{ue} \cap S^{\text{out}}$ which is in turn tangent to $\{u = 0\}$ at all points of $W_{\text{loc}}^u \cap S^{\text{out}}$ for a suitable choice of coordinates in the neighborhood of O, as detailed below [Ovsyannikov–Shil'nikov coordinates, see (2.14)–(2.16)]. Therefore, $(\text{Fix}(R))^-$ is transverse to the plane $\{u = 0\}$. Hence, a_1 in (2.1) is nonzero and we can rewrite (2.1) as

$$x = c(v - v^*)^2 + au + \cdots\,, \qquad (2.2)$$

Evidently, the sign of the coefficient c indicates how the set Fix(R) adjoins to W^u (positive or negative).

For nonzero μ, Eq. (2.2) perturbs to

$$x = c(\mu + (v - v^*)^2) + au + \cdots \qquad (2.3)$$

where the dots indicate higher order terms of the Taylor expansion in $(\mu,\ v - v^*,\ u)$. In a small neighborhood of the tangency μ, $(v - v^*)$, and u are small. Rescaling μ in (2.3), if necessary, we can assume the coefficients in front of μ and $(v - v^*)^2$ to be equal. By (2.3), the surface $(\text{Fix}(R))^-$ does not intersect $W_{\text{loc}}^u \cap S^{\text{out}} = \{x = 0,\ u = 0\}$ for $\mu > 0$. For $\mu < 0$, on the other hand, there are two intersection points, M_1^* and M_2^* with v-coordinates

$$v_{1,2} = v^* \pm \sqrt{|\mu|} + o(\sqrt{|\mu|}) \qquad (2.4)$$

through which the transverse homoclinic orbits Γ_1 and Γ_2 pass.

We describe $(\text{Fix}(R))^+$, next. We will prove below that this set is a C^1-surface bounded by $W_{\text{loc}}^u \cap S^{\text{out}}$ (Fig. 11). Moreover, for a suitable choice of coordinates $(x,\ y,\ u,\ v)$, this set is given by an equation of the form

$$u = \psi(x,\ v,\ \mu)\,, \quad x > 0 \qquad (2.5)$$

(we recall that S^{out} is the hyperplane $\{y = \varepsilon\}$, so the coordinates in S^{out} are $(x,\ u,\ v)$). The function ψ is C^1-smooth and

$$\psi \equiv 0\,, \quad \frac{\partial \psi}{\partial x} \equiv 0 \qquad (2.6)$$

at $x = 0$. In particular, $\psi = o(x)$. We will also prove below that

$$\frac{\partial \psi}{\partial(v,\ \mu)} = o(x)\,. \qquad (2.7)$$

Comparing formulas (2.5) and (2.3) for $(\text{Fix}(R))^+$ and $(\text{Fix}(R))^-$, respectively, one can easily see that the line $\mathcal{L} = (\text{Fix}(R))^+ \cap (\text{Fix}(R))^-$ is

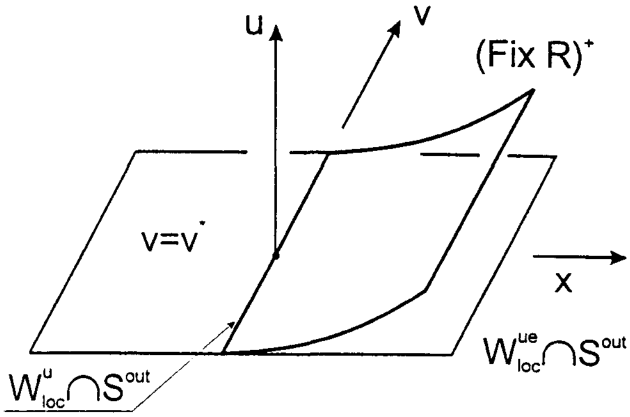

Fig. 11. The surface $(\mathrm{Fix}(R))^+$ is the exit set on which the forward flow applied to the small piece of $\mathrm{Fix}(R)$ near O intersects S^{out}. It is a C^1-surface bounded by $W^{u}_{\mathrm{loc}} \cap S^{\mathrm{out}}$ and tangent to $W^{ue}_{\mathrm{loc}} \cap S^{\mathrm{out}}$ at the points of $W^{u}_{\mathrm{loc}} \cap S^{\mathrm{out}}$.

given by the system

$$\begin{cases} x = c(\mu + (v - v^*)^2) + \varphi(\mu, v) \\ u = \psi(x, v, \mu) \\ x > 0 \end{cases} \tag{2.8}$$

where

$$\begin{aligned} &\varphi = o(|\mu| + (v - v^*)^2)\,, \\ &\frac{\partial \varphi}{\partial \mu} = o(1)\,, \\ &\frac{\partial \varphi}{\partial v} = O(|\mu|) + o(|v - v^*|)\,, \end{aligned} \tag{2.9}$$

for $\mu \to 0$, $v - v^* \to 0$.

System (2.8) is easily analyzed. First, we see that the points on $\mathcal{L}$ are parametrized over $(v - v^*)$. The dependence of x on $(v - v^*)$ is given by the first equation of (2.8). The graph of this dependence is a parabola-like curve. Throughout, we have to select the values of $(v - v^*)$ for which x is positive, in order to obtain principal periodic orbits. The results are summarized in Fig. 12. For example, principal periodic orbits do not exist for $\mu > 0$, in the negative case. The v-values of principal periodic orbits form an interval $v_1(\mu) < v - v^* < v_2(\mu)$, for $\mu < 0$. The bounds of this interval correspond to the homoclinic value $x = 0$. Indeed, the second equation of (2.8) and (2.6) imply $u = 0$. Therefore, these are the v-coordinates of the points of intersection of $(\mathrm{Fix}(R))^-$ with $W^{=}u_{\mathrm{loc}} \cap S^{\mathrm{out}} = \{x = 0,\, u = 0\}$, given by the homoclinic points (2.4). We also see that the set $\mathcal{L}$ which traces the principal periodic

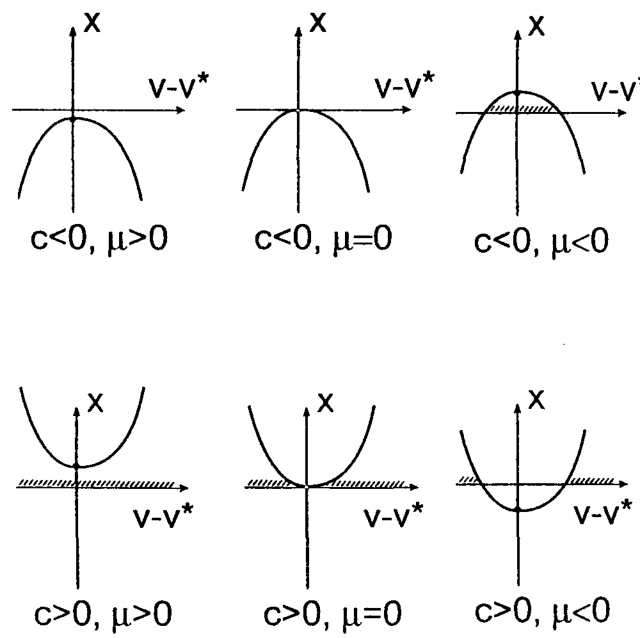

Fig. 12. The dependence of the x-coordinate on $(v - v^*)$ for the points of intersection of the principal periodic orbits with S^{out}. Throughout, only positive x correspond to principal periodic orbits [see (2.8)]. In the negative case ($c < 0$), the principal periodic orbits do not exist for $\mu \geq 0$. For $\mu < 0$, the v-values of principal periodic orbits form an interval $v_1(\mu) < v - v^* < v_2(\mu)$. The bounds of this interval correspond to the homoclinic value $x = 0$. In the positive case ($c > 0$), any small value of $(v - v^*)$ is admissible for $\mu > 0$ and, for $\mu \leq 0$, the admissible $(v - v^*)$ are $v - v^* < v_1(\mu)$ and $v - v^* > v_2(\mu)$.

sheet, is empty for $c < 0$, $\mu \geq 0$, it consists of one connected component for $c < 0$, $\mu < 0$, or $c > 0$, $\mu > 0$ and of two connected components for $c > 0$, $\mu \leq 0$ (Fig. 13). All this is in complete agreement with Theorem 1. To finish the proof it remains to prove expansions (2.5)–(2.7) for the exit set $(\mathrm{Fix}(R))^+$.

The structure of the set $(\mathrm{Fix}(R))^+$ is determined by the orbits of the system in a neighborhood of the equilibrium state O. We study this set, via the by now classical *Shil'nikov method* which is *the* powerful tool for studying local behavior. For simplicity of notation we suppress the parameter μ here.

We briefly review Shil'nikov's method, pertinent to our reversible problem. By Shil'nikov [1967], for small $\varepsilon > 0$, for any x_0, u_0, y_τ, v_τ such that $\|x_0\| \leq \varepsilon$, $\|u_0\| \leq \varepsilon$, $\|y_\tau\| \leq \varepsilon$, $\|v_\tau\| \leq \varepsilon$, and for any $\tau \geq 0$, there exists a unique solution of the so-called Shil'nikov problem: to find an orbit $x(t)$, $y(t)$, $u(t)$, $v(t)$ of system (1.24) that lies entirely in the ε-neighborhood of O and satisfies the boundary

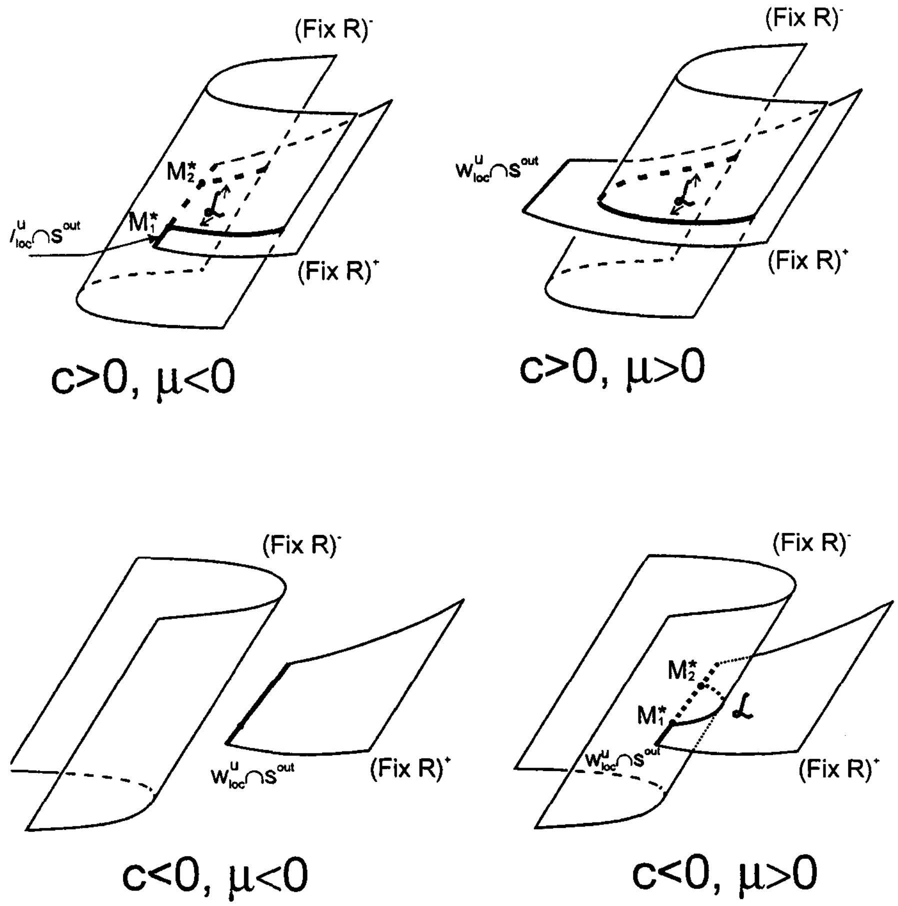

Fig. 13. The set $\mathcal{L} = (\text{Fix}(R))^+ \cap (\text{Fix}(R))^-$ which traces the principal periodic sheet in S^{out}, is empty for $c < 0$, $\mu \geq 0$, it consists of one connected component for $c < 0$, $\mu < 0$, or $c > 0$, $\mu > 0$ and of two connected components for $c > 0$, $\mu \leq 0$.

conditions:

$$\begin{aligned} x(0) &= x_0\,, \quad & u(0) &= u_0\,, \\ y(\tau) &= y_\tau\,, \quad & v(\tau) &= v_\tau\,, \end{aligned} \tag{2.10}$$

(see Fig. 14). Note that these boundary conditions mix (incoming) stable directions with (outgoing) unstable directions. The Shil'nikov variables $(x_0, u_0, y_\tau, v_\tau, \tau)$ effectively parametrize orbits, rather than points, near O. In fact, the solution depends smoothly on the initial data $(x_0, u_0, y_\tau, v_\tau, \tau)$. In other words, there exist C^r-functions (X, Y, U, V) such that an orbit of system (1.24) that starts with a point (x_0, y_0, u_0, v_0) in a small neighborhood of O reaches a point $(x_\tau, y_\tau, u_\tau, v_\tau)$ at time $t = \tau$ if, and only if,

$$\begin{aligned} x_\tau &= X(x_0, u_0, y_\tau, v_\tau, \tau)\,, \\ u_\tau &= U(x_0, u_0, y_\tau, v_\tau, \tau)\,, \\ y_0 &= Y(x_0, u_0, y_\tau, v_\tau, \tau)\,, \\ v_0 &= V(x_0, u_0, y_\tau, v_\tau, \tau)\,. \end{aligned} \tag{2.11}$$

Due to the reversibility of our system, relations (2.11) must be symmetric (equivariant) with respect to the transformation

$$(x_0, u_0, y_0, v_0) \leftrightarrow (y_\tau, v_\tau, x_\tau, u_\tau)\,.$$

This yields the following identities:

$$\begin{aligned} Y(x_0, u_0, y_\tau, v_\tau, \tau) &\equiv X(y_\tau, v_\tau, x_0, u_0, \tau)\,, \\ V(x_0, u_0, y_\tau, v_\tau, \tau) &\equiv U(y_\tau, v_\tau, x_0, u_0, \tau)\,. \end{aligned} \tag{2.12}$$

Suppressing the parameter μ, we claim that the set $(\text{Fix}(R))^+ \subset S^{\text{out}} = \{y = \varepsilon\}$ is given, locally, by the points $P = (x_\tau, y_\tau = \varepsilon, u_\tau, v_\tau)$ for which

$$\begin{cases} x_\tau = X(\varepsilon, v_\tau, \varepsilon, v_\tau, \tau) \\ u_\tau = U(\varepsilon, v_\tau, \varepsilon, v_\tau, \tau) \end{cases}, \tag{2.13}$$

where v_τ can take arbitrary values close to v^*, and τ must be taken large enough. In other words, the point P belongs to the surface $(\text{Fix}(R))^+$ if and only

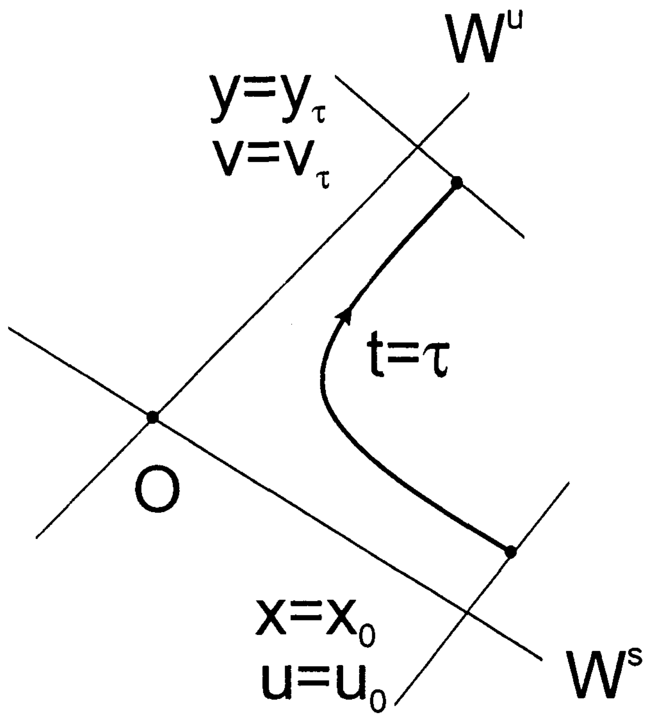

Fig. 14. For small $\varepsilon > 0$, for any x_0, u_0, y_τ, v_τ such that $\|x_0\| \le \varepsilon$, $\|u_0\| \le \varepsilon$, $\|y_\tau\| \le \varepsilon$, $\|v_\tau\| \le \varepsilon$, and for any $\tau \ge 0$ there exists a unique solution of the Shil'nikov problem: an orbit $x(t)$, $y(t)$, $u(t)$, $v(t)$ that lies entirely in the ε-neighborhood of O and satisfies the boundary conditions

$$x(0) = x_0\,, \quad u(0) = u_0\,,$$
$$y(\tau) = y_\tau\,, \quad v(\tau) = v_\tau\,.$$

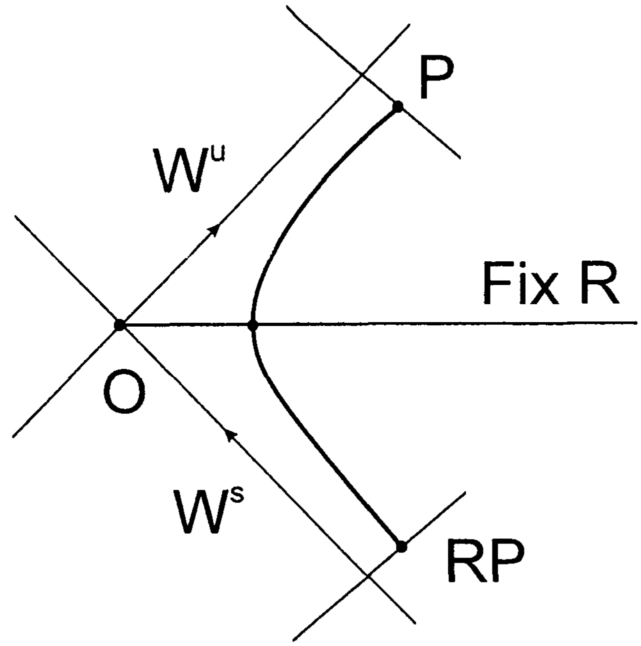

Fig. 15. A point $P \in S^{\text{out}}$ belongs to the surface $(\text{Fix}(R))^+$ if and only if the orbit starting with the point $RP \in S^{\text{in}}$ reaches P after some time $t = \tau$: the orbit intersects Fix(R) near O "halfway" between P and RP.

if the orbit starting with the point $RP(x_0 = \varepsilon,\, y_0 = x_\tau,\, u_0 = v_\tau,\, v_0 = u_\tau)$ in $S^{\text{in}} := \{x = \varepsilon\} = RS^{\text{out}}$ reaches $P \in S^{\text{out}}$ after some time $t = \tau$. Indeed, consider the forward orbit starting at some point of Fix(R) near O and intersecting S^{out} at P. Then, by reversibility, the backward orbit intersects the cross-section $S^{\text{in}} = RS^{\text{out}}$ at the point RP, after the same time (Fig. 15). Conversely, any orbit passing through points P and RP intersects Fix(R) "halfway" between P and RP. Now insert

$$\begin{aligned}(x_0, y_0, u_0, v_0) &= RP = R(x_\tau, y_\tau, u_\tau, v_\tau)\\ &= (y_\tau, x_\tau, v_\tau, u_\tau) = (\epsilon, x_\tau, v_\tau, u_\tau)\end{aligned}$$

into (2.11) to obtain (2.13).

To utilize (2.13) we need some estimates on the functions X and U. We obtain these estimates using results by Ovsyannikov & Shil'nikov [1992]. They show that, by a local C^{r-1}-smooth transformation of coordinates, near identity, the system takes the following Ovsyannikov–Shil'nikov form near O:

$$\begin{aligned}\dot{x} &= -x + g_{11}(x, y, v)x + g_{12}(x, y, u, v)u\,,\\ \dot{u} &= -\gamma u + g_{21}(x, y, v)x + g_{22}(x, y, u, v)u\,,\\ \dot{y} &= y + f_{11}(x, y, u)y + f_{12}(x, y, u, v)v\,,\\ \dot{v} &= \gamma v + f_{21}(x, y, u)y + f_{22}(x, y, u, v)v\,,\end{aligned} \tag{2.14}$$

where g_{ij} and f_{ij} are C^{r-1}-functions satisfying the identities:

$$\begin{aligned}g_{11}(x = 0, y, v) &\equiv 0\,,\\ g_{21}(x = 0, y, v) &\equiv 0\,,\\ g_{11}(x, y = 0, v = 0) &\equiv 0\,,\\ g_{12}(x, y = 0, u, v = 0) &\equiv 0\,,\end{aligned} \tag{2.15}$$

$$\begin{aligned}f_{11}(x, y = 0, u) &\equiv 0\,,\\ f_{21}(x, y = 0, u) &\equiv 0\,,\\ f_{11}(x = 0, y, u = 0) &\equiv 0\,,\\ f_{12}(x = 0, y, u = 0, v) &\equiv 0\,.\end{aligned} \tag{2.16}$$

In these coordinates the local stable and unstable manifolds are straightened. Identities (2.15), (2.16) imply that the equations for $\dot{x}$ and $\dot{y}$ are linear on W^s_{loc} and W^u_{loc}, respectively. Moreover, terms of the form $x\tilde{g}(y, v)$, linear in x, are eliminated in the equations for $\dot{x}$ and $\dot{u}$. Likewise, terms of the form $y\tilde{f}(x, u)$ are eliminated in the equations for $\dot{y}$ and $\dot{v}$.

Another feature of Ovsyannikov–Shil'nikov form is that the manifold $W^{ue}_{\rm loc}$ is tangent to the hyperplane $\{u = 0\}$ everywhere on $W^u_{\rm loc}$, in the coordinates (2.14)–(2.16). This fact was used for expansions (2.2), (2.3) above. To prove the claimed tangency, consider an arbitrary orbit $\{\hat{x}(t) = 0, \hat{y}(t), \hat{u}(t) = 0, \hat{v}(t)\}$ on $W^u_{\rm loc}$ together with the linearization of system (2.14) along that orbit. In other words, we consider the coordinate transformation $(x \to \hat{x}(t)+x, y \to \hat{y}(t)+y, u \to \hat{u}(t)+u, v \to \hat{v}(t)+v)$ and omit second (and higher) order terms in the Taylor expansion of the right-hand sides in powers of the "deviations" (x, y, u, v). Using that $g_{21}|_{\hat{x}=0} \equiv 0$, by (2.15), the equation for $\dot{u}$ in the linearization takes the form

$$\dot{u}=(-\gamma+g_{22}(\hat{x}=0, \hat{y}(t), \hat{u}=0, \hat{v}(t)))u\,. \qquad (2.17)$$

We see that $u = 0$ is a solution of this equation; i.e., the hyperplane $\{u = 0\}$ along $W^u_{\rm loc}$ is invariant with respect to the linearized flow. One can extract from Hirsch *et al.* [1977] that the family of tangent hyperplanes to the invariant manifold $W^{ue}_{\rm loc}$ at all points of $W^u_{\rm loc}$ is a unique family, which is transverse to the manifold $W^{ss}_{\rm loc}\{x = 0, y = 0, v = 0\}$ at O, and which is invariant with respect to the linearized flow. Therefore, it is the hyperplane $\{u = 0\}$ which is tangent to $W^{ue}_{\rm loc}$ at all points of $W^u_{\rm loc}$, indeed. Note that this observation completes the proof of (2.2), (2.3) concerning the expansion for $(\mathrm{Fix}(R))^-$.

The advantage of the reduction of the system near the saddle to Ovsyannikov–Shil'nikov form is that (2.15), (2.16) imply the following estimates for the solutions of the Shil'nikov problem (2.11) to hold (see Ovsyannikov & Shil'nikov [1992]):

$$\begin{aligned} X &= e^{-\tau}x_0 + o(e^{-\tau})\,,\\ Y &= e^{-\tau}y_\tau + o(e^{-\tau})\,,\\ U &= o(e^{-\tau})\,,\\ V &= o(e^{-\tau})\,. \end{aligned} \qquad (2.18)$$

Note that the Shilnikov variables (X, Y, U, V) in (2.18) are only C^{r-1} in $(x_0, u_0, y_\tau, v_\tau, \tau)$, the same smoothness as the transformed vector field near O. To determine the degree of smoothness with respect to the parameter μ we recall the proof in Ovsyannikov & Shil'nikov [1992]. The coordinate transformation bringing the vector field to form (2.14)–(2.16) appears in Ovsyannikov & Shil'nikov [1992] as a solution of some functional equation which, simultaneously, determines a strong stable manifold of an equilibrium state of some C^{r-1} vector field. In general, a strong stable manifold is known to be of the same smoothness with respect to phase variables as the associated vector field (C^{r-1} in our case) but the smoothness with respect to parameters decreases by 1 (from C^{r-1} to C^{r-2}). Thus, we may expect that the coordinate transformation is C^{r-1}-smooth with respect to (x, y, u, v) and C^{r-2}-smooth with respect to μ. The same smoothness is inherited by the functions f_{ij} and g_{ij} in (2.14): after one differentiation with respect to (x, y, u, v) they admit $(r - 2)$ continuous derivatives with respect to (x, y, u, v, μ). The analogous smoothness result holds true for the functions (X, Y, U, V) in (2.18). Note that differentiation preserves estimates (2.18) (see Ovsyannikov & Shil'nikov [1992]).

We will see that estimates (2.18) are sufficient in order to prove expansions (2.5)–(2.7) for $(\mathrm{Fix}(R))^+$ (and to finish the proof of the theorem in the part concerning principal periodic orbits). Note, however, that the estimates are proved for the general non-reversible situation. Specifically, the coordinate transformation need not preserve the linear involution R. Since the transformation is close to identity, the involution $(R^x, R^y, R^u, R^v) = R$, given by (1.25) can be written in the transformed coordinates as

$$R(x, y, u, v) = (y, x, v, u) + \cdots\,; \qquad (2.19)$$

dots indicate terms of higher order. Note that $RW^s_{\rm loc} = W^u_{\rm loc}$; that is, the plane $\{y = 0, v = 0\}$ is mapped locally onto the plane $\{R^x = 0, R^u = 0\}$ by R. This means that (R^x, R^u) vanishes simultaneously with (y, v). Using (x, u, R^x, R^u) as new coordinates (x, u, y, v), R retains its linear form. Moreover, the equations for $\dot{x}$ and $\dot{u}$ preserve their form. Since the new coordinates $(y, v) = (R^x, R^u)$ vanish simultaneously with the old (y, v), identities (2.15) persist in these new coordinates.

Since the vector field, accordingly transformed, is again reversible with respect to the again linear involution (1.25), it follows that the equations for $\dot{y}$ and $\dot{v}$ in (2.14) also preserve their form, and the functions f_{ij} satisfy the identities:

$$f_{ij}(x, y, u, v) = -g_{ij}(y, x, v, u)\,. \qquad (2.20)$$

Clearly (2.20) and (2.15) imply (2.16). To conclude, bringing the system to Ovsyannikov–Shil'nikov form

(2.14)–(2.16) can be achieved without destroying linearity of the involution R.

We are now ready to prove formulas (2.5)–(2.7) for the exit set $(\mathrm{Fix}(R))^+$. Estimates (2.18) allow us to rewrite Eqs. (2.13) for $(\mathrm{Fix}(R))^+$ in the form

$$\begin{aligned} x_\tau &= e^{-\tau}\varepsilon + o(e^{-\tau})\,, \\ u_\tau &= o(e^{-\tau})\,. \end{aligned} \tag{2.21}$$

From the first equation we have

$$x > 0 \tag{2.22}$$

and

$$\tau = -\ln\frac{x}{\varepsilon} + o(1)\,. \tag{2.23}$$

Here, $o(1)$ is a function of (x_τ, v_τ, μ) that tends to zero along with its first derivatives for $x_\tau \to +0$. Substituting (2.23) in the second equation of (2.21) we obtain the desired formulas (2.5)–(2.7). This completes the proof of our claims on reversible periodics in Theorem 1.

It remains to prove the absence of k-homoclinic orbits, for $k \geq 2$. Consider the *pass wedge* $W \subseteq S^{\mathrm{out}}$ of those points $z(\tau)$ in S_{out} which lie on orbits through points $z(0)$ in $S^{in} = RS^{\mathrm{out}}$ in a neighborhood U of the primary reversible homoclinic Γ, and which hence pass by O. By the Ovsyannikov–Shilnikov expansions (2.18) for the Shilnikov coordinatization (2.11), (2.12), the pass wedge W is indeed a wedge-shaped region, tangent to W^{ue}_{loc} along W^u_{loc} in S^{out}; it is given by free small coordinates u_0, v_τ, $e^{-\tau}$ and satisfies

$$\begin{aligned} x_\tau &= e^{-\tau}\epsilon + o(e^{-\tau})\,, \\ u_\tau &= o(e^{-\tau})\,. \end{aligned} \tag{2.24}$$

This proves tangency of W to $W^{ue}_{\mathrm{loc}} \cap S^{\mathrm{out}} = \{y_\tau = \epsilon,\ u_\tau = 0\}$.

Propagating W to S^{far} by the Poincaré map Π_{out}, we claim

$$z_1 - z_0 \in \mathrm{Fix}(-R) \Rightarrow z_1 = z_0\,, \tag{2.25}$$

for any two points z_0, z_1 in the closure of $\Pi_{\mathrm{out}}(W) \subseteq S^{\mathrm{far}}$. Indeed, this holds for points $z_1 - z_0$ in the tangent space $T_M W^{ue}$ by transversality of W^{ue} to R^-_M, that is, to $\mathrm{Fix}(-R)$ at the intersection M of the primary reversible homoclinic Γ with S^{far}. Consequently, (2.25) holds, locally, in $W^{ue} \cap S^{\mathrm{far}}$. By tangency to W^{ue}, in turn, (2.25) extends to z_0, z_1 in the closure of $\Pi_{\mathrm{out}}(W) \subseteq S^{\mathrm{far}}$. This proves our claim.

Now suppose $z_0 \in S^{\mathrm{far}} \cap W^s$ lies on any, not necessarily reversible, k-homoclinic orbit, $k \geq 2$. Recall that $z_0 \in W^s$ indicates that the forward orbit of z_0 immediately limits onto O, rather than passing by. Since z_0 lies on a k-homoclinic, its backward orbit must pass by O, hence z_0 also lies in the propagated pass wedge $\Pi_{\mathrm{out}}(W)$. Let $z_1 := Rz_0 \in S^{\mathrm{far}} \cap W^u$. Note that z_1 lies in the closure of the propagated pass wedge W. Applying (2.25), we obtain $z_0 = z_1 = Rz_0 \in \mathrm{Fix}(R)$. In particular,

$$z_0 \in W^s \cap \mathrm{Fix}(R)$$

is on a reversible 1-homoclinic, rather than a k-homoclinic with $k \geq 2$. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Before proving Theorem 2 on the presence of elliptic orbits in the family of principal periodics, we briefly review the underlying geometry. Consider the Poincaré map Π_{loc} defined by the orbits which start on the cross-section S^{in} near the point $RM^* = \Gamma \cap S^{\mathrm{in}}$ and reach the cross-section S^{out} near the point $M^* = \Gamma \cap S^{\mathrm{out}}$. The domain of definition of the map Π_{loc} is a wedge RW tangent to the plane $\{v = 0\}$ along the line $\{y = 0,\ v = 0\} = W^s_{\mathrm{loc}} \cap S^{\mathrm{in}}$. The wedge RW is the R-image of the pass wedge W which we introduced in the previous section when proving the absence of k-homoclinic orbits ($k \geq 2$). The pass wedge W is, in turn, the range of the map Π_{loc}.

Using the Ovsyannikov–Shil'nikov expansion (2.18), one can see that the map Π_{loc} acts as a strong contraction along the u-axis, and as a strong expansion along the v-axis. There are also neutral directions: the y-axis in S^{in} and, correspondingly, the x-axis in S^{out}. The flow along Γ defines the so-called global map $\Pi_{\mathrm{glo}} : W \to S^{\mathrm{in}}$. If the homoclinic orbit Γ was transverse, then the flow along Γ would map the pass wedge W into S^{in} as shown in Fig. 16, so that the image of $W^u_{\mathrm{loc}} \cap S^{\mathrm{out}} = \{x = 0,\ u = 0\}$ would have a nonzero projection onto the v-axis in S^{in}. In this case, the combined Poincaré map $\Pi := \Pi_{\mathrm{glo}} \circ \Pi_{\mathrm{loc}} : RW \to S^{\mathrm{in}}$ would be contracting along the u-axis and expanding along the v-axis, as is the local map. The neutral direction would be tangent to the curve of intersection of S^{in} with the surface of the principal periodic orbits adjoining to the homoclinic loop. All points of this curve are fixed points of Π; they are saddles and their strong

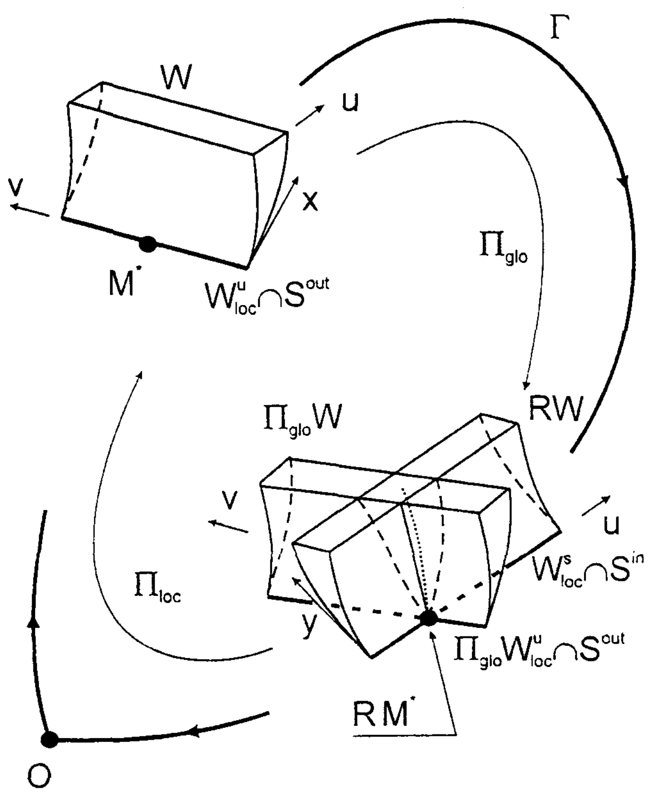

Fig. 16. For a transverse homoclinic loop Γ, the flow along Γ maps the pass wedge W into $S^{\rm in}$ in such a way that the image of $W^u_{\rm loc} \cap S^{\rm out} = \{x = 0,\, u = 0\}$ projects nontrivially onto the v-axis in $S^{\rm in}$.

stable and unstable invariant manifolds limit onto $W^s_{\rm loc} \cap S^{\rm in}$ and $\Pi_{\rm glo}(W^u_{\rm loc} \cap S^{\rm out})$, respectively.

Two possibilities can be distinguished in this case: the principal periodics can be Möbius or non-Möbius. In the non-Möbius case the Poincaré map Π preserves orientation of the strong unstable manifolds (strong stable as well), and in the Möbius case orientation is changed. The strong unstable manifolds possess a nonzero projection onto the v-axis, since they limit onto $\Pi_{\rm glo}(W^u_{\rm loc} \cap S^{\rm out})$. Note that the local map $\Pi_{\rm loc}$ preserves the orientation of projections onto the v-axis. Therefore, the global map $\Pi_{\rm glo}$ rotates $W^u_{\rm loc} \cap S^{\rm out}$ so that the image of a positively directed vector along $W^u_{\rm loc} \cap S^{\rm out}$ possesses a negative projection onto the v-axis in $S^{\rm in}$, in the Möbius case. In contrast, if the projection of the image is positive, then we are in the non-Möbius situation.

Now, return to our nontransverse homoclinic Γ. For definiteness, let Fix(R) adjoin to W^u from the negative side. In this case, the image $\Pi_{\rm glo}W$ of the pass wedge W looks like a horseshoe. For $\mu = 0$, the line $\Pi_{\rm glo}(W^u_{\rm loc} \cap S^{\rm out})$ is tangent to $W^s_{\rm loc} \cap S^{\rm in}$ and the domain RW of the Poincaré map Π does not intersect $\Pi_{\rm glo}W$ [Fig. 17(a)]. For $\mu < 0$ [Fig. 17(b)], the lines $\Pi_{\rm glo}(W^u_{\rm loc} \cap S^{\rm out})$ and $W^s_{\rm loc} \cap S^{\rm in}$ intersect at two points which correspond to transverse reversible homoclinic loops. According to Theorem 1, these points are connected by the line of fixed points of the Poincaré map Π which correspond to the principal periodic orbits. Since the line $\Pi_{\rm glo}(W^u_{\rm loc} \cap S^{\rm out})$ is folded, orientation with respect to the v-axis changes when moving from one of the intersection points to the other. Thus, the fixed points of Π are Möbius near one of the points of intersection of $\Pi_{\rm glo}(W^u_{\rm loc} \cap S^{\rm out})$ and $W^s_{\rm loc} \cap S^{\rm in}$ and they are non-Möbius near the other point.

We see that the nontrivial Floquet multiplier s must be negative near one end of the curve of fixed points of Π, and positive near the other end. So, $s \neq 0$ must leave the real axis somewhere in between to become complex: the corresponding principal periodic orbits are elliptic.

Rather than corroborating this underlying geometrical picture directly, we prove the claims of Theorem 2 concerning the elliptic periodic orbits algebraically. Our computations will prove, in particular, that the curves $v^{\pm}(\mu)$ which bound the cuspidal ellipticity region bifurcate along the same tangent.

We recall from (1.21) that the spectrum of Floquet multipliers of a principal periodic orbit has the form $\{s,\, s^{-1},\, 1,\, 1\}$. Therefore, the nontrivial multiplier s of a principal periodic orbit is determined by the trace equation

$$\text{tr } C := s + \frac{1}{s} + 2 \tag{3.1}$$

where C is the Floquet linearization matrix of the period map along the orbit. For fixed μ, the principal periodic orbits are parametrized by the coordinate v of their intersection point with $S^{\rm out}$. Therefore, the matrix C in (3.1) depends on v and μ, only.

By (3.1), the equations

$$\text{tr } C(v,\, \mu) = 4 \tag{3.2}$$

and

$$\text{tr } C(v,\, \mu) = 0 \tag{3.3}$$

correspond to Floquet multipliers $s = +1$ and $s = -1$, respectively. To prove Theorem 2 we must therefore solve (3.2), (3.3) by functions $v = v^+(\mu)$, $v = v^-(\mu)$, respectively, for μ near zero and v near v^*.

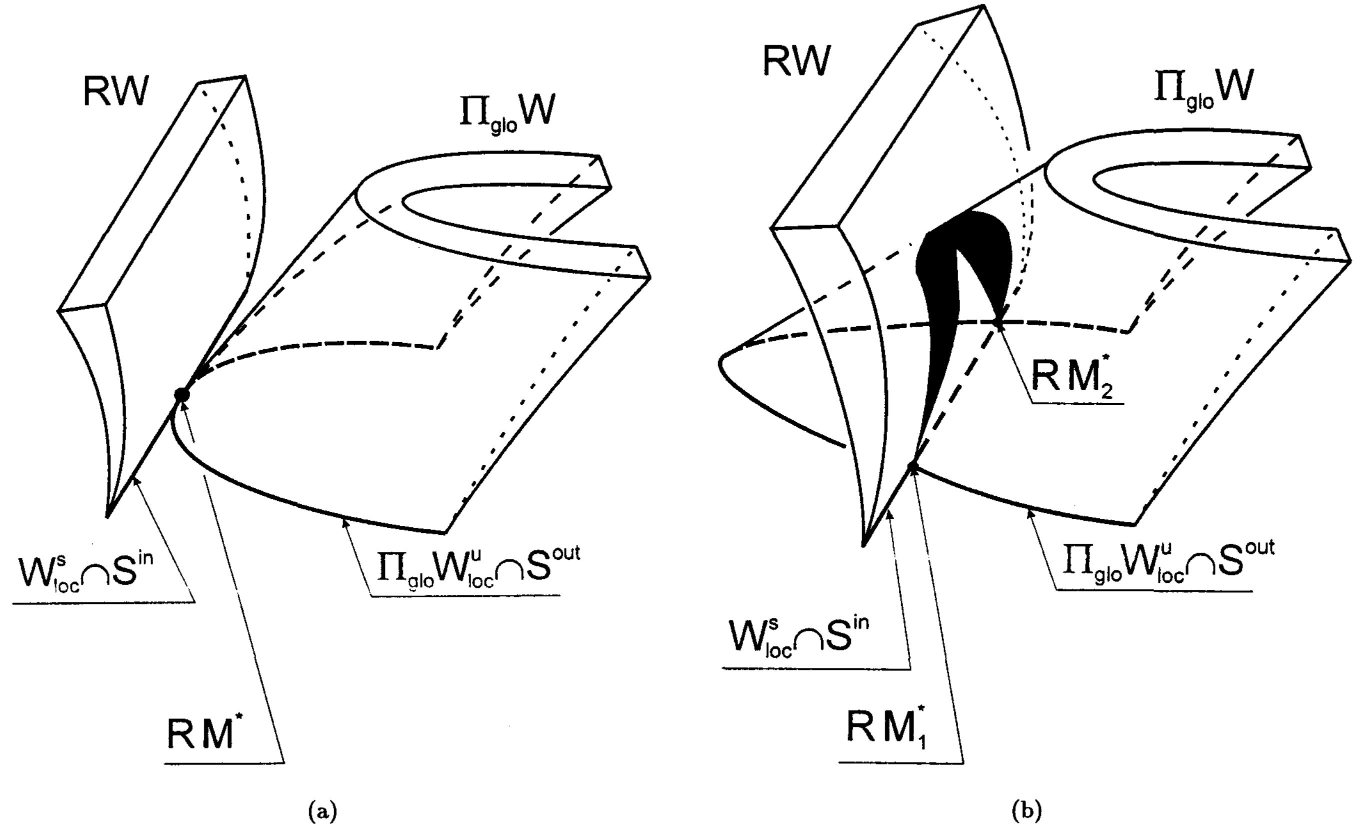

Fig. 17. For the nontransverse homoclinic Γ the image $\Pi_{glo}W$ of the pass wedge W has a horseshoe-like form. If Fix(R) adjoins to W^u from the negative side, then the domain RW of the Poincaré map Π does not intersect $\Pi_{glo}W$ for $\mu = 0$ [Fig. 17(a)]. For $\mu < 0$ [Fig. 17(b)], the lines $\Pi_{glo}(W^u_{loc} \cap S^{out})$ and $W^s_{loc} \cap S^{in}$ intersect at two points, RM_1^* and RM_2^*, which correspond to transverse reversible homoclinic loops. The intersection of RW and $\Pi_{glo}W$ is a "banana-like" region, shown black in the figure, with the ends at these two points. According to Theorem 1, the end points are connected by a line within this region which consists of fixed points of the Poincaré map Π, alias principal periodic orbits.

Let $P(x, y = \varepsilon, u, v)$ be the point of intersection of a principal periodic orbit with S^{out}. The reflected point $RP = (\varepsilon, x, v, u)$ is the point of intersection of this orbit with the incoming section $S^{in} = RS^{out}$ near the equilibrium O. The coordinates x and u are expressed in terms of v and μ by the following system

$$\begin{cases} x = c(\mu + (v - v^*)^2) + o(|\mu| + (v - v^*)^2) \\ u = o(x) \end{cases} \tag{3.4}$$

for $x \to 0$, $v - v^* \to 0$, $\mu \to 0$ [see (2.5)–(2.9)]. We will also use the expressions [see (2.21), (2.23)]

$$\begin{aligned} x &= e^{-\tau}\varepsilon + o(e^{-\tau}), \\ u &= o(e^{-\tau}), \end{aligned} \tag{3.5}$$

$$\tau = -\ln\frac{x}{\varepsilon} + o(1), \tag{3.6}$$

where τ is the flight time from RP to the exit point P in S^{out}. By (3.5), (3.6), we have $\tau \to \infty$ for $v \to v^*$, $\mu \to 0$.

The time τ orbit from RP to P remains inside the ε-neighborhood of O, whereas the orbit from P to RP lies outside. The flight time T from P to RP is bounded, uniformly for all small $(\mu, v - v^*)$, and it is close to the flight time of the homoclinic orbit Γ from M^* to RM^* at $\mu = 0$.

The Floquet matrix C in (3.1)–(3.3) correspondingly decomposes into the product of two matrices

$$C = AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}. \tag{3.7}$$

Here, A is the linearization at RP of the time τ flow map, and B is the linearization at P of the time T

flow map along Γ. The matrices A and B in (3.7) depend on v, μ. Below, we compute and estimate elements of these matrices.

We now give an outline of our algebraic approach to the ellipticity region. Below, we estimate elements of A using relations between the time τ flow map and solutions of the Shil'nikov problem [see (3.21), (3.22)]. The estimates obtained will allow us to rewrite the trace Eq. (3.1) in the form

$$b_{44} + (b_{43}\chi_1 + b_{34}\chi_2 + b_{33}\chi_3) = o(e^{-\tau})$$

for any given finite s [see (3.25), (3.18)]. Here, s is absorbed into the $o(e^{-\tau})$ term which is, in fact, a function of (s, v, μ). The functions χ_1, χ_2, and χ_3 of (v, μ) tend to zero, along with their derivatives, for $\tau(v, \mu) \to \infty$.

Next, using reversibility arguments and assumption (G) of the theorem, we obtain expansions (3.32)–(3.35) for b_{jk} which allow us to rewrite the trace equation in the form

$$\alpha(x, u, v, \mu) = o(e^{-\tau})\,.$$

Here, (x, u, v) are the coordinates of the point P where the principal periodic orbit intersects S^{out} [see (3.4), (3.5)], and τ is the flight time from RP to P [see (3.6)]. In (3.42), the function α is expanded as

$$\begin{aligned}\alpha(x, u, v, \mu) = K_1(v - v^*) + K_2\mu + K_3x + K_4u \\ + o(|v - v^*| + |\mu| + |x| + |u|)\,,\end{aligned}$$

where $K_1 \neq 0$. This will follow because the tangency is quadratic.

For $\mu \to 0$, estimates (3.4), (3.5), (3.6) finally reduce the trace equations to

$$v - v^* = -\frac{K_2 + K_3c}{K_1}\mu + o(\mu)\,,\ c\mu > 0$$

[see (3.46)] for any given finite s. As before, s enters only into the term $o(\mu)$. Inserting the values $s = \pm 1$ yields the curves $v = v^{\pm}(\mu)$ which separate elliptic and saddle regions in the bifurcation diagram of Fig. 9. Note that the tangents of these curves at $\mu = 0$, $v = v^*$ coincide, in agreement with Theorem 2. This completes our outline of the proof.

Now we give the details. We begin with evaluation of the matrix A, the linearization at RP of the time τ flow map $(x_0, y_0, u_0, v_0) \mapsto (x_\tau, y_\tau, u_\tau, v_\tau)$. Recall that this map is determined by the solution (2.11) of the Shil'nikov problem. To linearize the map we differentiate (2.11) at fixed τ:

$$\begin{aligned}\begin{pmatrix} dx_\tau \\ du_\tau \end{pmatrix} &= \begin{pmatrix} X_x & X_u \\ U_x & U_u \end{pmatrix}\begin{pmatrix} dx_0 \\ du_0 \end{pmatrix} + \begin{pmatrix} X_y & X_v \\ U_y & U_v \end{pmatrix}\begin{pmatrix} dy_\tau \\ dv_\tau \end{pmatrix},\\ \begin{pmatrix} dy_0 \\ dv_0 \end{pmatrix} &= \begin{pmatrix} Y_x & Y_u \\ V_x & V_u \end{pmatrix}\begin{pmatrix} dx_0 \\ du_0 \end{pmatrix} + \begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}\begin{pmatrix} dy_\tau \\ dv_\tau \end{pmatrix},\end{aligned} \tag{3.8}$$

with lower subscripts x, y, u, v indicating partial derivatives. Since we are interested in the linearization at RP, the derivatives should be evaluated at $x_0 = \varepsilon$, $y_\tau = \varepsilon$, $u_0 = v_\tau = v$-coordinate of P, and at τ given by (3.6) where $x = x_\tau$ in (3.6) is given by (3.4).

By definition of A, on the other hand,

$$\begin{aligned}\begin{pmatrix} dx_\tau \\ du_\tau \end{pmatrix} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} dx_0 \\ du_0 \end{pmatrix} + \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}\begin{pmatrix} dy_0 \\ dv_0 \end{pmatrix},\\ \begin{pmatrix} dy_\tau \\ dv_\tau \end{pmatrix} &= \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}\begin{pmatrix} dx_0 \\ du_0 \end{pmatrix} + \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}\begin{pmatrix} dy_0 \\ dv_0 \end{pmatrix}.\end{aligned} \tag{3.9}$$

Comparing (3.8) and (3.9) we have

$$\begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.10}$$

$$\begin{pmatrix} Y_x & Y_u \\ V_x & V_u \end{pmatrix} + \begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = 0\,, \tag{3.11}$$

$$\begin{pmatrix} X_y & X_v \\ U_y & U_v \end{pmatrix}\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}, \tag{3.12}$$

$$\begin{pmatrix} X_x & X_u \\ U_x & U_u \end{pmatrix} + \begin{pmatrix} X_y & X_v \\ U_y & U_v \end{pmatrix}\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{3.13}$$

By (3.10), the matrix $\begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}$ is invertible, that is,

$$\det\begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix} \neq 0\,. \tag{3.14}$$

From (3.10)–(3.13) we easily obtain A:

$$\begin{aligned}
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \begin{pmatrix} X_x & X_u \\ U_x & U_u \end{pmatrix} - \begin{pmatrix} X_y & X_v \\ U_y & U_v \end{pmatrix} \\
&\quad \times \begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}^{-1} \begin{pmatrix} Y_x & Y_u \\ V_x & V_u \end{pmatrix}, \\
\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} &= \begin{pmatrix} X_y & X_v \\ U_y & U_v \end{pmatrix} \begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}^{-1}, \\
\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} &= -\begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}^{-1} \begin{pmatrix} Y_x & Y_u \\ V_x & V_u \end{pmatrix}, \\
\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} &= \begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}^{-1}.
\end{aligned} \tag{3.15}$$

In this formula

$$\begin{aligned} X_x &= e^{-\tau} + o(e^{-\tau}) > 0, \\ Y_y &= e^{-\tau} + o(e^{-\tau}) > 0, \end{aligned} \tag{3.16}$$

and all the other derivatives are of order $o(e^{-\tau})$, by estimates (2.18) for the solution of the Shil'nikov problem. Recall that τ is the function of v and μ given by (3.6), (3.4) with $x = x_\tau$.

Since $Y_y \neq 0$, we can write

$$\det \begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix} = Y_y \Delta, \tag{3.17}$$

where

$$\Delta = V_v - V_y Y_y^{-1} Y_v = o(e^{-\tau}). \tag{3.18}$$

By (3.14), (3.17)

$$\Delta \neq 0. \tag{3.19}$$

One can easily verify that

$$\begin{aligned} \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} &\equiv \begin{pmatrix} Y_y & Y_v \\ V_y & V_v \end{pmatrix}^{-1} \\ &= Y_y^{-1}\Delta^{-1} \begin{pmatrix} V_v & -Y_v \\ -V_y & Y_y \end{pmatrix}. \end{aligned} \tag{3.20}$$

Since $Y_v Y_y^{-1} = o(1)$, $V_y Y_y^{-1} = o(1)$, $V_v Y_y^{-1} = o(1)$, for $\tau \to \infty$, we have

$$\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Delta^{-1} \end{pmatrix} + o(\Delta^{-1}). \tag{3.21}$$

Reinserting (3.21) into (3.15), all other elements of $A = (a_{jk})$ are of order

$$e^{-\tau} o(\Delta^{-1}), \tag{3.22}$$

for $\tau \to \infty$.

Using these estimates and

$$\begin{aligned} \operatorname{tr} AB = \operatorname{tr}&\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right] \\ &+ \operatorname{tr}\left[\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}\begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}\right] \\ &+ \operatorname{tr}\left[\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}\begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix}\right] \\ &+ \operatorname{tr}\left[\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}\begin{pmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{pmatrix}\right] \end{aligned} \tag{3.23}$$

we obtain the following estimate for the trace of the Floquet matrix:

$$\begin{aligned} \operatorname{tr} C = \operatorname{tr} AB = \Delta^{-1}(&b_{44} + b_{43}\chi_1 + b_{34}\chi_2 \\ &+ b_{33}\chi_3 + o(e^{-\tau})). \end{aligned} \tag{3.24}$$

Here, χ_1, χ_2, and χ_3 are functions of (v, μ) which tend to zero along with their derivatives for $\tau(v, \mu) \to \infty$. The values b_{jk} are uniformly bounded for v close to v^* and μ close to zero, by linearization of a time T map with uniformly bounded T.

Inserting (3.24) into (3.1), the nontrivial Floquet multiplier s of a principal periodic orbit satisfies

$$\begin{aligned} b_{44} + (&b_{43}\chi_1 + b_{34}\chi_2 + b_{33}\chi_3) \\ &= \Delta\left(s + \frac{1}{s} + 2\right) + o(e^{-\tau}), \end{aligned} \tag{3.25}$$

with χ_1, χ_2, χ_3 of order $o(1)$. The coefficients b_{jk} can be calculated as

$$\begin{aligned} b_{44} &= (e_v, Be_v), \quad b_{43} = (e_v, Be_y), \\ b_{34} &= (e_y, Be_v), \quad b_{33} = (e_y, Be_y), \end{aligned} \tag{3.26}$$

where (e_x, e_y, e_u, e_v) are unit vectors along the corresponding coordinate axes at the points P and RP; $(\cdot, \cdot)$ denotes the standard scalar product. We compute asymptotic expansions for these $b_{j,k}$ next.

Let $B_{1/2}$ be the linearization of the time $T/2$ flow map at the point P. This flow map moves the point P to some point Q on $\mathrm{Fix}(R)$ near $M = \Gamma \cap \mathrm{Fix}(R)$. By reversibility,

$$B = RB_{1/2}^{-1} RB_{1/2} \tag{3.27}$$

(see Fiedler & Heinze [1996a, 1996b]). To facilitate the calculation of $B_{1/2}$, we introduce suitable coordinates $(\xi_1, \xi_2, \eta, \zeta)$ near M such that the vector

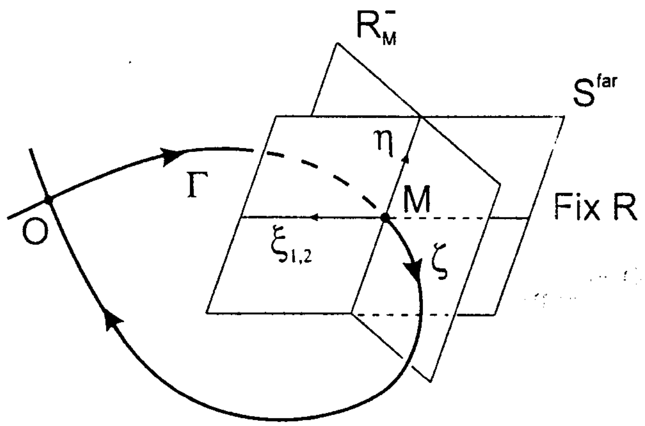

Fig. 18. The coordinates $(\xi_1, \xi_2, \eta, \zeta)$ near M are introduced so that the vector field is parallel to the ζ-axis near M, the cross-section S^{far} is given by $\zeta = 0$, the plane Fix(R) near M is $\{\eta = 0, \zeta = 0\}$, and the plane $R_M^- = M + \text{Fix}(-R)$ is $\{\xi_1 = 0, \xi_2 = 0\}$.

field is parallel to the ζ-axis near M, the cross-section S^{far} is given by $\zeta = 0$, the plane Fix(R) near M is $\{\eta = 0, \zeta = 0\}$, the plane R_M^- is $\{\xi_1 = 0, \xi_2 = 0\}$; see Fig. 18. We recall that $R_M^- = M + \text{Fix}(-R)$, by definition, is the two-dimensional subspace through M where R acts as $-id$. In these coordinates, the involution R acts as

$$(\xi_1, \xi_2, \eta, \zeta) \to (\xi_1, \xi_2, -\eta, -\zeta). \tag{3.28}$$

Time can be scaled so that the cross-section $S^{\text{out}} = \{y = \varepsilon\}$ is mapped to $S^{\text{far}} = \{\zeta = 0\}$ by a Poincaré time identically $T/2$ and, thus, the time $T/2$ map restricted to S^{out} coincides with the Poincaré map Π_{out}. Since the vector e_v lies in S^{out}, the image $B_{1/2}e_v$ lies in S^{far}. Therefore, we can write

$$B_{1/2}e_v = \alpha e_\eta + \beta_1 e_{\xi_1} + \beta_2 e_{\xi_2}, \tag{3.29}$$

where α and $\beta_{1,2}$ are functions of μ and of the coordinates (x, u, v) of P. Here, the vectors $(e_{\xi_1}, e_{\xi_2}, e_\eta, e_\zeta)$ are unit vectors along the corresponding coordinate axes attached to Q. Below, we estimate α. By (3.28), (3.29)

$$RB_{1/2}e_v = B_{1/2}e_v - 2\alpha e_\eta \tag{3.30}$$

and

$$B_{1/2}^{-1}RB_{1/2}e_v = e_v - 2\alpha B_{1/2}^{-1}e_\eta, \tag{3.31}$$

Now we get by (3.26), (3.27)

$$\begin{aligned} b_{44} &= (e_v, Be_v) \\ &= (Re_v, B_{1/2}^{-1}RB_{1/2}e_v) \\ &= (e_u, e_v - 2\alpha B_{1/2}^{-1}e_\eta) \\ &= -2\alpha(e_u, B_{1/2}^{-1}e_\eta). \end{aligned} \tag{3.32}$$

Analogously,

$$b_{34} = -2\alpha(e_x, B_{1/2}^{-1}e_\eta). \tag{3.33}$$

Note that the factor $(e_u, B_{1/2}^{-1}e_\eta)$ in the right-hand side of (3.32) is nonzero. Indeed, otherwise the vector $B_{1/2}^{-1}e_\eta$, tangent to the line $\Pi_{\text{out}}^{-1}(R_M^- \cap S^{\text{far}})$, would belong to the hyperplane $\{u = 0\}$ tangent to W_{loc}^{ue}, contradicting the transversality of W^{ue} and R_M^-.

The elements b_{43} and b_{33} can be estimated as follows:

$$b_{43} = -\frac{\dot v}{\dot y}b_{44} + O(e^{-\tau}), \tag{3.34}$$

$$b_{33} = -\frac{\dot v}{\dot y}b_{34} + O(e^{-\tau}). \tag{3.35}$$

Indeed, the linearization of the flow map moves the time derivative at P onto the time derivative at Q:

$$B_{1/2}(\dot x e_x + \dot y e_y + \dot u e_u + \dot v e_v) = e_\zeta \tag{3.36}$$

where $(\dot x, \dot y, \dot u, \dot v)$ at the point P are given in Ovsyannikov–Shil'nikov form by (2.14) with (x, u) as in (3.4), (3.5) and with $y = \varepsilon$. Note that $\dot x$ and $\dot u$ are of order $O(e^{-\tau})$ and $\dot y$ does not vanish. By (3.28), (3.36)

$$\begin{aligned} &B_{1/2}^{-1}RB_{1/2}(\dot x e_x + \dot y e_y + \dot u e_u + \dot v e_v) \\ &\quad = B_{1/2}^{-1}Re_\zeta \\ &\quad = -B_{1/2}^{-1}e_\zeta - (\dot x e_x + \dot y e_y + \dot u e_u + \dot v e_v). \end{aligned} \tag{3.37}$$

Evaluating the component of $e_u = Re_v$ yields

$$\begin{aligned} &\dot y(e_u, B_{1/2}^{-1}RB_{1/2}e_y) + \dot v(e_u, B_{1/2}^{-1}RB_{1/2}e_v) \\ &\quad + \dot x(e_u, B_{1/2}^{-1}RB_{1/2}e_x) \\ &\quad + \dot u(e_u, B_{1/2}^{-1}RB_{1/2}e_u) = -\dot u. \end{aligned} \tag{3.38}$$

Since $\dot x$, $\dot u$ are of order $O(e^{-\tau})$ and $\dot y \neq 0$, we obtain

$$\begin{aligned} (e_u, B_{1/2}^{-1}RB_{1/2}e_y) &= -\frac{\dot v}{\dot y}(e_u, B_{1/2}^{-1}RB_{1/2}e_v) \\ &\quad + O(e^{-\tau}). \end{aligned} \tag{3.39}$$

Comparing with (3.26), (3.27) we obtain the claimed estimate (3.34). Analogously, estimate (3.35) follows from (3.37) by evaluation of the component of $e_x = Re_y$.

We can now substitute expansions (3.32)–(3.35) for b_{jk} into condition (3.25) for the presence of a nontrivial multiplier s. Then (3.25) takes the form

$$\alpha(x,\, u,\, v,\, \mu) = o(e^{-\tau}) \tag{3.40}$$

for any given finite s. Here, (x, u, v) are the coordinates of the point P where the principal periodic orbit intersects S^{out} [see (3.4), (3.5)], the value τ is the flight time from RP to P [see (3.6)], and the function α is the η-component of the vector $B_{1/2}e_v$ [see (3.29)]. Note that s is absorbed into the $o(e^{-\tau})$ term, with s entering only via

$$\Delta(s + 1/s)\,,$$

and $\Delta = o(e^{-\tau})$ by (3.18).

We claim that

$$\begin{aligned} &\alpha(x = 0,\, u = 0,\, v = v^*,\, \mu = 0) = 0\,, \\ &\frac{\partial \alpha}{\partial v}(x = 0,\, u = 0,\, v = v^*,\, \mu = 0) \neq 0\,. \end{aligned} \tag{3.41}$$

Indeed, the vector $B_{1/2}e_v$ becomes tangent to $\Pi_{\text{out}}(W^u_{\text{loc}} \cap S^{\text{out}})$ for P tending to M^*, that is, for $\mu \to 0$ and $v \to v^*$. Since $W^u \cap S^{\text{far}}$ is quadratically tangent to the $\{\eta = 0\}$ plane, for $\mu = 0$, this implies (3.41).

By (3.41) we can now expand

$$\begin{aligned} &\alpha(x,\, u,\, v,\, \mu) \\ &\quad = K_1(v - v^*) + K_2\mu + K_3 x + K_4 u \\ &\qquad + o(|v - v^*| + |\mu| + |x| + |u|) \end{aligned} \tag{3.42}$$

where $K_1 \neq 0$. Estimates (3.5) for x and u allow us to rewrite the expression for α in the form

$$\begin{aligned} \alpha(x,\, u,\, v,\, \mu) &= K_1(v - v^*) + K_2\mu + K_3\varepsilon e^{-\tau} \\ &\quad + o(|\mu| + |v - v^*| + e^{-\tau})\,. \end{aligned} \tag{3.43}$$

Thus, Eq. (3.40) can be rewritten as

$$v - v^* = -\frac{K_2}{K_1}\mu - \frac{K_3}{K_1}e^{-\tau}\varepsilon + o(e^{-\tau} + |\mu|) \tag{3.44}$$

Substituting (3.44) into expansions (3.4), (3.5) we obtain

$$x \sim e^{-\tau}\varepsilon \sim c\mu \tag{3.45}$$

for $\mu \to 0$.

Finally, we have from (3.44) and (3.45) that, for $\mu \to 0$, the condition of the presence of the nontrivial Floquet multiplier s reduces asymptotically to

$$v - v^* = -\frac{K_2 + K_3 c}{K_1}\mu + o(\mu),\ c\mu > 0 \tag{3.46}$$

for any given finite s. As before, s enters only into the term $o(\mu)$. Inserting the values $s = \pm 1$ yields the curves $v = v^{\pm}(\mu)$ which separate elliptic and saddle regions in the bifurcation diagram of Fig. 9. By (3.46), these curves are tangent at $\mu = 0$, $v = v^*$ in agreement with Theorem 2.

Note that the curves given by (3.46) do not intersect each other for different s, since the unique principal periodic orbit has only one pair of nontrivial Floquet multiplier s. Moreover, for any finite s and for any small μ, there is only one v satisfying (3.46). Thus, the real part, $\operatorname{Re} s$, depends monotonically on v for each fixed μ. This implies that $|\operatorname{Re} s| < 1$, for $v \in (v^-(\mu),\, v^+(\mu))$, and $|\operatorname{Re} s| > 1$ for $v \notin (v^-(\mu),\, v^+(\mu))$. Therefore, the region $v \in (v^-(\mu),\, v^+(\mu))$ corresponds to elliptic periodic orbits and the region $v \notin (v^-(\mu),\, v^+(\mu))$ corresponds to saddle periodic orbits. This completes the proof of Theorem 2.

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