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Theme Issue (Part 1)

ON DIMENSION OF NON-LOCAL BIFURCATIONAL PROBLEMS

D. TURAEV
Department of Differential Equations,
Institute for Applied Mathematics and Cybernetics,
10 Ul'janova St., Nizhny Novgorod 603005, Russia

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An analogue of the center manifold theory is proposed for non-local bifurcations of homo- and heteroclinic contours. In contrast with the local bifurcation theory it is shown that the dimension of non-local bifurcational problems is determined by the three different integers: the geometrical dimension d_g which is equal to the dimension of a non-local analogue of the center manifold, the critical dimension d_c which is equal to the difference between the dimension of phase space and the sum of dimensions of leaves of associated strong-stable and strong-unstable foliations, and the Lyapunov dimension d_L which is equal to the maximal possible number of zero Lyapunov exponents for the orbits arising at the bifurcation. For a wide class of bifurcational problems (the so-called semi-local bifurcations) these three values are shown to be effectively computed. For the orbits arising at the bifurcations, effective restrictions for the maximal and minimal numbers of positive and negative Lyapunov exponents (correspondingly, for the maximal and minimal possible dimensions of the stable and unstable manifolds) are obtained, involving the values d_c and d_L. A connection with the problem of hyperchaos is discussed.

1. Introduction

It is well known that a large number of models of chaotic behavior in dynamical systems has been provided by the theory of global bifurcations. We could mention, for instance, a saddle-focus homoclinic loop (Fig. 1) under the Shil'nikov conditions, the transition from quasiperiodicity to chaos through an invariant torus breakdown, the homoclinic butterfly bifurcation (Fig. 2) leading to the Lorenz attractor, etc.

The investigation of global bifurcations of multi-dimensional systems was pioneered by Shil'nikov in the 1960s. It began with simple bifurcations of a homoclinic loop of a saddle equilibrium state (Fig. 3) and of a homoclinic loop of a saddle-node (Fig. 4); for both cases a single stable periodic orbit was proved to be generated [Shil'nikov, 1962, 1963]. More complex situations where a *saddle* periodic orbit is born were considered in Shil'nikov [1966, 1968]. Then, the complex structure (hyperbolic sets) in a neighborhood of a homoclinic loop of a saddle-focus [Shil'nikov, 1965, 1967a, 1970] and in a neighborhood of a homoclinic bunch of a saddle-saddle equilibrium state (Fig. 5) [Shil'nikov, 1969] were discovered and investigated [Shil'nikov, 1967b]. The hyperbolic structure in a neighborhood of *a structurally stable Poincaré homoclinic orbit*[1] and the hyperbolic subsets and bifurcations near *a structurally unstable Poincaré*

[1]This is an orbit of transverse intersection of the stable and unstable invariant manifolds of a saddle periodic orbit (Fig. 6); if the manifolds are tangent at the points of some orbit, such a homoclinic orbit is structurally unstable (Fig. 7).

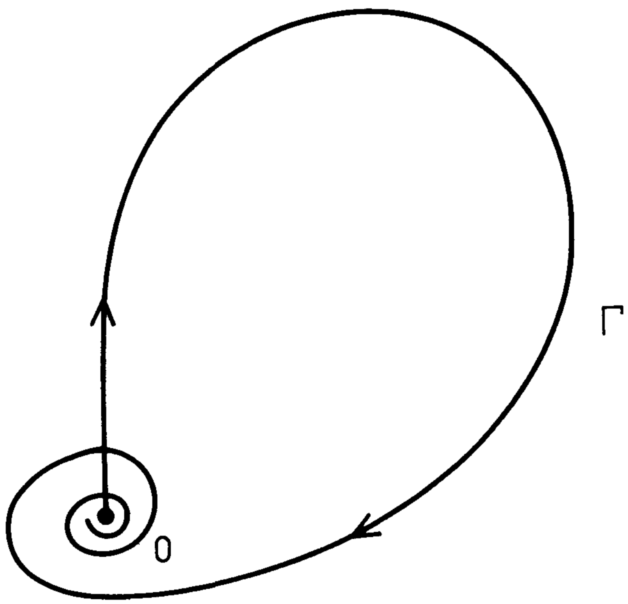

Fig. 1. A homoclinic loop Γ of a saddle-focus O.

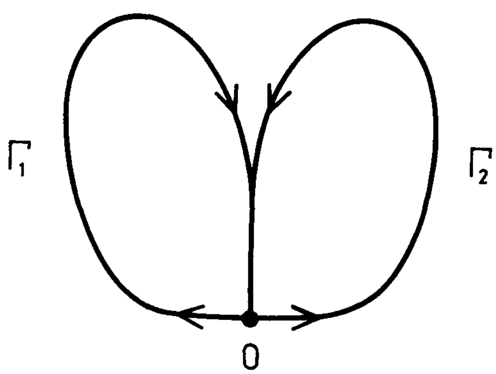

Fig. 2. A homoclinic butterfly composed of two loops Γ_1 and Γ_2 to a saddle O.

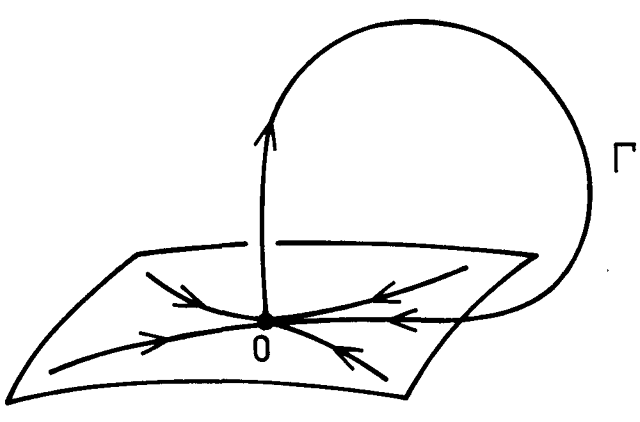

Fig. 3. A homoclinic loop Γ of a saddle O.

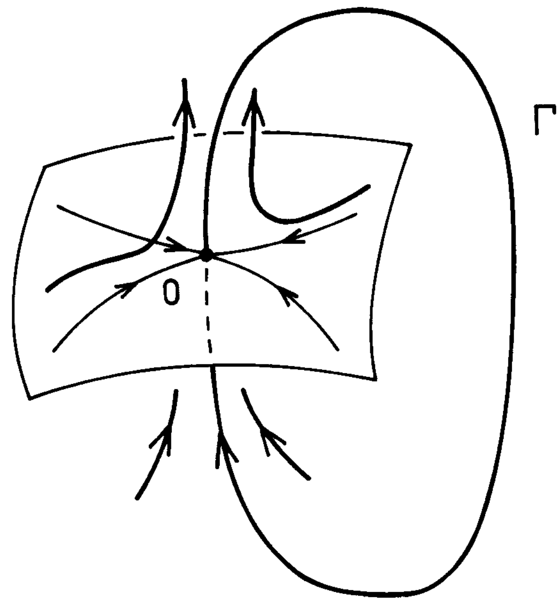

Fig. 4. A homoclinic loop Γ of a saddle-node O.

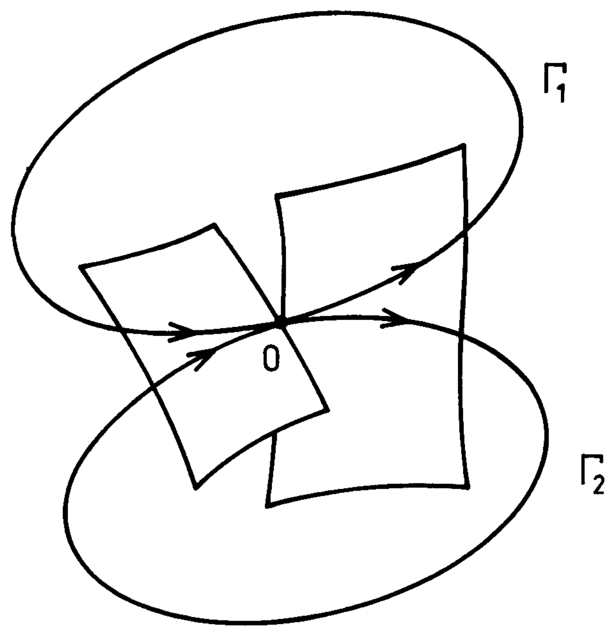

Fig. 5. A pair of homoclinic loops Γ_1 and Γ_2 of a saddle-saddle equilibrium state O.

homoclinic orbit (the latter with Gavrilov [Gavrilov & Shil'nikov, 1972]) had also been studied.

These studies were continued in the 1970s by Shil'nikov and coworkers as well as by other researchers (see, partly, references in Afraimovich *et al.* [1989]) and by now the theory has been intensively developed.

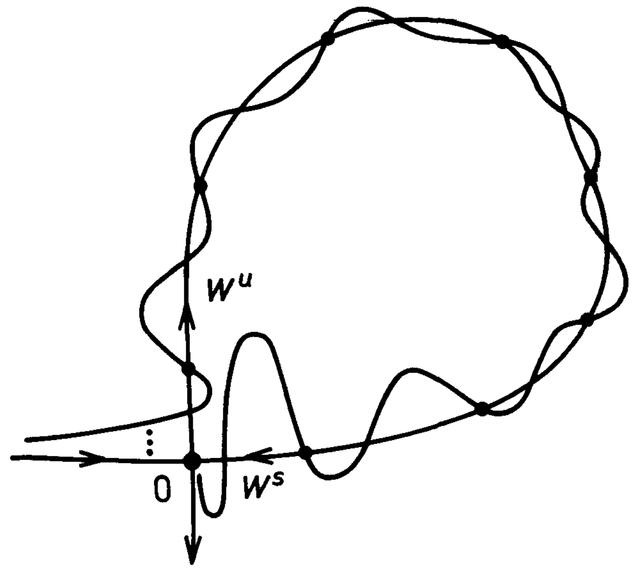

Fig. 6. An example of a homoclinic structure: the unstable and stable manifolds W^u and W^s of a saddle fixed point O of a diffeomorphism of a plane have a transverse intersection at the points of a structurally stable homoclinic orbit (boldly printed points in the figure).

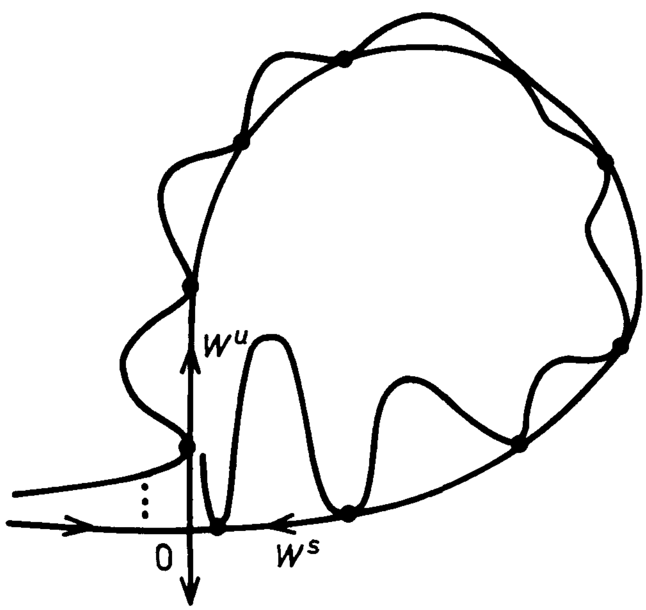

Fig. 7. A structurally unstable Poincaré homoclinic orbit at the points of which the unstable and stable manifolds W^u and W^s of a saddle fixed point O of a diffeomorphism of a plane have a tangency (boldly printed points in the figure).

The substantial part of the theory of global bifurcations relates to the study of *homo-* and *heteroclinic contours*. Recall that a contour is a union of a finite number of periodic orbits and (or) equilibrium states and a finite number of orbits asymptotic to them. We will denote such a contour as C, the periodic orbits and equilibrium states belonging to C will be denoted as $L_1, L_2, \ldots$, and the orbits that are asymptotic to the orbits L_i will be denoted as $\Gamma_1, \Gamma_2, \ldots$:

$$C = L_1 \cup L_2 \cup \cdots \cup \Gamma_1 \cup \Gamma_2 \cup \cdots$$

Each orbit Γ_s is an intersection of the unstable $W^u(L_i)$ and stable $W^s(L_j)$ manifolds of some orbits L_i and L_j. If $i = j$, then Γ_s is called *homoclinic*, and it is called *heteroclinic* if $i \neq j$.

The examples enumerated above and shown on Figs. 1–7 are homoclinic contours with one equilibrium state or one periodic orbit. Figures 8 and 9 show heteroclinic contours with two equilibria which were studied by Bykov [1978, 1980, 1993] and Chow *et al.* [1990] and Shashkov [1991a, 1992].

In the present paper, we intend to consider the most general properties of dynamical systems possessing a contour of one of these or, maybe, other types. Specifically, we find effective restrictions for possible dimensions of stable and unstable manifolds (in other words, for the possible numbers of positive and negative Lyapunov exponents) for any orbit which may lie in a small neighborhood of the contour or which may be born under a small perturbation of the system.

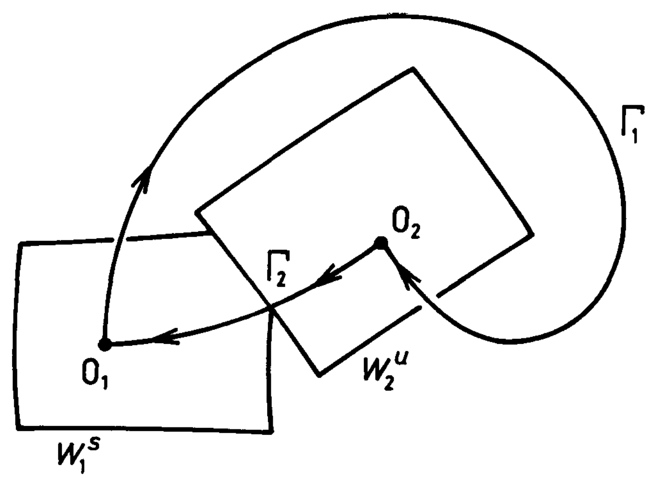

Fig. 8. A three-dimensional example of a heteroclinic contour containing two saddle equilibrium states: the orbit Γ_2 is the orbit of transverse intersection of the two-dimensional unstable manifolds W_2^u of the point O_2 and the two-dimensional stable manifold W_1^s of the point O_1; the orbit Γ_1 is a common orbit for the one-dimensional unstable manifold of O_1 and the one-dimensional stable manifold of O_2.

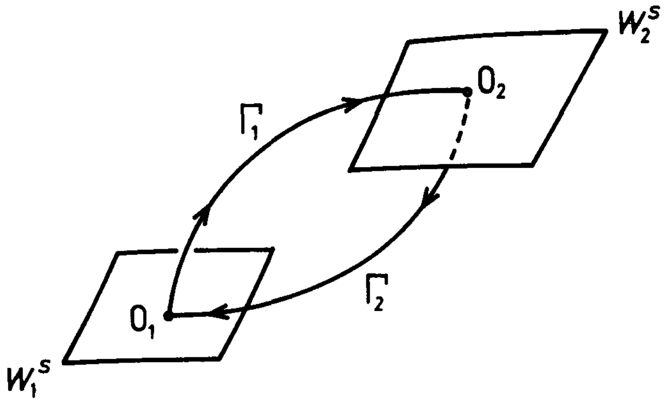

Fig. 9. Another three-dimensional example of a heteroclinic contour with two saddle equilibrium states: the one-dimensional unstable manifold of O_1 intersects the two-dimensional stable manifold of O_2 and the one-dimensional unstable manifold of O_2 intersects the two-dimensional stable manifold of O_1.

Such restrictions are well-known in the local bifurcation theory studying bifurcations in a neighborhood of a single structurally unstable equilibrium state or periodic orbit. If some dynamical systems have such an orbit with k characteristic exponents on the imaginary axis, n characteristic exponents to the right and m characteristic exponents to the left of the imaginary axis, then the following "center manifold theorem" holds, going back to Kelley [1967] and Pliss [1964] and then to Hirsch *et al.* [1977] and to Shoshitaishvili [1975]:

For any nearby system, in a small neighborhood of the equilibrium state (respectively, of the periodic orbit) there exists a locally invariant k-dimensional smooth center manifold such that any orbit, staying in the neighborhood for all times, belongs to the center manifold.

The center manifold is an intersection of two smooth invariant manifolds: the center-stable $(m + k)$-dimensional manifold that contains all orbits not leaving the neighborhood for all positive times and the center-unstable $(n + k)$-dimensional manifold that contains all orbits staying in the neighborhood for all negative times.

On the center-stable (center-unstable) manifold there exists a strong-stable (respectively, strong-unstable) invariant foliation with m-dimensional (respectively, n-dimensional) leaves transverse to the center manifold. There is a contraction (exponential) along the strong-stable leaves and an expansion along the strong-unstable leaves and the contraction and expansion are stronger than those which may take place along the directions tangential to the center manifold (Fig. 10).

Hence, the relevant dynamics is preserved if one restricts the system onto the center manifold. Note that the dimension of the reduced system equals here to the number k of the characteristic exponents on the imaginary axis and it does not correlate with the dimension of the initial system. The last [equal to $(m+n+k)$] can, in principle, be arbitrarily high and even infinite in some cases (see, for instance, Marsden & McCracen [1976] where a center manifold theorem is given for some classes of PDEs). Thus, the restriction onto the center manifold gives one an essential reduction of the dimension of the problem: exactly from $(m + n + k)$ to k.

We point out another consequence of the center manifold theorem (particularly, of the presence of the strong-stable and strong-unstable foliations):

For any orbit $\mathcal{L}$ lying entirely in a small neighborhood of the bifurcating equilibrium state (the periodic orbit), the dimensions of its stable and unstable manifolds are restricted by the following inequalities:

$$\begin{aligned} n \le \dim W^u(\mathcal{L}) \le n + k \\ m \le \dim W^s(\mathcal{L}) \le m + n \end{aligned} \tag{1}$$

In other words, the orbit $\mathcal{L}$ cannot have more than $(n + k)$ positive or $(m + k)$ negative Lyapunov exponents and less than n positive and m negative Lyapunov exponents.

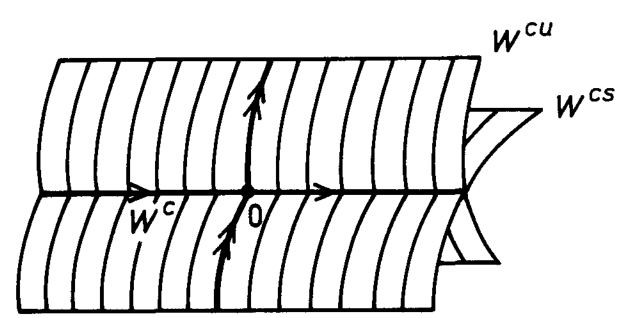

Fig. 10. An illustration to the center manifold theorem: O is a structurally unstable equilibrium state, W^{cu} and W^{cs} are the center-unstable and center-stable manifolds, the intersection $W^{cu} \cap W^{cs}$ is the center manifold W^c, on W^{cu} and W^{cs} there exist, respectively, strong-unstable and strong-stable foliations transverse to W^c.

The number k is called *the dimension of the bifurcational problem* and its determining is a standard preliminary step when studying any local bifurcation.

The scope of the present paper is to give a sketch of an analogous theory for the bifurcations of homo- and heteroclinic contours. Actually, we will consider a more general situation, namely, bifurcations in a small neighborhood of an arbitrary closed connected invariant set composed of a finite number of orbits. Such bifurcational problems are called *semi-local*, to emphasize that the set whose bifurcations are studied contains a *finite* number of orbits. We will also call any such set *a contour.*

Evidently, this setting includes the study of homo- and heteroclinic contours as a partial case. In fact, there are very few other semi-local bifurcations which have been studied to date. As an example, we mention the bifurcation considered in Homburg [1993]: a saddle equilibrium state with one-dimensional unstable manifold one separatrix which forms a homoclinic loop and the other has the loop as an ω-limit set (see Fig. 11). Such a bifurcation may take place on a two-dimensional torus (Fig. 12), producing a Cherry flow [Cherry, 1937], and it may also accompany the homoclinic butterfly (Fig. 2) bifurcation [Turaev & Shil'nikov, 1986; Gambaudo *et al.*, 1988]. A more sophisticated example (an orbit homoclinic to a bunch consisting of four homoclinic loops) can be found in Eleonsky *et al.* [1989].

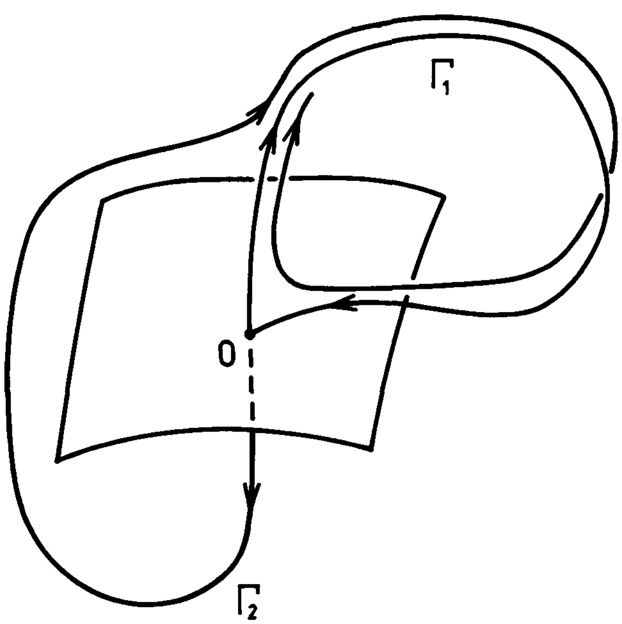

Fig. 11. An example of a second-level contour: one separatrix Γ_1 of a saddle equilibrium state O_1 forms a homoclinic loop and the other separatrix Γ_2 tends to the loop as $t \to +\infty$.

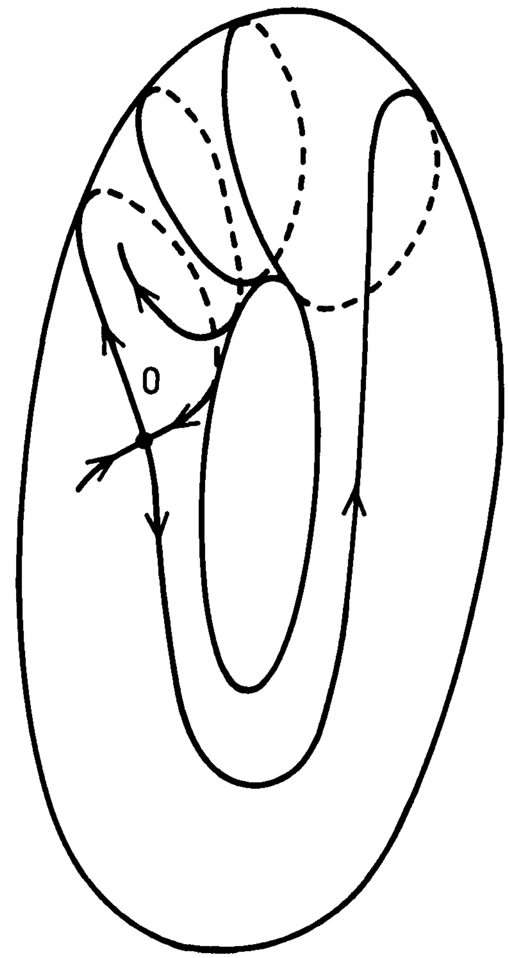

Fig. 12. The second-level contour on a two-dimensional torus.

It can be easily shown that the structure of an arbitrary closed invariant set C composed of a finite number of orbits is as follows. First, the set C cannot contain non-trivial recurrent orbits (otherwise, it would be infinite: the closure of a non-trivial recurrent orbit is well known to contain an infinite number of other recurrent orbits). Therefore, the only recurrent orbits in C are periodic orbits and equilibrium states in a finite number. We will call them *orbits of the zero level.* Next, there may be a finite number of orbits asymptotic to the orbits of the zero level as $t \to \pm\infty$. They will be called *orbits of the first level.* The orbits whose α- and ω-limit sets belong to the union of the orbits of zero and first levels are called *orbits of the second level,* and so on.

Each set C contains a finite number of levels. In these terms, homo- and heteroclinic contours, examples of which are shown in Figs. 1–9, are first-level contours; the contours in Figs. 11 and 12 are second-level contours. Single periodic orbits and equilibrium states (bifurcations of which are studied by the local theory) could be considered as zero-level contours.

In contrast with the case of local bifurcations, we will show that the dimension of semi-local bifurcational problems is determined by three different integers. One of them which we call *the geometrical dimension* d_g, is equal to the dimension of a non-local analogue of the center manifold. The second, *the critical dimension* d_c, is connected with the strong-stable and strong-unstable foliations: it is equal to the difference between the dimension of the phase space and the sum of dimensions of strong-stable and strong-unstable leaves. The third integer, *the Lyapunov dimension* d_L, relates to the estimates [similar to estimates (1)] on possible dimensions of stable and unstable manifolds of the orbits that may be born at the bifurcation and it is equal to the maximal number of zero Lyapunov exponents possible for these orbits.

For the local bifurcational problems, these three definitions give the same number k. In the non-local case, the three dimensions may be different and the following relation holds:

$$d_g \geq d_c \geq d_L \tag{2}$$

The main results of the paper are given by the three theorems in Sec. 3. Theorem 1 establishes that for any finite contour C, there are defined the numbers m, n and k ($m + n + k$ = the dimension of the phase space) such that:

> *For any orbit $\mathcal{L}$ of any nearby system, if the orbit lies entirely in a small neighborhood U of C, then the flow linearized along $\mathcal{L}$ admits an exponential trichotomy; i.e., there exists an m-dimensional strong-stable subspace for which the linearized flow is exponentially contracting, an n-dimensional strong-unstable subspace for which the linearized flow is expanding, and a k-dimensional center subspace for which contraction or expansion are weaker than those on the strong-stable and strong-unstable subspaces.*

We emphasize that this result is fulfilled for any orbit which lies near C or which is born at the bi-

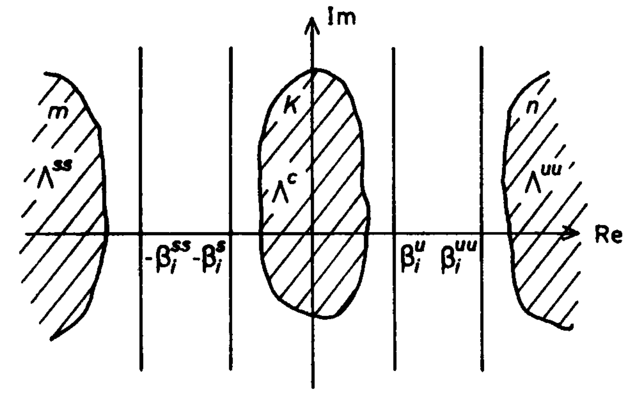

Fig. 13. The trichotomy property: there exists non-negative integers $k \geq 1$, m and n such that for each periodic orbit or equilibrium state L_i in the contour the spectrum of characteristic exponents of L_i is divided into the three parts; exactly k characteristic exponents belong to the center part Λ^c that lies in the strip $-\beta_i^s < \Re e\lambda < \beta_i^u$ on the complex plane, m characteristic exponents belong to the strong-stable part Λ^{ss} that lies to the left of the line $\Re e\lambda = -\beta_i^{ss}$ and n characteristic exponents belong to the strong-unstable part Λ^{uu} that lies to the right of the line $\Re e\lambda = \beta_i^{uu}$, where β_i^{uu}, β_i^u, β_i^{ss} and β_i^s are some positive values ($\beta_i^{uu} > \beta_i^u$, $\beta_i^{ss} > \beta_i^s$) which may be different for different orbits L_i.

furcation. If P is a point on the orbit $\mathcal{L}$, then $\mathcal{N}_P^{ss}$, $\mathcal{N}_P^{uu}$ and $\mathcal{N}_P^{c}$ denote, respectively, the strong-stable, strong-unstable and center subspaces at the point P. By Theorem 1, the system of these subspaces is invariant with respect to the linearized flow, and they depend continuously on the point P. Note also that $\mathcal{N}_P^{c}$ contains the vector of phase velocity at P.

If we take P on a recurrent orbit $L_i \subset C$ (a periodic orbit or an equilibrium state), then by Theorem 1 the spectrum of characteristic exponents[2] of L_i is decomposed into three parts: *center*, or *critical*, part Λ^c corresponding to the eigenspace $\mathcal{N}_P^{c}$, *right* (or *strong-unstable*) part Λ^{uu} corresponding to the eigenspace $\mathcal{N}_P^{uu}$, and *left* (or *strong-stable*) part Λ^{ss} corresponding to the eigenspace $\mathcal{N}_P^{ss}$. Schematically, the trichotomy property is reflected by the following relation (see Fig. 13)

$$\Re e\Lambda^{ss} < -\beta_i^s < \Re e\Lambda^c < \beta_i^u < \Re e\Lambda^{uu}$$

for some positive β_i^u and β_i^s. The separating values β_i can be different for different orbits L_i. It is important, nevertheless, that the numbers k, m, n of

[2]Roots of characteristic equation if L_i is an equilibrium state, logarithms of the Floquet multipliers (the eigenvalues of the linearized period map) if L_i is a periodic orbit.

characteristic exponents belonging to each part of the spectrum do not depend on L_i. Evidently, the presence of such a decomposition for each recurrent orbit L_i in C is a necessary condition for Theorem 1 (but it is not sufficient, see details in Secs. 3 and 4).

Evidently, Theorem 1 shows that, for any orbit staying in the neighborhood U for all times, *the number of negative Lyapunov exponents cannot be greater than* $(m+k)$ *and less than* m *and the number of positive Lyapunov exponents cannot be greater than* $(n+k)$ *and less than* n.

In particular, we have that for any periodic orbit $\mathcal{L}$ lying in U entirely

$$\begin{aligned} m+1 \le \dim W^s(\mathcal{L}) \le m+k \\ n+1 \le \dim W^u(\mathcal{L}) \le n+k \end{aligned} \tag{3}$$

(by the theorem, the orbit $\mathcal{L}$ has at least m multipliers inside the unit circle and at least n multipliers outside the unit circle).

Inequality (3) is analogous to inequality (1) following from the "center manifold theorem" in the case of local bifurcations. Note that the numbers k, m and n are not defined uniquely. We call the lowest possible value of k for which Theorem 1 remains valid *the critical dimension of the problem* and denote it as d_c (an algorithm of finding d_c is given in Sec. 4).

In contrast with what we have for local bifurcations, estimates (3) based on considerations involving the exponential trichotomy of the linearized flow are not final. Indeed, suppose the so-called sequential divergences are less than zero for all recurrent orbits L_i in the contour C; i.e., suppose that for each orbit L_i, the sum of its *critical* characteristic exponents is negative. This means that the flow linearized along L_i is volume contracting in restriction onto the spaces $\mathcal{N}^c$. Since any orbit of C (and, therefore, any orbit lying in a small neighborhood of C) spends most of the time in a small neighborhood of the union of the orbits L_i, the linearized flow applied to the space $\mathcal{N}^c$ is volume contracting along any orbit $\mathcal{L}$ lying near C. This implies that, in addition to m negative Lyapunov exponents corresponding to the invariant subspace $\mathcal{N}^{ss}$, the orbit $\mathcal{L}$ has at least one more negative Lyapunov exponent corresponding to the restriction of the linearized flow onto the space $\mathcal{N}^c$. In particular, if $\mathcal{L}$ is a periodic orbit, it has at least one additional Floquet multiplier less than unity in absolute value which corresponds to an eigen-direction belonging to $\mathcal{N}^c$. Thus, in this case (we call it *volume contracting*)

$$\begin{aligned} n+1 \le \dim W^u(\mathcal{L}) \le n+d_c-1 \\ m+2 \le \dim W^s(\mathcal{L}) \le m+d_c \end{aligned} \tag{4}$$

and these estimates are stronger than inequalities (3).

Analogously, if the sequential divergence is greater than zero for each recurrent orbit in C (*the volume expanding case*), then an additional restriction arises for the maximal possible dimension of the stable manifolds.

In Sec. 4 we define the Lyapunov dimension of the problem, d_L, such that, in the volume contracting case, the linearized flow applied to the space $\mathcal{N}^c$ contracts exponentially (d_L+1)-dimensional volumes along each recurrent orbit $L_i \subset C$, and in the volume expanding case the (d_L+1)-dimensional volumes are expanded. Using simple arguments, one can prove the following result (Theorem 2 of Sec. 4):

> *In the volume contracting case, any orbit $\mathcal{L}$ that stays in the neighborhood U for all times cannot have more than $(n+d_L)$ non-negative Lyapunov exponents; in the volume expanding case, it cannot have more than $(m+d_L)$ non-positive Lyapunov exponents.*

By this theorem, for any periodic orbit $\mathcal{L}$ in a small neighborhood of C, the following estimates hold in the volume contracting case:

$$\begin{aligned} n+1 \le \dim W^u(\mathcal{L}) \le n+d_L \\ m+1+(d_c-d_L) \le \dim W^s(\mathcal{L}) \le m+d_c \end{aligned} \tag{5}$$

Analogously, in the volume expanding case,

$$\begin{aligned} n+1+(d_c-d_L) \le \dim W^u(\mathcal{L}) \le n+d_c \\ m+1 \le \dim W^s(\mathcal{L}) \le m+d_L \end{aligned} \tag{6}$$

In the other cases (we call them *the case of divergence of indefinite sign*) we get no additional information in comparison with inequalities (3) which we have rewritten as

$$\begin{aligned} m+1 \le \dim W^s(\mathcal{L}) \le m+d_c \\ n+1 \le \dim W^u(\mathcal{L}) \le n+d_c \end{aligned} \tag{7}$$

In this case one can assign $d_L \equiv d_c$ and inequalities (7) would formally coincide with (5) and (6).

Inequalities (5)–(7) constitute the main result of the paper. They seem to be final; namely, we propose the following "realization conjecture":[3]

Conjecture 1. *Let all recurrent orbits L_i in the contour C be structurally stable. Then, by a small perturbation of the system, a periodic orbit can be born in a small neighborhood of C, having d_L zero characteristic exponents.*

The "perturbation" means variation of the vector field that defines the system; "structurally stable" means not having zero characteristic exponents for an equilibrium state and not having multipliers on the unit circle for a periodic orbit (except one trivial Floquet multiplier which is always equal to unity).

In all cases known to the author, this conjecture is valid. If not considered a trivial Floquet multiplier, the periodic orbit with d_L zero characteristic exponents has $(d_L - 1)$ multipliers on the unit circle. By a small perturbation, one can move all these multipliers inside or outside the unit circle and thereby, obtain a periodic orbit for which the dimension of the unstable manifold achieve the left or, respectively, right bound given by (5)–(7). This means that if the conjecture is valid, then estimates (5)–(7) cannot be improved.

For most of the bifurcational problems solved to date, the Lyapunov dimension is small (maximum 4 as, for instance, in Gonchenko *et al.* [1993b, 1995] where the multi-dimensional case of bifurcations of a structurally unstable Poincaré homoclinic orbit is studied). Nevertheless, examples of *codimension one*[4] contours for which the Lyapunov dimension is arbitrarily high can be easily constructed. In conclusion, we consider such an example (see Fig. 14) in detail because it exhibits rather non-trivial dynamical phenomena which could, presumably, model the general situation in high-dimensional dynamical systems.

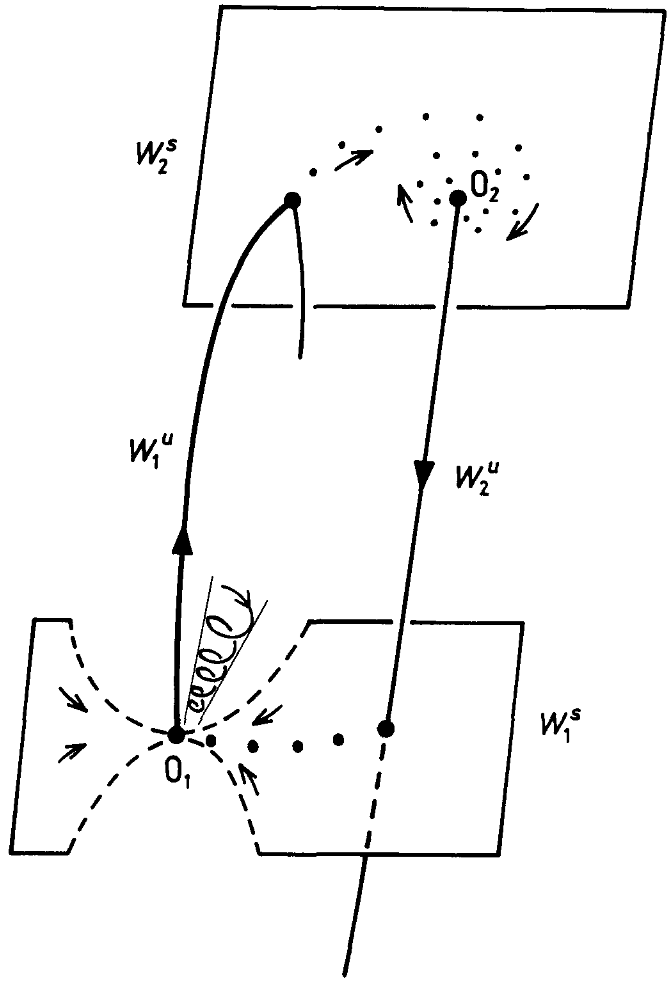

Fig. 14. An example of a contour which produces hyperchaos: O_1 and O_2 are the points of intersection of saddle periodic orbits L_1 and L_2 with some cross sections (O_1 and O_2 are the saddle fixed points of the Poincaré map); the point O_1 has only one multiplier (real) greater than unity and all other multipliers lie inside the unit circle and, among them, the multiplier nearest to the unit circle is real and the others have non-zero imaginary parts; the point O_2 has also only one real multiplier greater than unity and all other multipliers lie inside the unit circle, all of them have non-zero imaginary parts except the least multiplier which is real; the one-dimensional unstable manifold of O_1 have a tangency with the stable manifold of O_2 and the one-dimensional unstable manifold of O_2 intersects the stable manifold of O_1 transversely; the product of the multipliers of O_1 is less than unity and the product of the multipliers of O_2 is greater than unity. At the bifurcation of such a contour, infinitely many coexisting sinks, sources and saddles of all possible types can simultaneously be born.

For the contour of this example, the validity of the realization conjecture can be proved. This implies, in particular, that the study of the bifurcations of this contour requires at least $(d_L - 1)$

[3]Here, for more definiteness, we assume that the contour is *indecomposable*; i.e., for any two recurrent orbits $L_i \subset C$ and $L_j \subset C$ there exists a sequence of non-recurrent orbits $\Gamma_{s_1}, \Gamma_{s_2}, \ldots \subset C$ such that the α-limit set of the first orbit in this sequence contains L_i, the ω-limit set of the last orbit in this sequence contains L_j and, for each orbit in this sequence, its ω-limit set has a non-empty intersection with the α-limit set of the next orbit.

[4]That is, those which may occur in general one-parameter families of dynamical systems.

independent control parameters: it is the number of parameters required for the study of the bifurcations of the periodic orbits with $(d_L - 1)$ non-trivial multipliers on the unit circle, which may appear in a neighborhood of the contour according to the realization theorem. As mentioned, the examples can be constructed with arbitrarily high values of the Lyapunov dimension d_L. Therefore, the study of the bifurcations in a neighborhood of such a contour is a multi-parameter problem, the detailed analysis of which is, evidently, impossible, though the contours under consideration occur in general one-parameter families of dynamical systems. Note that this phenomenon does not take place in the theory of local bifurcations: the local bifurcational problems of high dimensions correspond to high degenerations of the linearized system at an equilibrium state or a periodic orbit and they cannot occur in general low-parameter families. For instance, an equilibrium state with s pairs of complex conjugate pure imaginary characteristic exponents (the dimension of the problem is $2s$) may occur in at least s-parameter families of dynamical systems.

As we have seen, the restrictions for the dimensions of the stable and unstable manifolds of orbits involved in semi-local bifurcations do not relate to the presence of non-local analogues of the center manifold. Nevertheless, the question on the existence of such global invariant manifolds is interesting itself. In a small neighborhood of a homoclinic loop of a saddle equilibrium state, the existence of a non-local center manifold was earlier established in Turaev [1984, 1991]; Homburg [1993] and Sandstede [1994] (the last reference gives a most general result for this case), for a kind of heteroclinic contours a similar result was proved in Shashkov [1991b], for systems with a homoclinic tangency the center manifold theorem was given in Gonchenko *et al.* [1993b, 1995]. The present paper solves this question for all finite contours. Namely, Theorem 3 in Sec. 4 gives necessary and sufficient conditions under which in a small neighborhood of the contour, there exists an invariant manifold which contains all orbits staying in the neighborhood for all times and which is tangent to the center subspace $\mathcal{N}^c$ at the points of the contour.

Note that the presence of the "center manifold" requires some additional conditions in comparison with Theorem 1; i.e., in contrast with the local bifurcation theory, the trichotomy of the linearized flow does not directly imply the existence of the center manifold: in different cases it may exist or it may not. As we have mentioned, the decomposition onto three subspaces is not defined uniquely. We call the lowest possible dimension k of the center subspace for which both Theorem 1 is valid and conditions of Theorem 3 are fulfilled (i.e., not only the tangent space is decomposed into the direct sum of $\mathcal{N}^{ss}$, $\mathcal{N}^{uu}$ and $\mathcal{N}^c$, but also the invariant manifold tangent to $\mathcal{N}^c$ exists) *the geometrical dimension of the problem* and denote it as d_g.

By definition $d_g \geq d_c$, and these quantities do not coincide in general. Notice another distinction with the case of local bifurcations: in the non-local case, the smoothness of the center manifold is not high. In fact, its smoothness does not correlate with the smoothness of the system and, in general, the center manifold is only C^1.

Therefore, when studying concrete non-local bifurcational problems, one cannot use the reduction on the center manifold directly: usually, subtle questions require calculations involving the derivatives of the orders higher than the first. If the geometrical dimension is rather high, especially, with respect to the critical and Lyapunov dimensions, the presence of the high-dimensional center manifold gives practically no additional information in comparison with the presence of the strong-stable and strong-unstable invariant foliations and with the estimates of the contraction or expansion of volumes. At the same time, if d_g is low ($d_g = 2, 3, 4$), the established presence of the d_g-dimensional invariant manifold, that captures all the orbits not leaving the neighborhood, can essentially simplify the understanding of the dynamics of the system, though the center manifold is only C^1-smooth. In this case, one can, at least, consider a model d_g-dimensional situation assuming the smoothness which is necessary, in order to work out conjectures that are, then, to be verified for the initial non-reduced system.

2. Local Structures and Transversality Conditions

Consider a C^r-smooth ($r \geq 1$) dynamical system X possessing a closed connected invariant set C composed of a finite number of orbits. There is a finite number of recurrent orbits (periodic orbits and equilibrium states) in C which we denote as $L_1, L_2, \ldots$. The non-recurrent orbits of C are denoted as $\Gamma_1, \Gamma_2, \ldots$. The orbits L_i are also called zero-level orbits. Those orbits Γ_s whose α- and ω-limit sets belong to the union of the zero-level orbits are the first-level orbits; those for which the α- and ω-limit sets belong to the union of the zero-level and first-level orbits are the second-level orbits and so on.

A common assumption for the three theorems giving the main results of the present paper is that for each recurrent orbit L_i, its characteristic exponents are grouped into three parts: *center* (or *critical*) part Λ^c, *right* (or *strong-unstable*) part Λ^{uu} and *left* (or *strong-stable*) part Λ^{ss}. Namely, we suppose that the following "trichotomy property" holds:

There exist non-negative integers $k \geq 1$, m, n ($k+m+n =$ the dimension of the phase space) such that for each orbit L_i, for some positive β_i^u and β_i^s, exactly k characteristic exponents λ lie in the strip

$$-\beta_i^s < \Re e\lambda < \beta_i^u$$

(*this is the center part of the spectrum*); *n characteristic exponents lie to the right of this strip*:

$$\Re e\lambda > \beta_i^u$$

and m characteristic exponents lie to the left of this strip:

$$\Re e\lambda < -\beta_i^s$$

Schematically, we can write (see Fig. 13)

$$\Re e\Lambda^{ss} < -\beta_i^s < \Re e\Lambda^c < \beta_i^u < \Re e\Lambda^{uu}$$

To be more accurate, we take into account the gap between the center part and the strong-stable and strong-unstable parts and write

$$\Re e\Lambda^{ss} < -\beta_i^{ss} < -\beta_i^s < \Re e\Lambda^c < \beta_i^u < \beta_i^{uu} < \Re e\Lambda^{uu} \tag{8}$$

where $\beta_i^{uu} > \beta_i^u > 0$, $\beta_i^{ss} > \beta_i^s > 0$.

The separating values β_i can be different for different orbits L_i. The important requirement is that the numbers k, m, n of characteristic exponents belonging to each part of the spectrum do not depend on L_i. Note that the numbers k, m, n are not uniquely determined by the system. For instance, if the contour contains only one recurrent orbit, namely, a saddle periodic orbit L, one may, in principle, consider all characteristic exponents of L as critical and in this case $m = n = 0$ and k equals the dimension of the phase space, or one may consider all characteristic exponents with negative real parts as strong-stable, the characteristic exponents with positive real parts as strong-unstable and only a trivial characteristic exponent equal to zero is critical in this case (i.e., $k = 1$); other variants corresponding to intermediate values of k are also allowed.

Implicitly, when studying concrete multidimensional bifurcational problems, such a kind of separation of the spectrum of characteristic exponents was always done. Usually (see Gonchenko & Shil'nikov [1986, 1992]; Gonchenko *et al.* [1993a, 1993b, 1995]; Ovsyannikov & Shil'nikov [1986, 1991] for example), the so-called *leading* characteristic exponents are taken as critical and *non-leading* as strong-stable and (or) strong-unstable. Recall that the characteristic exponents nearest to the imaginary axis from the left-hand side are called *leading stable* and those nearest to the imaginary axis from the right-hand side are called *leading unstable*; the rest of the characteristic exponents are *non-leading*, respectively, stable and unstable exponents.

We restrict the freedom in the choice of the trichotomy decomposition by an additional requirement. Namely, we suppose that for each non-recurrent orbit Γ_s in the contour C a pair of the so-called *transversality conditions* is fulfilled. Below we will give a rule by which an $(m+k)$-dimensional plane N^{cs} can be constructed at each point of Γ_s such that the family of planes N^{cs} is invariant with respect to the system linearized along Γ_s (if a vector field is given by the system $\dot{x} = f(x)$ and $\{x = \varphi(t)\}$ is a solution, then the linearized system, or *the variational equation*, along $\{x = \varphi(t)\}$ is $\dot{y} = \frac{\partial f}{\partial x}|_{x=\varphi(t)} \cdot y$).

Analogously, there is constructed a family of m-dimensional planes N^{ss}, a family of $(n+k)$-dimensional planes N^{cu} and a family of n-dimensional planes N^{uu}. Each family is invariant with respect to the variational equation along Γ_s. These planes (better say subspaces of the tangent space) are uniquely defined, and $N^{ss} \subset N^{cs}$, $N^{uu} \subset N^{cu}$. The transversality conditions are:

At each point of each orbit Γ_s the space N^{cs} is transverse to N^{uu} and the space N^{cu} is transverse to N^{ss}.

Due to the invariance of the subspaces with respect to the variational equation, the transversality is to be verified in one point on each orbit Γ_s. Since N^{cs} and N^{uu} (N^{cu} and N^{ss}) have complementary dimensions [$(m+k)$ and n and $(n+k)$ and m, respectively], the transversality conditions are well posed.

These conditions are effectively verified for the first-level orbits Γ_s. A preliminary step is to consider the local structure near recurrent orbits L_i. Take, first, a periodic orbit (we omit the index i and denote the orbit as L). Let the multipliers of L be divided into three groups: Λ^{ss}, Λ^{uu}, Λ^c, such that

$$|\Lambda^{ss}| < \alpha^{ss} < \alpha^s < |\Lambda^c| < \alpha^u < \alpha^{uu} < |\Lambda^{uu}|$$

for some $\alpha^{uu} > \alpha^u > 1 > \alpha^s > \alpha^{ss} > 0$. Let the numbers of strong-stable, strong-unstable and critical multipliers be, respectively, m, n and k as before. We also make a more subtle separation: those critical multipliers which lie strictly inside the unit circle will be called critical-stable, those outside the unit circle will be called critical-unstable and the multipliers lying on the unit circle will be called center-critical. We denote the numbers of these multipliers as k_s, k_u and k_c, respectively, $k_s + k_u + k_c = k$. We take into account a trivial Floquet multiplier which is always equal to unity, so $k_c \geq 1$.

Let us construct a small cross section S to L and consider the Poincaré map $T_0 : S \to S$. The point $O = L \cap S$ is a fixed point for the map T_0. The eigenvalues of the linearization of T_0 at the point O are the non-trivial multipliers of L. Let (x, y, z, u, v) be the coordinates on S: here x corresponds to the k_s-dimensional critical-stable eigenspace, y corresponds to the k_u-dimensional critical-unstable eigenspace, z to the $(k_c - 1)$-dimensional center-critical eigenspace, u to the m-dimensional strong-stable eigenspace and v corresponds to the n-dimensional strong-unstable eigenspace. The map T_0 is written in the following form near the point O

$$\begin{aligned}
\bar{x} &= B^{c-}x + g_{11}(x, y, z, v)x + g_{12}(x, y, z, u, v)u\,,\\
\bar{u} &= B^{ss}u + g_{21}(x, y, z, v)x + g_{22}(x, y, z, u, v)u\,,\\
\bar{y} &= B^{c+}y + h_{11}(x, y, z, u)y + h_{12}(x, y, z, u, v)v\,,\\
\bar{v} &= B^{uu}v + h_{21}(x, y, z, u)y + h_{22}(x, y, z, u, v)v\\
\bar{z} &= B^{c0}z + f_1(z)z + f_2(x, y, z)x + f_3(x, y, z)y\\
&\quad + f_4(x, y, z, u, v)u + f_5(x, y, z, u, v)v
\end{aligned} \tag{9}$$

where $Spectr B^{c-} = \Lambda^{c-}$, $Spectr B^{ss} = \Lambda^{ss}$, $Spectr B^{c+} = \Lambda^{c+}$, $Spectr B^{uu} = \Lambda^{uu}$, $Spectr B^{c0} = \Lambda^{c0} \subset \{|\Lambda| = 1\}$; the functions g_{ij}, h_{ij} and f_j are C^{r-1} functions vanishing at the origin. For convenience, we have locally straightened the center-stable and center-unstable invariant manifolds w^{s0} and w^{u0} of the point O; i.e., we assume $w^{s0} = \{y = 0, v = 0\}$ and $w^{u0} = \{x = 0, u = 0\}$ near O.

Note that the stable and unstable sets of the point O lie, respectively, in the manifolds w^{s0} and w^{u0}. If the periodic orbit L is structurally stable (i.e., if there are no multipliers on the unit circle and the coordinate z should be eliminated in this case), then w^{s0} and w^{u0} are the usual stable and unstable invariant manifolds.

It follows from Gonchenko & Shil'nikov [1992] that the coordinates can be chosen near O such that

$$\begin{aligned}
&g_{12}|_{y=0,v=0} \equiv 0\,, \quad f_4|_{y=0,v=0} \equiv 0\,, \quad g_{21}|_{x=0} \equiv 0\\
&h_{12}|_{x=0,u=0} \equiv 0\,, \quad f_5|_{x=0,u=0} \equiv 0\,, \quad h_{21}|_{y=0} \equiv 0
\end{aligned} \tag{10}$$

Here, $\bar{x}$ and $\bar{z}$ in Eqs. (9) do not depend on u for $\{y = 0, v = 0\}$; i.e., the system of the leaves $\{x = \text{const.}, z = \text{const.}, y = 0, v = 0\}$ composes an invariant foliation ϕ^{ss} on the manifold w^{s0}. Analogously, the leaves $\{y = \text{const.}, v = \text{const.}, x = 0, u = 0\}$ compose an invariant foliation ϕ^{uu} on the manifold w^{u0}. The presence of these foliations can also be extracted from Hirsch *et al.* [1977]. The leaf of the foliation ϕ^{ss} which contains the point O is the invariant m-dimensional strong-stable manifold w^{ss}, and that the leaf of ϕ^{uu} which contains O is the invariant n-dimensional strong-unstable manifold w^{uu} (Figs. 15 and 16).

These invariant manifolds are defined uniquely. It can be shown that if there are no non-trivial multipliers on the unit circle, then the smooth invariant foliations ϕ^{ss} and ϕ^{uu} containing, respectively, w^{ss} and w^{uu} are also defined uniquely on the stable and unstable manifolds. Even in the case where the number of non-trivial center-critical multipliers is greater than zero, for each point of w^{s0} whose orbit tends to O with the iterations of the Poincaré map T_0, the leaf of the foliation ϕ^{ss} that passes through this point is uniquely defined by the condition that the limit (in smooth topology) of the iterations of this leaf is the manifold w^{ss} (see Fig. 17 for an illustration). Analogously, for each point of w^{u0} whose orbit tends to O with the iterations of the inverse Poincaré map T_0^{-1}, the leaf of the foliation ϕ^{uu} that passes through this point is uniquely defined (see Sec. 5).

It follows easily from identities (10), that the field of $(n + k - 1)$-dimensional planes $\{u = 0\}$

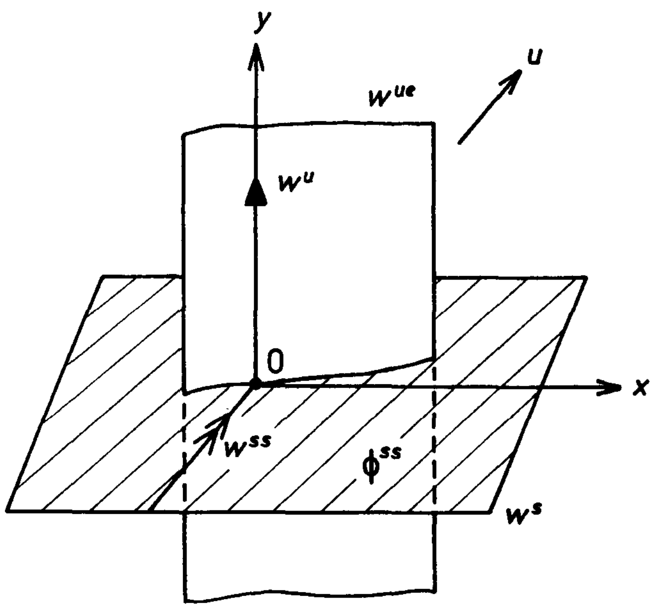

Fig. 15. If, for instance, a saddle fixed point O of a three-dimensional map has the multipliers $\{\gamma, \lambda_1, \lambda_2\}$ where $\gamma > 1 > \lambda_1 > \lambda_2$, then there exist: the stable manifold w^s which is tangent at O to the eigenspace corresponding to the multipliers (λ_1, λ_2), the strong-stable manifold w^{ss} which is tangent at O to the eigenspace corresponding to the multiplier λ_2, the strong-stable foliation ϕ^{ss} on w^s which is transverse to the leading eigen-direction corresponding to the multiplier λ_1, the unstable manifold w^u which is tangent at O to the eigenspace corresponding to the multiplier γ, the extended unstable manifold w^{ue} (non-unique) which contains w^u and which is tangent at O to the eigenspace corresponding to the multipliers (γ, λ_1).

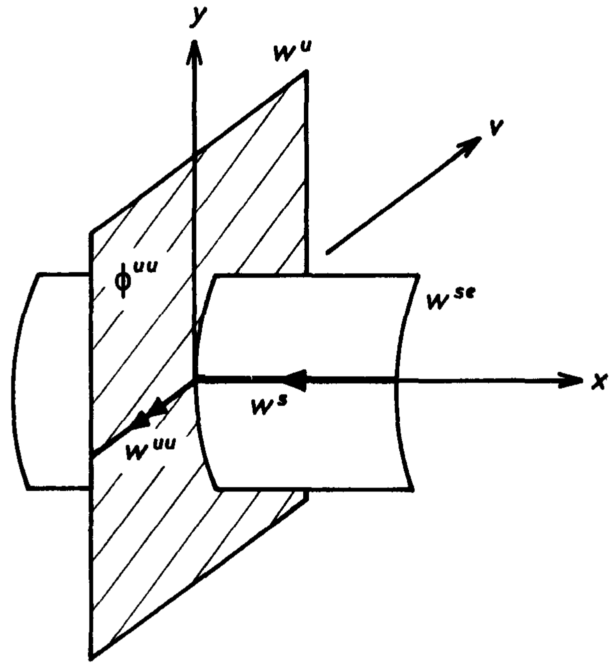

Fig. 16. Another example: if a saddle fixed point O of a three-dimensional map has the multipliers $\{\gamma_1, \gamma_2, \lambda\}$ where $\gamma_2 > \gamma_1 > 1 > \lambda$, then there exist the unstable manifold w^u which is tangent at O to the eigenspace corresponding to the multipliers (γ_1, γ_2), the strong-unstable manifold w^{uu} which is tangent at O to the eigenspace corresponding to the multiplier γ_2, the strong-unstable foliation ϕ^{uu} on w^u which is transverse to the leading eigen-direction corresponding to the multiplier γ_1, the stable manifold w^s which is tangent at O to the eigenspace corresponding to the multiplier λ, the extended stable manifold w^{se} (non-unique) which contains w^s and which is tangent at O to the eigenspace corresponding to the multipliers (γ_1, λ).

constructed at the points of the manifold w^{u0} is invariant with respect to the derivative of the map T_0: when $\{x = 0,\ u = 0\}$ we have by virtue of (10)

$$d\bar{u} = (B^{ss} + g_{22}(0,\ y,\ z,\ 0,\ v))du$$

We see that if $du = 0$, then $d\bar{u} = 0$ which means the invariance of the field $\{u = 0\}$.

It can be shown that this field of planes is uniquely defined at the points of the unstable set of O; i.e., if the backward semi-orbit of a point on w^{u0} tends to O, then the plane $\{u = 0\}$ constructed at this point is the unique m-dimensional plane whose iterations by the derivative of the inverse Poincaré map T_0^{-1} have, as a limit, an $(n+k-1)$-dimensional plane transverse to w^{ss} at O.

Besides, it can also be shown that w^{u0} is embedded into the so-called *extended unstable* invariant $(n + k - 1)$-dimensional manifold w^{ue} (Fig. 15) which is tangent to $\{u = 0\}$ everywhere on w^{u0}. This is a C^1-manifold[5]; it is not unique but any two of such manifolds have a common tangent at each point of the unstable set of O.

Analogously, the manifold w^{s0} is embedded into the *extended unstable* $(m+k-1)$-dimensional manifold w^{se} of the smoothness equal to the greatest integer less than $(\ln \alpha_{uu}/ \ln \alpha_u)$ and not greater than r. This manifold is transverse to w^{uu} at O; it is not unique but any of such manifolds is tangent to the invariant field of $(m + k - 1)$-dimensional planes $\{v = 0\}$ everywhere on the stable set of O (Fig. 16).

Denote $W^{ue}(L)$ and $W^{se}(L)$ as the sets composed of the orbits of the system X which pass,

[5]In general, the smoothness of w^{ue} is equal to the greatest integer which is less that $(\ln \alpha_{ss}/ \ln \alpha_s) > 1$ (of course, if this quantity does not exceed the smoothness r of the system).

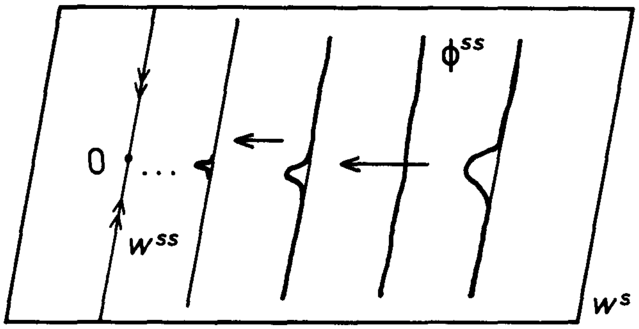

Fig. 17. An illustration to the uniqueness of the strong-stable foliation: a small bubble on the strong-stable leaf would not disappear at the iterations of the map because the contraction along the strong-stable direction is stronger than the contraction in the transverse (leading) direction.

respectively, through w^{ue} and w^{se}. These sets are invariant manifolds of codimensions m and n. Any orbit which tends to L as $t \to +\infty$ or $t \to -\infty$ belongs, respectively, to $W^{se}(L)$ or $W^{ue}(L)$.

Without loss of generality, we can rescale time near L so that all orbits starting on the cross section S return to S at the same time. This allows one to extend the foliation ϕ^{ss} along the orbits of X onto the whole stable set of L so that through each point whose orbit tends to L as $t \to +\infty$, there would pass a unique m-dimensional leaf. Analogously, the foliation ϕ^{uu} can be extended onto the unstable set of L so that a unique n-dimensional leaf would pass through each point whose orbit tends to L as $t \to -\infty$. We denote the obtained invariant foliations as $F^{ss}(L)$ and $F^{uu}(L)$.

The analogous structure exists in a neighborhood of an equilibrium state. Thus, if an equilibrium state L has m strong-stable characteristic exponents lying to the left of the line $\Re e(\cdot) = -\beta^{ss}$ on the complex plane, n strong-unstable characteristic exponents lying to the right of the line $\Re e(\cdot) = \beta^{uu}$ and k critical characteristic exponents lying in the strip $-\beta^s < \Re e(\cdot) < \beta^u$ where $\beta^{ss} > \beta^s > 0$ and $\beta^{uu} > \beta^u > 0$, then the equilibrium state has extended stable and unstable invariant manifolds $W^{se}(L)$ and $W^{ue}(L)$ of dimensions $(m+k)$ and $(n+k)$ respectively. The smoothness of $W^{se}(L)$ is equal to the maximal integer less than $(\Re e\beta_{uu}/\Re e\beta_u)$ and not greater than r, and the smoothness of $W^{ue}(L)$ is equal to the maximal integer less than $(\Re e\beta_{ss}/\Re e\beta_s)$ and not greater than r. The manifold $W^{se}(L)$ is tangent at L to the eigenspace corresponding to the strong-stable and critical characteristic exponents. It contains the stable set of L and, though the manifold $W^{se}(L)$ is not uniquely defined, any two of such manifolds have a common tangent at each point whose orbit tends to L as $t \to +\infty$. The manifold $W^{ue}(L)$ is tangent at L to the eigenspace corresponding to the strong-unstable and critical characteristic exponents. It contains the unstable set of L and any two of such manifolds have a common tangent at each point whose orbit tends to L as $t \to -\infty$.

Besides, there exist m-dimensional strong-stable and n-dimensional strong-unstable invariant manifolds $W^{ss}(L)$ and $W^{uu}(L)$ which are tangent at L to the eigenspaces corresponding to the strong-stable and strong-unstable characteristic exponents, respectively. These manifolds are embedded into invariant C^{r-1}-foliations $F^{ss}(L)$ and $F^{uu}(L)$. The foliation $F^{ss}(L)$ is defined on the center-stable manifold W^{s0} which is tangent at L to the eigenspace corresponding to the characteristic exponents whose real parts are not greater than zero (these are strong-stable, critical-stable and center-critical characteristic exponents). The foliation $F^{uu}(L)$ is defined on the center-unstable manifold W^{u0} which is tangent at L to the eigenspace corresponding to the characteristic exponents whose real parts are not less than zero (strong-unstable, critical-unstable and center-critical characteristic exponents). For each point belonging to the stable set of L, the leaf of F^{ss} that contains this point is uniquely defined by the condition that the limit (in smooth topology) of the shift of this leaf by the forward flow is the strong-stable manifold $W^{ss}(L)$. Analogously, the leaves of the strong-unstable foliation are uniquely defined at the points of the unstable set of L.

Like near periodic orbits, in a neighborhood of the equilibrium state L, there exists a coordinate transformation which straightens the manifolds and foliations. In fact, the coordinates can be chosen such that the vector field near L is written in the form (see Ovsyannikov & Shil'nikov [1991])

$$
\begin{aligned}
\dot{x} &= B^{c-}x + g_{11}(x, y, z, v)x + g_{12}(x, y, z, u, v)u\,,\\
\dot{u} &= B^{ss}u + g_{21}(x, y, z, v)x + g_{22}(x, y, z, u, v)u\,,\\
\dot{y} &= B^{c+}y + h_{11}(x, y, z, u)y + h_{12}(x, y, z, u, v)v\,,\\
\dot{v} &= B^{uu}v + h_{21}(x, y, z, u)y + h_{22}(x, y, z, u, v)v\,,\\
\dot{z} &= B^{c0}z + f_1(z)z + f_2(x, y, z)x + f_3(x, y, z)y\\
&\quad + f_4(x, y, z, u, v)u + f_5(x, y, z, u, v)v
\end{aligned}
\tag{11}
$$

where the eigenvalues of B^{c-} are the critical-stable characteristic exponents, the eigenvalues of B^{ss} are the strong-stable characteristic exponents, the eigenvalues of B^{c+} are the critical-unstable characteristic exponents, the eigenvalues of B^{uu} are the strong-unstable characteristic exponents and the eigenvalues of B^{c0} are the center-critical characteristic exponents; the functions g_{ij}, h_{ij} and f_j are C^{r-1} functions vanishing at the origin and satisfying identities (10). In these coordinates, the invariant manifolds W^{s0} and W^{u0} are locally straightened: $W^{s0} = \{y = 0, v = 0\}$ and $W^{u0} = \{x = 0, u = 0\}$ near L. As it was shown for the case where L is a periodic orbit, identities (10) imply that the foliation $F^{ss}(L)$ has the form $\{x = \text{const.}, z = \text{const.}, y = 0, v = 0\}$, and the foliation $F^{uu}(L)$ has the form $\{y = \text{const.}, z = \text{const.}, x = 0, u = 0\}$; the manifold $W^{ue}(L)$ is tangent to the invariant field of $(k+m)$-dimensional planes $\{u = 0\}$ everywhere on $W^{u0}(L)$ and the manifold $W^{se}(L)$ is tangent to the invariant field of $(k+n)$-dimensional planes $\{v = 0\}$ everywhere on $W^{s0}(L)$.

Let Γ be a first-level orbit of the contour C. As $t \to +\infty$, the orbit Γ tends to some recurrent orbit L_j and it also tends to some recurrent orbit, L_i, as $t \to -\infty$. This means that Γ belongs to the stable set of L_j and to the unstable set of L_i.[6] According to the above considerations, the orbit Γ lies in the manifolds $W^{ue}(L_i)$ and $W^{se}(L_j)$ of dimensions $(n+k)$ and $(m+k)$, respectively, and through each point of Γ, an n-dimensional leaf of the foliation $F^{uu}(L_i)$ and an m-dimensional leaf of the foliation $F^{ss}(L_j)$ pass (recall that the quantities n, m and k must not vary for the recurrent orbits in C). We denote the tangent to W^{ue} as N^{cs}, the tangent to W^{se} as N^{cu}, the tangent to the leaf of the strong-stable foliation as N^{ss} and the tangent to the leaf of the strong-unstable foliation as N^{uu}. By construction, these subspaces of the tangent space are uniquely defined at each point of Γ and they are invariant with respect to the linearized flow. Note also that $N^{ss} \subset N^{cs}$ and $N^{uu} \subset N^{cu}$ and that the spaces N^{cs} and N^{cu} contain the phase velocity vector.

The subspaces N^{ss}, N^{uu}, N^{cs} and N^{cu} are exactly those participating in the transversality conditions for the first-level orbits. We have seen that the subspaces N^{ss} and N^{cs} are effectively computed at the points of Γ lying near the orbit L_j to which Γ tends as $t \to +\infty$. In suitable coordinates, they are parallel to the spaces $\{x = 0, y = 0, z = 0, v = 0\}$ and $\{v = 0\}$, respectively (note that the presence of such coordinates is not a pure fact of existence: the proof in Ovsyannikov & Shil'nikov [1991]; Gonchenko & Shil'nikov [1992] reduces the problem of finding the corresponding coordinate transformation to the solution of some functional equations which can be done by expansion in a converging power series). Also, the subspaces N^{uu} and N^{cu} are effectively computed at the points of Γ lying near the orbit L_i to which Γ tends as $t \to -\infty$ (these subspaces are parallel to $\{x = 0, y = 0, z = 0, u = 0\}$ and $\{u = 0\}$, respectively).

For the first-level contours (i.e., for usual homo- and heteroclinic contours) each orbit Γ_s is asymptotic to the recurrent orbit as $t \to \pm\infty$. Thus, in this case, the transversality conditions can be verified in the following way: for each orbit Γ_s, take a point $P^+ \in \Gamma_s$ near the orbit L_j to which Γ_s tends as $t \to +\infty$, and a point $P^- \in \Gamma_s$ near the orbit L_i to which Γ_s tends as $t \to -\infty$. Then, construct subspaces N^{ss} and N^{cs} at the point P^+ and subspaces N^{uu} and N^{cu} at the point P^-. The subspaces N^{uu} and N^{cu} at the point P^+ are obtained as a shift of the subspaces N^{uu} and N^{cu} from the point P^- by the linearized flow defined by the variational equation along Γ_s. Therefore, the verification of the transversality conditions at the point P^+ is reduced to some calculations with the fundamental matrix of the solutions of the variational equation integrated along the piece of Γ_s that connects P^- and P^+ (Fig. 18).

As an example, consider a homoclinic loop of a saddle equilibrium state. Let an equilibrium state L have one unstable (positive) characteristic exponent γ and $(m+1)$ stable characteristic exponents

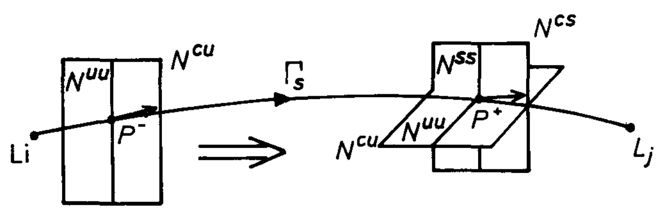

Fig. 18. A schematic illustration to the transversality conditions: N^{ss} is transverse to N^{cu} and N^{uu} is transverse to N^{cs}.

[6] If L_i and L_j were structurally stable we would say "the stable and unstable manifolds", but we consider a more general situation and, in principle, the stable and unstable sets of structurally unstable periodic orbits or equilibrium states may be quite complicated objects.

$\lambda_1, \ldots, \lambda_{m+1}$ with negative real parts; moreover, the leading (i.e., nearest to the imaginary axis) stable exponent λ_1 is also real:

$$\gamma > 0 > \lambda_1 > \Re e \lambda_i$$

The unstable manifold of L is one-dimensional and it is divided by the point L into two separatrices. Suppose that one of the separatrices, Γ returns to L as $t \to +\infty$, forming a homoclinic loop (see Fig. 3).

Let us separate the spectrum of the characteristic exponents of L in the following way: the exponents γ and λ_1 will be considered as critical and all the other exponents as strong-stable. The strong-unstable part of the spectrum is empty in this case. According to Eq. (11), the vector field near L can be written in the form

$$\begin{aligned} \dot{x} &= \lambda_1 x + g_{11}(x, y)x + g_{12}(x, y, u)u \\ \dot{y} &= \gamma y + h(x, y, u)y \\ \dot{u} &= B^{ss}u + g_{21}(x, y)x + g_{22}(x, y, u)u \end{aligned} \tag{12}$$

where the spectrum of the matrix B^{ss} is $\{\lambda_2, \ldots, \lambda_{m+1}\}$. Identities (10) take the form

$$\begin{aligned} g_{12}|_{y=0} &\equiv 0 \\ g_{21}|_{x=0} &\equiv 0 \end{aligned} \tag{13}$$

Locally, in these coordinates, the stable manifold W^s is $\{y = 0\}$, the unstable manifold is $\{x = 0, u = 0\}$, the strong-stable manifold W^{ss} is $\{x = 0, y = 0\}$, the strong-stable foliation F^{ss} on W^s is $\{x = \text{const.}, y = 0\}$, the extended unstable manifold W^{ue} is tangent to the two-dimensional plane $\{u = 0\}$ at the points of W^u_{loc}. The extended stable manifold coincide in our case with the whole phase space. At the same time, the leaves of the strong-unstable foliation are just points on W^u. Thus, the space N^{cs} coincides with the tangent space and N^{uu} is zero; these spaces are transverse automatically.

Therefore, in the case under consideration, the transversality condition to be verified is that at each point of Γ, the space N^{cu} is transverse to the space N^{ss}. The necessary condition for this is that the separatrix Γ returns to L not lying in the strong-stable manifold (otherwise, everywhere on Γ, the phase velocity vector would belong to the tangent space to the strong-stable manifold which is, by definition, the space N^{ss}; at the same time, the phase velocity vector always lies in N^{cu}, so we would have a contradiction with the transversality of N^{ss} and N^{cu}). Thus, we should require that Γ tends to L tangent to the leading direction (i.e., to the x-axis).

Take a pair of points P^+ and P^- on Γ such that $P^+ \in W^s_{\text{loc}}$ and $P^- \in W^u_{\text{loc}}$: $P^- = (x = 0, y = y^-, u = 0)$ and $P^+ = (x = x^+, y = 0, u = u^+)$ for some non-zero x^+ and y^- (see Fig. 19). Construct cross sections $\Pi^+ : \{x = x^+\}$ and $\Pi^- : \{y = y^-\}$ to Γ. Let $T_1 : \Pi^- \to \Pi^+$ be the map defined by the orbits of the system which lie near Γ (the so-called *global map*). At the point P^-, the two-dimensional space N^{cu} is $\{u = 0\}$ and it is spanned on the phase velocity vector e_y parallel to the y-axis and on the vector e_x parallel to the x-axis which lies in the intersection with Π^-. At the point P^+, the space N^{cu} is obtained as the shift of the space N^{cu} from point P^- by the flow linearized along Γ. Evidently, this space is spanned on the phase velocity vector transverse to Π^+ and on the image $T_1' e_x$ of the vector e_x by the map T_1 linearized at the point P^-. Since the leaf $\{x = x^+, y = 0\}$ of the foliation F^{ss} that passes through the point P^+ lies in Π^+ as a whole and it coincides with the intersection of Π^+ with the stable manifold W^s_{loc}, the transversality of N^{cu} and N^{ss} at the point P^+ is, therefore, equivalent to the transversality of the vector $T_1' e_x$ to the stable manifold in intersection with Π^+.

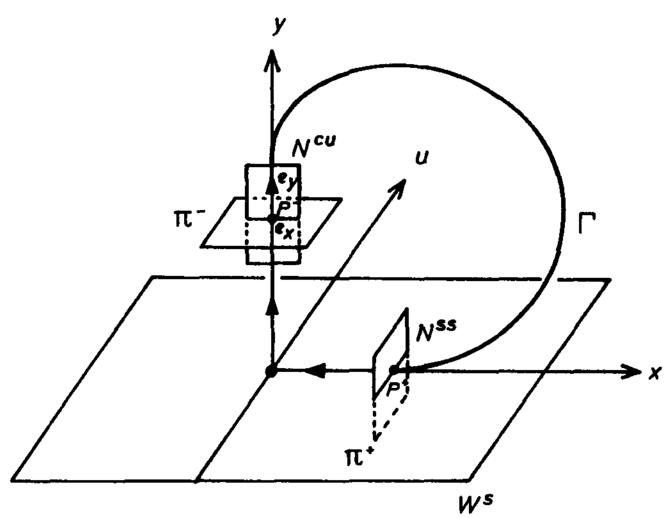

Fig. 19. A homoclinic loop Γ. The points P^+ and P^- on Γ are taken, respectively, on W^u_{loc} and W^s_{loc}. The planes Π^+ and Π^- are cross sections at the points P^+ and P^-. The space N^{cu} at the point P^- is spanned on the vector e_y parallel to the phase velocity vector and on the vector $e_x \in \Pi^-$. The space N^{ss} at the point P^+ coincides with the intersection of Π^+ with W^s.

The global map T_1 is written in the form

$$\begin{aligned} \bar{y} &= a_{11}x + a_{12}u + \cdots \\ \bar{u} - u^+ &= a_{21}x + a_{22}u + \cdots \end{aligned} \tag{14}$$

where (x, u) are the coordinates on Π^- and $(\bar{y}, \bar{u})$ are the coordinates on Π^+; the dots stand for nonlinear terms. The vector $e_x = (1, 0)$ is mapped by the linearization of T_1 into the vector $T_1' e_x = (a_{11}, a_{21})$. The transversality of this vector to the intersection $\{y = 0\}$ of W^s_{loc} with Π^- is given by the condition

$$a_{11} \neq 0$$

Actually, this condition is standard when studying bifurcations of homoclinic loops. Usually, this condition is introduced in other connections: it can be shown [Ovsyannikov & Shil'nikov, 1986, 1991] that if identities (13) hold, then the Poincaré map from Π^+ to Π^+ is written in the form

$$\begin{aligned} \bar{y} &= Ay^\nu + \cdots \\ \bar{u} - u^+ &= a_{21}x^+(y/y^-)^\nu + \cdots \end{aligned} \tag{15}$$

where $A = a_{11}x^+/(y^-)^\nu$, $\nu = |\lambda_1/\gamma|$ and the dots stand for terms of order $o(y^\nu)$. Thus, the nonvanishing of the value a_{11} is equivalent to the nonvanishing of the first term in the asymptotic of the Poincaré map.

The value A is called *the separatrix value.* As we have shown, for the bifurcations of a homoclinic loop of a saddle with one-dimensional unstable manifold, our transversality conditions are equivalent to the well-known condition of the non-vanishing of the separatrix value A together with the requirement (which is also standard) that the separatrix does not lie in the strong-stable manifold.

Of course, this is true for our particular choice of the separation of the spectrum: only leading characteristic exponents were considered as critical. Other kinds of separation of the spectrum of characteristic exponents into the three parts lead to other transversality conditions. For instance, in Shil'nikov [1968] where the bifurcations of a homoclinic loop of a saddle with the multi-dimensional unstable manifold were studied, the spectrum of characteristic exponents was divided into two parts: $\gamma_1, \lambda_1, \ldots, \lambda_{k-1}$ and $\gamma_2, \ldots, \gamma_{n+1}$ (notations are slightly changed) where $\Re e\lambda_i < 0$ and $\Re e\gamma_j > \gamma_1 > 0$. In our terms, all stable characteristic exponents λ_i and the leading unstable characteristic exponent γ_1 are grouped as the critical part of the spectrum and the non-leading unstable exponents are considered as strong-unstable; the strong-stable part is empty here. The transversality conditions appeared as the requirement that the separatrix would not lie in the strong-unstable manifold W^{uu} and the determinant of some matrix composed by a part of the coefficients of the linearization matrix of the corresponding global map would not be equal to zero. Apparently, it was in that paper where the transversality conditions were (implicitly) introduced for the first time.

For the bifurcations of a homoclinic loop of a saddle-focus, different analogues of our transversality conditions can be found in Shil'nikov [1970]; Ovsyannikov & Shil'nikov [1986, 1991]. Note that if the equilibrium state has no characteristic exponents on the imaginary axis, then at least one characteristic exponent with negative real part and one with positive real part must be contained in the critical strip. Indeed, otherwise, either the strong-stable foliation would consist of the only leaf coinciding with the whole stable manifold or the strong-unstable foliation would consist of the only leaf coinciding with the unstable manifold. If, for instance, the strong-unstable leaf coincides with the unstable manifold, this implies that the space N^{uu} at each point of the unstable manifold contains the phase velocity vector. If the contour is indecomposable (see the previous section), then there is at least one non-recurrent orbit in the contour that belongs to the unstable manifold of the equilibrium state under consideration. At the points of this orbit, the transversality of N^{uu} with the associated subspace N^{cs} cannot hold because N^{cs} also contains the phase velocity vector.

Consider now an example of a contour containing a periodic orbit. This is a contour with a Poincaré homoclinic orbit (i.e., an orbit homoclinic to a saddle periodic orbit). Here, different kinds of the trichotomy decomposition also lead to different variants of the transversality conditions. Thus, let L be a saddle periodic orbit with the multipliers $\lambda_1, \ldots, \lambda_m$ and $\gamma_1, \ldots, \gamma_n$ where λ_i lies inside the unit circle and γ_j lies outside the unit circle (there is also one trivial Floquet multiplier equal to unity). Suppose the system has an orbit Γ homoclinic to L [i.e., an orbit of intersection of $W^s(L)$ and $W^u(L)$]. If we consider the logarithms of stable multipliers λ_i as the strong-stable characteristic exponents, the logarithms of unstable multipliers as the strong-unstable characteristic exponents and zero as the unique critical characteristic exponent, then the extended stable and unstable manifolds

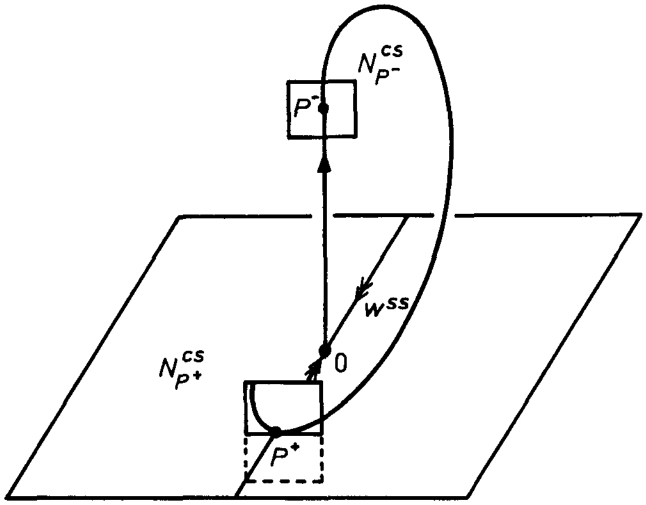

Fig. 20. A saddle fixed point O with a homoclinic tangency. Though the point of tangency P^+ belongs to the strong-stable manifold w^{ss}, the transversality conditions are fulfilled: the space N_{P+}^{cs} is transverse to the strong-stable manifold. At the same time, the non-coincidence conditions of Theorem 3 are not fulfilled: all points of the orbit of the point P^+ belong to the same leaf, namely, to the manifold w^{ss}.

W^{se} and W^{ue} will coincide with W^s and W^u, respectively. At the same time, the strong-stable leaf of any point on W^s will coincide with the stable set of this point[7] and the strong-unstable leaf of any point of W^u will coincide with the unstable set of the point. For an appropriately chosen cross section, the strong-stable leaf will coincide with the intersection of W^s with the cross section. Therefore, the transversality of W^{ue} and F^{ss} (and, analogously, the transversality of W^{se} and F^{uu}) is equivalent to the transversality of W^s and W^u in this case.

We see that for the given choice of the separation of the spectrum, the transversality conditions are equivalent to the requirement that the Poincaré homoclinic orbit is transverse (or structurally stable) as in Fig. 6. If this requirement is not satisfied (i.e., in the case of homoclinic tangency, as in Fig. 7), one must consider separation of another kind. For instance, the critical strip can be enlarged to include the leading characteristic exponents. For this case, the transversality conditions are explicitly written in Gonchenko *et al.* [1993b] as the condition for the determinant of some matrix to be not equal to zero. Note that, in contrast with the case of a homoclinic loop of an equilibrium state, the transversality of W^{ue} and F^{ss} does not mean here that the homoclinic orbit is prohibited to lie in the strong-stable manifold (see Fig. 20 for an explanation).

In a general case, for the orbits of the second and higher levels, construction of the subspaces N^{ss}, N^{uu}, N^{cs} and N^{cu} is more abstract. It follows from Sec. 5 that if for all first-level orbits in the contour C the transversality conditions hold, then for each orbit Γ whose ω-limit set belongs to the union of the first- and zero-level orbits the invariant spaces N^{ss} and N^{cs} are uniquely defined by the following "continuity condition":

> *For any point P^* belonging to the union of the orbits of the first and zero levels, if a sequence P_i of points of Γ tends to P^*, then the sequence $N_{P_i}^{ss}$ tends to $N_{P^*}^{ss}$ and the sequence $N_{P_i}^{cs}$ tends to $N_{P^*}^{cs}$.*

The analogous continuity condition defines in a unique way the invariant spaces N^{uu} and N^{cu} for each orbit Γ whose α-limit set belongs to the union of the first- and zero-level orbits. Thus, for all second-level orbits, the four invariant subspaces are uniquely defined and the transversality conditions have a sense. If they hold for all second-level orbits in the contour, the invariant subspaces N^{ss}, N^{uu}, N^{cs} and N^{cu} are defined in the same way for the third-level orbits and so on.

3. Main Results

Hereafter, we suppose that for each recurrent orbit in the contour C, the trichotomy property holds with some integers m, n and k, and for each nonrecurrent orbit, the corresponding transversality conditions are fulfilled. Let U be a small neighborhood of the contour.

Theorem 1. *If the neighborhood U of C is sufficiently small, then for any system close to X, the set Ω of the orbits lying in U entirely is pseudohyperbolic in the sense that at each point P, whose*

[7]Due to the presence of the trivial Floquet multiplier, the phase shift between points on W^s is not changed with time. Accordingly, the stable manifold is foliated by the surfaces of equal phase so that the distance between any two points starting with the same surface tends to zero as $t \to +\infty$ whereas the distance between points starting with different phases will never approach zero. In other words, such a surface is the stable set for any point belonging to it. On the other hand, these surfaces compose the strong-stable foliation in our sense.

orbit stays in U for all times, the tangent space is uniquely decomposed into the direct sum of the three subspaces $\mathcal{N}^{ss} \oplus \mathcal{N}^{c} \oplus \mathcal{N}^{uu}$, $\dim \mathcal{N}^{ss} = m$, $\dim \mathcal{N}^{c} = k$, $\dim \mathcal{N}^{uu} = n$. This decomposition is invariant with respect to the variational equation and the linearized flow is exponentially contracting along the directions belonging to $\mathcal{N}^{ss}$ and it is expanding along the directions belonging to $\mathcal{N}^{uu}$; moreover, the contraction or expansion is stronger than that which may take place along the directions belonging to $\mathcal{N}^{c}$. The subspaces $\mathcal{N}^{ss}$, $\mathcal{N}^{uu}$ and $\mathcal{N}^{c}$ depend continuously on the point P and on the system. At the points of the recurrent orbits $L_i \subset C$, the subspaces $\mathcal{N}^{ss}$, $\mathcal{N}^{uu}$ and $\mathcal{N}^{c}$ coincide, respectively, with the eigenspaces corresponding to the left, right and center part of the spectrum of characteristic exponents. At the points of the non-recurrent orbits $\Gamma_s \subset C$, the subspaces $\mathcal{N}^{ss}$, $\mathcal{N}^{uu}$ and $\mathcal{N}^{c}$ coincide with N^{ss}, N^{uu} and $N^{cs} \cap N^{cu}$, respectively.

This theorem gives a rough description of the behavior of the linearized flow: it shows that for any orbit staying in U, for all times the number of negative Lyapunov exponents cannot be greater than $(m+k)$ and less than m and the number of positive Lyapunov exponents cannot be greater than $(n+k)$ and less than n.

The notion of *a pseudohyperbolic set* was introduced by Hirsch *et al.* [1977] as a generalization of a hyperbolic set: a non-trivial pseudohyperbolic set is hyperbolic if it does not contain equilibrium states and the space N^c is one-dimensional and coinciding with the direction of the phase velocity vector. Whereas the hyperbolicity is known to be a strong restriction for the structure of invariant sets, the pseudohyperbolicity is, in general, not a restriction at all: any closed invariant set can be considered as pseudohyperbolic with $m = n = 0$ ($\mathcal{N}^{uu}$ and $\mathcal{N}^{ss}$ are empty). The basic result concerning the pseudohyperbolic sets can be extracted from Hirsch *et al.* [1977]. It establishes the existence of strong-stable and strong-unstable manifolds for the orbits of the pseudohyperbolic set. In our situation we can formulate this result in the following way:

Corollary 1. *For any point $P \in \Omega$, there exist uniquely defined smooth strong-stable and strong-unstable manifolds $\mathcal{W}_P^{ss}$ and $\mathcal{W}_P^{uu}$ which are tangent, respectively, to N^{ss} and N^{uu} at P and which are such that the collections of manifolds $\{\mathcal{W}_P^{ss} | P \in \Omega\}$ and $\{\mathcal{W}_P^{uu} | P \in \Omega\}$ are invariant with respect to the flow. The manifolds $\mathcal{W}_P^{ss}$ and $\mathcal{W}_P^{uu}$ depend continuously on P and on the dynamical system; they are homeomorphic, respectively, to R^m and R^n and their size is separated from zero (in the sense that there exists some $\varepsilon_0 > 0$ independent of P and such that each manifold $\mathcal{W}_P^{ss}$ and $\mathcal{W}_P^{uu}$ contains the ball with the center P and radius ε_0 in the inner metrics induced by the metrics of the phase space). The manifolds $\mathcal{W}_P^{ss}$ and $\mathcal{W}_P^{uu}$ belong, respectively, to the stable and unstable sets of P : $\mathcal{W}_P^{ss} \subseteq \mathcal{W}_P^{s}$, $\mathcal{W}_P^{uu} \subseteq \mathcal{W}_P^{u}$.*

Recall, that the stable set $\mathcal{W}_P^{s}$ is the set of such points P' that

$$\mathrm{dist}(P'_t, P_t) \to 0$$

as $t \to +\infty$, where P_t and P'_t are the points of the orbits of P and P' which correspond to the time t. Analogously, the unstable set $\mathcal{W}_P^{u}$ is the union of the points P' such that

$$\mathrm{dist}(P'_t, P_t) \to 0$$

as $t \to -\infty$. By the corollary, for any point $P \in \Omega$ we have

$$\begin{aligned} \dim \mathcal{W}_P^{s} &\geq \dim \mathcal{W}_P^{ss} = m \\ \dim \mathcal{W}_P^{u} &\geq \dim \mathcal{W}_P^{uu} = n \end{aligned} \tag{16}$$

Actually, Theorem 1 implies even more; namely, it can be shown that for any point $P \in \Omega$

$$\begin{aligned} m \leq \dim \mathcal{W}_P^{s} \leq m + k \\ n \leq \dim \mathcal{W}_P^{u} \leq n + k \end{aligned} \tag{17}$$

Without any additional considerations, it follows from Theorem 1 that for any periodic orbit $\mathcal{L}$ lying in U entirely

$$\begin{aligned} m + 1 \leq \dim W^s(\mathcal{L}) \leq m + k \\ n + 1 \leq \dim W^u(\mathcal{L}) \leq n + k \end{aligned} \tag{18}$$

(by the theorem, the orbit $\mathcal{L}$ has at least m multipliers inside the unit circle and at least n multipliers outside the unit circle).

Inequalities (17) and (18) are analogous to those which follow from the "center manifold theorem" in the case of local bifurcations. Note that the numbers k, m and n are not defined uniquely, because there are different possibilities to divide the spectra of characteristic exponents of the recurrent orbits in the contour into three parts. Unless something different is said, we will assume that the number

k of characteristic exponents lying in the critical strip is taken as low as possible. The important requirement is that the transversality conditions must hold: the spaces N^{cu} must be transverse to N^{ss}, and N^{cs} must be transverse to N^{uu}. Due to the uniqueness of these subspaces, we have that N^{cu} and N^{ss} do not depend on where we choose the right boundary of the critical strip to be, and N^{cs} and N^{uu} do not depend on the position of the left boundary; i.e., the spaces N^{cu} and N^{ss} stay the same for different choices of the numbers β_i^u and β_i^{uu}, and the spaces N^{cs} and N^{uu} stay the same for different choices of β_i^s and β_i^{ss}.

Therefore, the algorithm of finding the lowest possible k may be as follows: take β_i^s and β_i^{ss} such that, for some m, exactly m characteristic exponents of each recurrent orbit L_i in the contour C lie to the left of the line $\Re e(\cdot) = -\beta_i^{ss}$ on the complex plane and the rest of characteristic exponents lie to the right of the line $\Re e(\cdot) = -\beta_i^s$ ($\beta_i^{ss} > \beta_i^s > 0$). Then, check the transversality of N^{cu} and N^{ss} at each non-recurrent orbit $\Gamma_s \subset C$. If the transversality condition holds, the values β_i^s and β_i^{ss} are appropriate and one can choose among the appropriate β_i^s and β_i^{ss} which give the maximal possible value of m (i.e., the maximal possible value of strong-stable characteristic exponents). Analogously, one can move the right boundary of the critical strip to the left (decreasing the values β_i^u and β_i^{uu}), provided the transversality of N^{cs} and N^{uu} holds for each orbit $\Gamma_s \subset C$; this procedure gives the maximal possible value n of strong-unstable characteristic exponents. The rest of the characteristic exponents forms, by definition, the center part of the spectrum and the number k of the center characteristic exponents is now taken as low as possible.

We will call this number *the critical dimension of the problem* and denote it as d_c. According to Corollary 1 it is equal to the difference between the dimension of the phase space and the sum of the dimensions of strong-stable and strong-unstable manifolds of the points of the set Ω.

As we mentioned in Sec. 1, estimates (17) and (18) of dimensions of the stable and unstable manifolds of orbits staying in U for all times are not final. To improve the estimates, we introduce the notion of *the Lyapunov dimension of the problem.*

Let $\lambda_1^i, \ldots, \lambda_k^i$ be the critical exponents of a recurrent orbit $L_i \subset C$ (a periodic orbit or an equilibrium state). Suppose the critical exponents are ordered so that

$$\Re e\lambda_1^i \geq \Re e\lambda_2^i \geq \cdots \geq \Re e\lambda_k^i$$

Let the sequential divergence S_i be less than zero:

$$S_i = \Re e\lambda_1^i + \cdots + \Re e\lambda_k^i < 0 \tag{19}$$

Let k_i' be such that

$$\Re e\lambda_1^i + \cdots + \Re e\lambda_{k_i'}^i \geq 0$$

and

$$\Re e\lambda_1^i + \cdots + \Re e\lambda_{k_i'+1}^i < 0$$

[$k_i' < k$ by virtue of (19)]. If condition (19) holds for all recurrent orbits $L_i \subset C$ (the volume contracting case), then the number

$$d_L = \max_{L_i \subset C} k_i'$$

will be called the Lyapunov dimension.

According to the definition, the linearized flow applied to the space $\mathcal{N}^c$ contracts exponentially $(d_L + 1)$-dimensional volumes along each recurrent orbit $L_i \subset C$. Since any orbit of C (and, therefore, any orbit lying in a small neighborhood U of C) spends most of the time in a small neighborhood of the union of the orbits L_i, the linearized flow applied to the space $\mathcal{N}^c$ contracts exponentially $(d_L + 1)$-dimensional volumes along any orbit $\mathcal{L}$ lying in U entirely.

If the sequential divergences are positive for each orbit L_i:

$$S_i = \Re e\lambda_1^i + \cdots + \Re e\lambda_k^i > 0\,, \tag{20}$$

then there are defined integers $k_i' < k$ such that

$$\Re e\lambda_{k_i'}^i + \cdots + \Re e\lambda_k^i \leq 0$$

and

$$\Re e\lambda_{k_{i-1}'}^i + \cdots + \Re e\lambda_k^i > 0\,.$$

The value

$$d_L = \max_{L_i \subset C}(k + 1 - k_i')$$

is called Lyapunov dimension in this case (the volume expanding case). By definition, the linearized flow applied to the space N^c expands $(d_L + 1)$-dimensional volumes.

Evidently, the property of expansion or contraction of volumes imposes a restriction for the maximal possible number of negative or, respectively, positive Lyapunov exponents. Namely, we arrive at the following result.

Theorem 2. *If for each recurrent orbit $L_i \subset C$ the sequential divergence S_i is negative, then any orbit $\mathcal{L} \subseteq \Omega$ cannot have more than $(n + d_L)$ non-negative Lyapunov exponents. If all the sequential divergences S_i are positive, then any orbit $\mathcal{L} \subseteq \Omega$ cannot have more than $(m + d_L)$ non-positive Lyapunov exponents.*

By this theorem, for any periodic orbit $\mathcal{L} \subseteq \Omega$

$$\begin{aligned} n+1 &\leq \dim W^u(\mathcal{L}) \leq n + d_L \\ m + d_c - d_L + 1 &\leq \dim W^s(\mathcal{L}) \leq m + d_c \end{aligned} \tag{21}$$

in the volume-contracting case (all $S_i < 0$), and

$$\begin{aligned} n + 1 + (d_c - d_L) &\leq \dim W^u(\mathcal{L}) \leq n + d_c \\ m+1 &\leq \dim W^s(\mathcal{L}) \leq m + d_L \end{aligned} \tag{22}$$

in the volume-expanding case (all $S_i > 0$). In the other cases, we get no additional information in comparison with inequalities (18) and we do not introduce the notion of Lyapunov dimension in this case (or, one can assign $d_L \equiv d_c$ and inequalities (21), (22) and (18) would formally coincide with one another).

Note a connection between the Lyapunov dimension and the well-known Shil'nikov conditions for the chaotic dynamics near a homoclinic loop of a saddle-focus. According to Shil'nikov [1965], if a three-dimensional system has a saddle-focus equilibrium state with the characteristic exponents $\gamma > 0$ and $-\lambda \pm i\omega$ ($\lambda > 0$, $\omega > 0$) and if it has a homoclinic loop to the saddle-focus (Fig. 1), then under the condition

$$\gamma > \lambda$$

the system has hyperbolic sets in any neighborhood of the loop. One can easily calculate that, for this problem, $d_L \geq 2$ if $\gamma > \lambda$, and $d_L = 1$ if $\gamma < \lambda$. So, the transition from simple dynamics (that takes place for $\gamma < \lambda$; see Shil'nikov [1963]) to chaotic dynamics happens when the Lyapunov dimension jumps from 1 to 2.

The next theorem solves the question on the existence of the non-local analogue of the center manifold near the contour C. In contrast with the case of local bifurcations, the existence of a d_c-dimensional

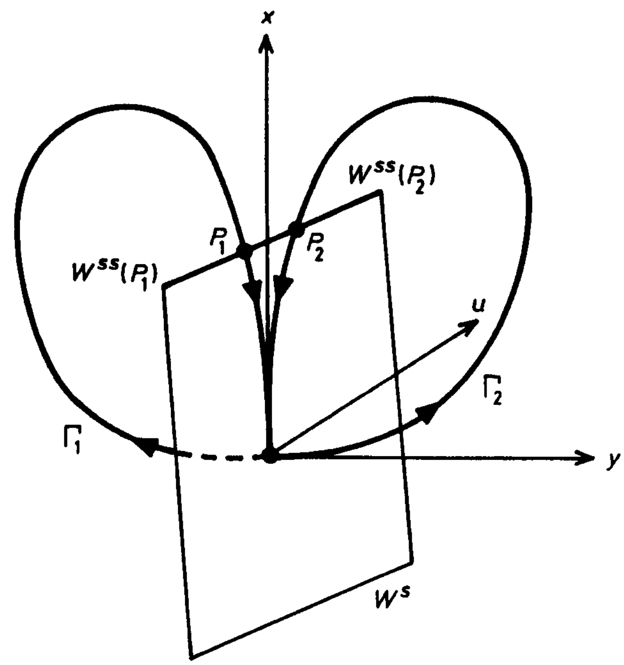

Fig. 21. The homoclinic butterfly composed of two loops Γ_1 and Γ_2 which does not satisfy the non-coincidence conditions: the strong-stable leaf of an arbitrary point $P_1 \in \Gamma_1$ lying near the equilibrium state coincides with the strong-stable leaf of some point P_2 where $P_2 \in \Gamma_2$.

smooth invariant manifold containing C and tangent to the space $\mathcal{N}^c$ at each point of C does not always take place here.

An example is given in Fig. 21: the homoclinic butterfly of a saddle equilibrium state with the characteristic exponents $(\gamma, -\lambda_1, -\lambda_2)$: $\gamma > 0 > -\lambda_1 > -\lambda_2$. Both separatrices form homoclinic loops returning to the equilibrium state and being tangent to each other and tangent to the leading direction which corresponds to the characteristic exponent $-\lambda_1$. If both separatrix values do not equal zero (see previous section), then the critical dimension is 2: the critical exponents are γ and $-\lambda_1$. Nevertheless, there is no two-dimensional smooth manifold containing both loops and tangent to the critical subspace at the equilibrium state. Indeed, the critical subspace is spanned onto the unstable eigen-direction and leading stable eigen-direction, whence the manifold is transverse to the strong-stable direction and it could intersect the strong-stable leaves at one point each. At the same time, the pair of homoclinic loops intersect the leaves at two points. Thus, the manifold cannot contain both loops.

Analogous considerations can be carried out for the general case. Suppose that the trichotomy property holds and the transversality conditions are fulfilled for some values k, m and n. Note that we do not assume here that k is taken as low as possible. Theorem 1 holds and, according to Corollary 1, for each point P of the contour C, there exist strong-stable and strong-unstable manifolds $\mathcal{W}_P^{ss}$ and $\mathcal{W}_P^{uu}$. We will say that *the non-coincidence conditions* are fulfilled:

> *If for any $P \in C$, the only point of intersection of the manifolds $\mathcal{W}_P^{ss}$ and $\mathcal{W}_P^{uu}$ with C is the point P itself.*

Let $p \leq r$ and $q \leq r$[8] be such integers that for each recurrent orbit $L_i \subset C$ the inequalities

$$\beta_i^s p < \beta_i^{ss} \tag{23}$$

and

$$\beta_i^u q < \beta_i^{uu} \tag{24}$$

hold where the β's are the values separating the spectrum of the characteristic exponents of L_i onto the three parts [see (8)].

Theorem 3. *The non-coincidence conditions are necessary and sufficient for the presence of a smooth k-dimensional invariant manifold $\mathcal{W}^c$ containing the contour C and tangent at each point of C to the critical subspace $\mathcal{N}^c$. The manifold $\mathcal{W}^c$ is the intersection of two invariant manifolds: C^p-smooth manifold $\mathcal{W}^{cu}$ and C^q-smooth manifold $\mathcal{W}^{cs}$ of dimensions $(n+k)$ and $(m+k)$, respectively. The invariant manifolds $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ exist for all nearby systems and depend on the system continuously. For any system close to X, the manifold $\mathcal{W}^{cu}$ contains all orbits staying in the neighborhood U for all negative times and the manifold $\mathcal{W}^{cs}$ contains all orbits staying in U for all positive times; the manifold $\mathcal{W}^c = \mathcal{W}^{cu} \cap \mathcal{W}^{cs}$ contains the set Ω of the orbits lying in U entirely. At each point of Ω, the manifold $\mathcal{W}^c$ is tangent to $\mathcal{N}^c$, the manifold $\mathcal{W}^{cu}$ to $\mathcal{N}^c \oplus \mathcal{N}^{uu}$ and the manifold $\mathcal{W}^{cs}$ to $\mathcal{N}^c \oplus \mathcal{N}^{ss}$.*

The non-coincidence conditions are easily verified for the first level contours. In this case, the manifolds $\mathcal{W}_P^{ss}$ and $\mathcal{W}_P^{uu}$ are the strong-stable and strong-unstable manifolds for the points on the recurrent orbits $L_i \subset C$ and they are the leaves of the strong-stable and strong-unstable foliations for the points on non-recurrent orbits $\Gamma_s \subset C$ (see Sec. 3). Thus, the non-coincidence conditions require, first,

[8]Recall that r is the smoothness of the system.

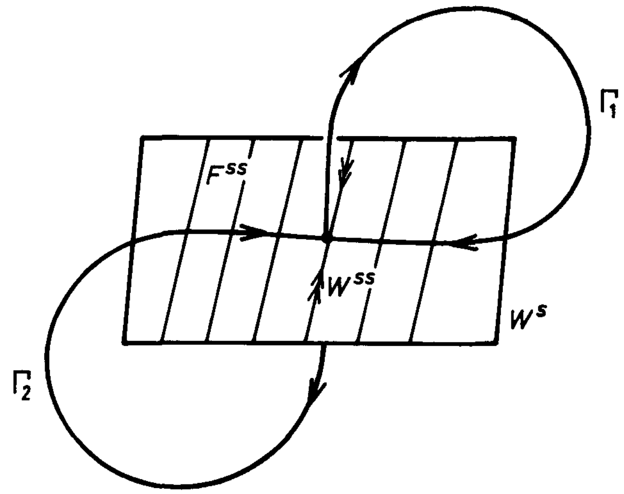

Fig. 22. The homoclinic figure-eight for which the non-coincidence conditions are fulfilled: the separatrix Γ_1 intersects only those strong-stable leaves which are not intersected by the separatrix Γ_2.

that none of the orbits Γ_s tends to its α- or ω-limit orbit L_i lying in the strong-unstable or, respectively, strong-stable manifold of L_i. Besides, if there is more than one non-recurrent orbit tending to a recurrent orbit L_i as $t \to +\infty$ or as $t \to -\infty$, they must intersect different leaves of the strong-stable or, respectively, the strong-unstable foliations. For example, the contours shown in Figs. 20 and 21 do not satisfy the non-coincidence conditions, and the contour in Fig. 22 (the homoclinic figure-eight) do satisfy.

In the coordinates which we have introduced in the previous section, the last condition can be reformulated as follows:

> *If L_i is a periodic orbit in the contour and the orbits $\Gamma_{s_1}, \Gamma_{s_2}, \ldots$ tend to L_i as $t \to +\infty$ (as $t \to -\infty$), then the different points of intersection of the orbits $\Gamma_{s_1}, \Gamma_{s_2}, \ldots$ with some small cross section to L_i must have different coordinates (x, z) [respectively, coordinates (y, z)]; if L_i is an equilibrium state, the coordinates (x, z) [respectively, (y, z)] must be different for the intersections of the orbits Γ_{s_1}, $\Gamma_{s_2}, \ldots$ with the cross section $\|(x, z)\| = const.$ ($\|(y, z)\| = const.$).*

4. Pseudohyperbolic Sets

The proof of Theorems 1 and 3 of the previous section is based on the notion of a pseudohyperbolic set. We use the following definition:

An invariant closed set Ω of a smooth flow on a smooth manifold is called *pseudohyperbolic* if

(1) at each point $P \in \Omega$ the tangent space is decomposed into the direct sum of three subspaces $\mathcal{N}_P^{ss} \oplus \mathcal{N}_P^{c} \oplus \mathcal{N}_P^{uu}$ depending continuously on P (the dimensions of the subspaces are assumed to be independent of P);

(2) the system of these subspaces is invariant with respect to the linearized flow: if P_t is the orbit of the point P, then

$$D(P)_t \mathcal{N}_P^{ss} = \mathcal{N}_{P_t}^{ss}, \quad D(P)_t \mathcal{N}_P^{c} = \mathcal{N}_{P_t}^{c}, \quad D(P)_t \mathcal{N}_P^{uu} = \mathcal{N}_{P_t}^{uu}$$

where $D(P)_t$ is the shift operator of the flow linearized along the orbit of the point P for the time t;

(3) there exists $T > 0$, independent of P, such that

$$\sup_{P\in\Omega} \|D_T^{ss}\| < 1, \quad \sup_{P\in\Omega} \|D_T^{ss}\|\,\|(D_T^{c})^{-1}\| < 1$$
$$\sup_{P\in\Omega} \|D_{-T}^{uu}\| < 1, \quad \sup_{P\in\Omega} \|D_{-T}^{uu}\|\,\|(D_{-T}^{c})^{-1}\| < 1 \tag{25}$$

where D_t^{ss}, D_t^{c} and D_t^{uu} denote the restrictions of the shift operator of the linearized flow onto the spaces $\mathcal{N}^{ss}$, $\mathcal{N}^{c}$ and $\mathcal{N}^{uu}$, respectively.

Item (3) of this definition gives a formal expression for the fact that the linearized flow is exponentially contracting on $\mathcal{N}^{ss}$, expanding on $\mathcal{N}^{uu}$ and expansion or contraction that may take place on $\mathcal{N}^{c}$ are weaker than those along directions belonging to $\mathcal{N}^{ss}$ and $\mathcal{N}^{uu}$.

Indeed, take an arbitrary point P and vectors $u \in \mathcal{N}_P^{ss}$, $v \in \mathcal{N}_P^{uu}$, $w \in \mathcal{N}_P^{c}$. For the time $t = sT$ where s is a natural number, the point P and vectors u, v and w are moved by the linearized flow into the vectors u_t, v_t, w_t:

$$u_t = D_t^{ss}(P)u, \quad v_t = (D_{-t}^{uu}(P_t))^{-1}v, \quad w_t = D_t^{c}(P)w = (D_{-t}^{c}(P_t))^{-1}w \tag{26}$$

Note that

$$D_t(P) = D_T(P_{(s-1)T}) \circ D_T(P_{(s-2)T}) \circ \cdots \circ D_T(P)$$
$$D_{-t}(P_t) = D_{-T}(P_T) \circ D_{-T}(P_{2T}) \circ \cdots \circ D_{-T}(P_{t=sT}) \tag{27}$$

By (26) and (27)

$$\|u_t\| \le (\sup\|D_T^{ss}\|)^s\|u\| \to 0,$$
$$\frac{\|u_t\|}{\|w_t\|} \le (\sup\|D_T^{ss}\|\,\|(D_T^{c})^{-1}\|)^s \frac{\|u\|}{\|w\|} \to 0$$
$$\|v_t\| \ge (\sup\|D_{-T}^{uu}\|)^{-s}\|v\| \to \infty,$$
$$\frac{\|v_t\|}{\|w_t\|} \ge (\sup\|D_{-T}^{uu}\|\,\|(D_{-T}^{c})^{-1}\|)^{-s} \frac{\|v\|}{\|w\|} \to \infty \tag{28}$$

as $s = \frac{t}{T} \to \infty$. We see that inequalities (25) imply that the norm of any vector $u \in \mathcal{N}^{ss}$ tends exponentially to zero under the action of the linearized flow; moreover, it tends to zero faster than the norm of any vector $w \in \mathcal{N}^{c}$. Analogously, the norm of any vector $v \in \mathcal{N}^{uu}$ grows, as $t \to -\infty$, exponentially and faster than the norm of any vector $w \in \mathcal{N}^{c}$. (Emphasize that the vectors u, v, w are erected at the same point and we do not compare the exponents characterizing the growth or decay of norms of vectors related to different points.)

Note that if to redefine the norm of vectors $u \in \mathcal{N}^{ss}$ in the following way

$$\|u\|_{new} = \int_0^\infty e^{\lambda^{ss}\tau}\|u_\tau\|_{old}d\tau \tag{29}$$

where $0 < \lambda^{ss} < -\frac{1}{T}\ln\sup\|D_T^{ss}\|$, then in the new norm

$$\|u_t\| \le e^{\lambda^{ss}t}\|u\| \tag{30}$$

for all $t \ge 0$ [the integral on the right-hand side of (30) converges since $\|u_\tau\| \le Ke^{-(\frac{1}{T}\ln\sup\|D_T^{ss}\|)t}$ for some $K > 0$ according to the first inequality of (28)]. Analogously, in the new norm

$$\|v\|_{new} = \int_0^\infty e^{\lambda^{uu}\tau}\|v_{-\tau}\|_{old}d\tau \tag{31}$$

for vectors $v \in \mathcal{N}^{uu}$ we have

$$\|v_t\| \ge e^{\lambda^{uu}t}\|v\| \tag{32}$$

(here $0 < \lambda^{uu} < -\frac{1}{T}\ln\sup\|D_{-T}^{uu}\|$).

For the vectors in $\mathcal{N}^{c}$, there can be shown to exist a pair of equivalent norms $\|w\|_1$ and $\|w\|_2$ such that

$$\sup_{u\in\mathcal{N}_P^{ss}} \frac{\|u_t\|}{\|u\|} \le e^{-\delta t} \inf_{w\in\mathcal{N}_P^{c}} \frac{\|w_t\|_1}{\|w\|_1} \tag{33}$$

and

$$\inf_{v\in\mathcal{N}_P^{uu}} \frac{\|v_t\|}{\|v\|} \ge e^{\delta t} \sup_{w\in\mathcal{N}_P^{c}} \frac{\|w_t\|_2}{\|w\|_2} \tag{34}$$

where δ is some positive constant. The presence of such a system of norms for which inequalities (30), (32), (33), (34) hold at each point $P \in \Omega$ can be considered as an alternative for item 3 of our definition of a pseudohyperbolic set. Note that inequalities (33) and (34) can be refined:

$$\sup_{u\in\mathcal{N}_P^{ss}} \frac{\|u_t\|}{\|u\|} \le e^{-\delta t} \inf_{u\in\mathcal{N}_P^c} \left(\frac{\|w_t\|_1}{\|w\|_1}\right)^p \tag{35}$$

$$\inf_{u\in\mathcal{N}_P^{uu}} \frac{\|v_t\|}{\|v\|} \ge e^{\delta t} \sup_{u\in\mathcal{N}_P^c} \left(\frac{\|w_t\|_2}{\|w\|_2}\right)^q \tag{36}$$

where $p \ge 1$ and $q \ge 1$ are integers such that

$$\sup_{P\in\Omega} \|D_T^{ss}\| \cdot \|(D_T^c)^{-1}\|^p < 1 \tag{37}$$

$$\sup_{P\in\Omega} \|D_{-T}^{ss}\| \cdot \|(D_{-T}^c)^{-1}\|^q < 1 \tag{38}$$

[compare with inequalities (25)].

In the present paper we do not develop the theory of pseudohyperbolic sets with formal proofs but we restrict ourselves to the presentation of a logical scheme. The central result is given by the following theorem.

Theorem 4. *Let a system X have a pseudohyperbolic invariant set C. Then there exists a neighborhood U of C such that, for the system X itself and for any system close to X, the set Ω of all orbits lying in U entirely is also pseudohyperbolic (with the same dimensions of the subspaces $\mathcal{N}^{ss}$, $\mathcal{N}^c$ and $\mathcal{N}^{uu}$ as for the set C).*

According to this theorem, in order to establish when the set of orbits lying in a small neighborhood of some contour is pseudohyperbolic (Theorem 1), it is sufficient to find out when such a contour is pseudohyperbolic itself. Since the contour consists of a finite number of orbits, checking its pseudohyperbolicity is fairly easy. We use the two following lemmas.

Lemma 1. *If C is a pseudohyperbolic invariant set and if Γ is an orbit whose ω-limit set is contained in C, then at each point $P \in \Gamma$, in the tangent space, there exists a pair of subspaces N_P^{ss} and N_P^{cs} ($N_P^{ss} \subset N_P^{cs}$, $\dim N_P^{ss} = m$, $\dim N_P^{cs} = m + k$) invariant with respect to the linearized flow. The subspaces N^{ss} and N^{cs} are uniquely defined by the "continuity condition":*

> *If $P^* \in C$ is a limit of a sequence of points $\{P_i\}$ of the orbit Γ, then the spaces $N_{P_i}^{ss}$ and $N_{P_i}^{cs}$ accumulate, respectively, on the space $\mathcal{N}_{P^*}^{ss}$ and on the space $\mathcal{N}_{P^*}^{ss} \oplus \mathcal{N}_{P^*}^{c}$.*

We will say that the spaces $N_{P_i}^{ss}$ and $N_{P_i}^{cs}$ are *taken out* of C. The subspaces N_P^{ss} and N_P^{cs} are composed exactly of those vectors which, shifted by the linearized forward flow, do not approach the spaces $\mathcal{N}_{P^*}^c \oplus \mathcal{N}_{P^*}^{uu}|_{P^*\in C}$ and $\mathcal{N}_{P^*}^{uu}|_{P^*\in C}$, respectively. The existence and uniqueness of these subspaces is a standard result for the theory of normal hyperbolicity and we do not discuss the proof here.

The analogous statement also takes place for the orbits which tend to C as $t \to -\infty$: if Γ is an orbit whose α-limit set is contained in C, then at each point $P \in \Gamma$, in the tangent space there exists an n-dimensional subspace N_P^{uu} and $(n+k)$-dimensional subspace N_P^{cu} ($N_P^{uu} \subset N_P^{cu}$) such that the system of these subspaces is invariant with respect to the linearized backward flow. The corresponding continuity condition defines N_P^{uu} and N_P^{cu} uniquely.

These results can be improved in the sense that the existence of uniquely defined subspaces N^{ss} and N^{cs} (or N^{uu} and N^{cu}) can be established not only for the orbits which tend to C as $t \to +\infty$ or $t \to -\infty$, but also for the orbits which stay in some sufficiently small neighborhood U of C for all positive (or, respectively, negative) times. In this case, the invariant subspaces N^{ss} and N^{cs} consist of the vectors which, shifted by the linearized forward flow do not come too close (in some definite sense) to the subspaces, respectively, $\mathcal{N}^{uu} \oplus \mathcal{N}^c$ and $\mathcal{N}^{uu}$ on C. The invariant subspaces N^{uu} and N^{cu} consist of the vectors which remain, as $t \to -\infty$, not very close to the subspaces, respectively, $\mathcal{N}^{ss} \oplus \mathcal{N}^c$ and $\mathcal{N}^{ss}$ on C.

This construction is a main element of the proof of Theorem 4: at each point P of any orbit Γ lying in a sufficiently small neighborhood U of C for all (positive and negative) times, there exist four subspaces N_P^{ss}, N_P^{cs}, N_P^{uu} and N_P^{cu} and one can check that the system of subspaces $\{\mathcal{N}_P^{ss} = N_P^{ss}$, $\mathcal{N}_P^c = N_P^{cs} \cap N_P^{cu}$, $\mathcal{N}_P^{uu} = N_P^{uu}\}$ gives a decomposition of the tangent space for which pseudohyperbolicity condition (25) holds (maybe for some longer T).

Let C_1 and C_2 now be a pair of pseudohyperbolic sets with the same values of k, m and n and

Γ be an orbit α-limit to C_1 and ω-limit to C_2. At any point P of Γ, there exists a pair of subspaces N_P^{ss} and N_P^{cs} taken out of C_2 and a pair of subspaces N_P^{uu} and N_P^{cu} taken out of C_1.

Lemma 2. *If at any point P of Γ the space N_P^{uu} is transverse to N_P^{cs} and the space N_P^{ss} is transverse to N_P^{cu}, then the set $C_1 \cup C_2 \cup \Gamma$ is pseudohyperbolic with the same m, n and k as the sets C_1 and C_2.*

This lemma is proved by the direct construction of the corresponding decomposition of the tangent space: for the points of the sets C_1 and C_2, the subspaces $\mathcal{N}^{ss}$, $\mathcal{N}^c$ and $\mathcal{N}^{uu}$ are taken without change, and for the points on Γ the decomposition is $\{\mathcal{N}^{ss} = N^{ss}, \mathcal{N}^c = N^{cs} \cap N^{cu}, \mathcal{N}^{uu} = N^{uu}\}$. This is a decomposition into the direct sum indeed, because by the transversality of N^{cs} with $N^{uu} \subset N^{cu}$ and N^{cu} with $N^{ss} \subset N^{cs}$, any vector of the tangent space is uniquely represented as a sum of three vectors belonging, respectively, to N^{ss}, $N^{cs} \cap N^{cu}$ and N^{uu}.

To prove Lemma 2, one should verify the continuity of the decomposition constructed, but this is rather easy. For instance, if we take the subspace N^{ss}, it tends, by definition, to the spaces $\mathcal{N}^{ss}$ on C_2 as $t \to +\infty$. As $t \to -\infty$, it tends to the spaces $\mathcal{N}^{ss}$ on C_1, because all the vectors which do not tend, as $t \to -\infty$, to vectors of the spaces $\mathcal{N}^{ss}$ on C_1, belong to the subspace N^{cu} which is transverse to N^{ss} and it has not, therefore, non-zero intersections with N^{ss}.

The pseudohyperbolicity conditions are fulfilled (by definition) for the points of C_1 and C_2. By the continuity of decomposition, this implies that pseudohyperbolicity conditions (25) are also fulfilled for the points on Γ close to C_1 or C_2. Since Γ spend only a finite time outside small neighborhoods of C_1 and C_2, one can now achieve the fulfillment of conditions (25) for each point on Γ, taking T large enough.

This lemma gives us a tool for establishing the pseudohyperbolicity of contours composed of a finite number of orbits. If for some contour, the trichotomy property holds (see Sec. 3), then each recurrent orbit L_i in the contour is easily seen to be a pseudohyperbolic set. For the first-level orbits Γ_s which are asymptotic to the recurrent orbits as $t \to \pm\infty$, the subspaces N^{ss}, N^{cs}, N^{cu} and N^{uu} are uniquely defined by continuity conditions (we discussed this in Sec. 3 in greater detail). Applying Lemma 2 to each first-level orbit Γ_s one can see that if the transversality conditions hold for all these orbits, then the union of the recurrent and the first-level orbits is a pseudohyperbolic set.

Now, we can derive from Lemma 1 (and from its analogue for the case $t \to -\infty$) the existence and uniqueness of the spaces N^{ss}, N^{cs}, N^{cu} and N^{uu} for each second-level orbit in the contour. Again, applying Lemma 2, one can see that if the transversality conditions hold for all second-level orbits, then the union of zero-, first- and second-level orbits is pseudohyperbolic. This procedure can be continued from level to level and it stops after a finite number of steps because the number of orbits in the contour is finite. We see that:

> *If the trichotomy property holds for each recurrent orbit in the contour C and if the transversality conditions are satisfied for each non-recurrent orbit in C, then the contour C is a pseudohyperbolic set.*

According to Theorem 4, this statement gives Theorem 1.

Theorem 3 of the previous section is based on the notion of a *properly situated* pseudohyperbolic set. We saw in Sec. 4 for example of the homoclinic butterfly that an arbitrary pseudohyperbolic set may not be embedded into a k-dimensional smooth manifold tangent to the spaces $\mathcal{N}^c$. In this connection we introduce the following definition.

Definition. A pseudohyperbolic set C is called *properly situated* if it lies in a smooth k-dimensional manifold $\mathcal{M}^c$ which is tangent to the space $\mathcal{N}_P^c$ at each point $P \in C$.

The manifold $\mathcal{M}^c$ is not assumed to be invariant, but the presence of at least one such manifold is a trivial necessary condition for the presence of the invariant "center" manifold. The following theorem shows that this condition is sufficient also.

Theorem 5. *Let a C^r-smooth ($r \geq 1$) dynamical system X have a properly situated pseudohyperbolic set C. Let $p \leq r$ and $q \leq r$ be integers for which inequalities (37) and (38) are fulfilled. Then, for the system X itself and for any nearby system, there exists a pair of invariant C^q-manifold W^{cs} and C^p-manifold W^{cu} of dimensions $(m+k)$ and $(n+k)$, respectively. The manifold W^{cs} contains all orbits staying in a small neighborhood U of C for all positive times and the manifold W^{cu} contains all orbits* *staying in U for all negative times. The intersection $W^c = W^{cs} \cap W^{cu}$ is a k-dimensional $C^{\min(p,q)}$-manifold containing the set Ω of all orbits lying in U entirely. At each point of Ω, the manifold W^{cs} is tangent to $\mathcal{N}^c \oplus \mathcal{N}^{ss}$, the manifold W^{cu} is tangent to $\mathcal{N}^c \oplus \mathcal{N}^{uu}$ and W^c is tangent to $\mathcal{N}^c$.*

This theorem is a generalization of the main theorem of Hirsch *et al.* [1977], where the analogous result was proved for the case where the set C is a smooth manifold. Nevertheless, the machinery of Hirsch *et al.* [1977] still works in our case where C is a compact subset of an appropriate smooth manifold.

In very few words, the scheme of the proof is as follows. Due to the compactness of C, one can take a sufficiently dense finite set $\tilde{C}$ of points in C and then construct a smooth manifold $\mathcal{M}^{cs}$ containing the points of the set $\tilde{C}$ and tangent to $\mathcal{N}^c \oplus \mathcal{N}^{ss}$ at these points.

Using inequalities (30), (32), and (36) one could prove with the standard technique that the shift $\mathcal{M}^{cs}_{-t}$ of the manifold $\mathcal{M}^{cs}$ by the backward flow has, as a C^q-limit, an invariant manifold W^{cs}, if to prevent the possible "shrinking" of the manifold $\mathcal{M}^{cs}_{-t}$.

A simple example of such shrinking is given in Fig. 23: this is an unstable equilibrium state on the plane with one positive characteristic exponent and one exponent equal to zero. The first Lyapunov value is taken equal to zero and the second Lyapunov exponent is positive. The vector field for this example is

$$\dot{y} = y\,, \quad \dot{x} = x^3$$

The straight line $\{y = 0\}$ is a center-stable manifold W^{cs}. If we take here a piece of a curve tangent to $\{y = 0\}$ at the origin, its iterations with the backward flow will become smaller and smaller, not approaching a finite piece of W^{cs} but merging into the origin.

To avoid such behavior of the iterations $\mathcal{M}^{cs}_{-t}$, it is convenient to redefine the system outside a small neighborhood of C. First, by a small variation of the vector field one can make the initial manifold $\mathcal{M}^{cs}$ be invariant with respect to the flow outside a small neighborhood of C. Then, multiplying the vector field on a scalar function that vanishes on $\mathcal{M}^{cs}$ outside the small neighborhood of C, points of that part of $\mathcal{M}^{cs}$ can be made equilibrium states. This implies that the iteration $\mathcal{M}^{cs}_{-t}$ will coincide with $\mathcal{M}^{cs}$ outside the small neighborhood of C and this prevents the shrinking. In this way, the existence of W^{cs} is proved and the existence of W^{cu} is proved analogously, due to the symmetry of the problem.

To apply Theorem 5 for the proof of Theorem 3 of Sec. 4, we use the construction analogous to Lemma 2. First, we call a pseudohyperbolic set $(p,\, q)$-*pseudohyperbolic* if conditions (37) are fulfilled for given integers p and q. Note that if in Lemma 2 the sets C_1 and C_2 are $(p,\, q)$-pseudohyperbolic, then the set $C_1 \cup C_2 \cup \Gamma$ is $(p,\, q)$-pseudohyperbolic too.

Suppose the sets C_1 and C_2 and the orbit Γ are as in Lemma 2. If the transversality conditions hold for the orbit Γ, then the set $C_1 \cup C_2 \cup \Gamma$ is pseudohyperbolic and the orbit Γ possesses a uniquely defined strong-stable and strong-unstable invariant manifolds $\mathcal{W}^{ss}(\Gamma)$ and $\mathcal{W}^{uu}(\Gamma)$ (see Corollary 1 in Sec. 4).

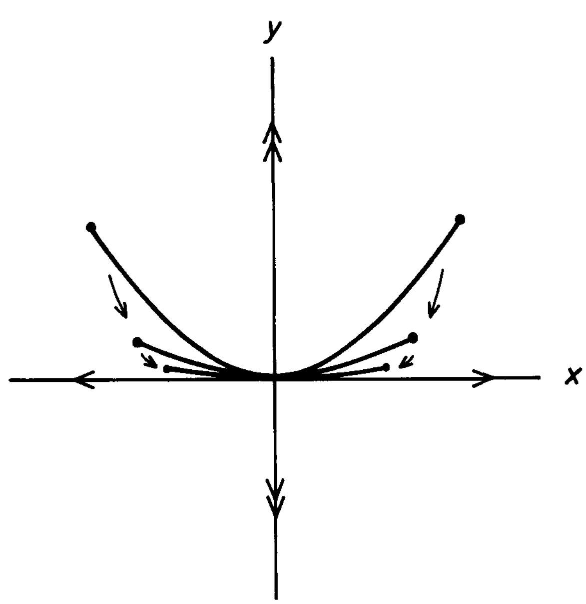

Fig. 23. An example of shrinking of the iterations of the manifold $\mathcal{M}^{cs}$: as $t \to \infty$ any finite piece of $\mathcal{M}^{cs}_{-t}$ merges into the origin.

Lemma 3. *Suppose the sets C_1 and C_2 are properly situated. In order for the set $C_1 \cup C_2 \cup \Gamma$ to be properly situated, it is necessary and sufficient that $\mathcal{W}^{ss}(\Gamma)$ would not contain orbits of C_2 and $\mathcal{W}^{uu}(\Gamma)$ would not contain orbits of C_1.*

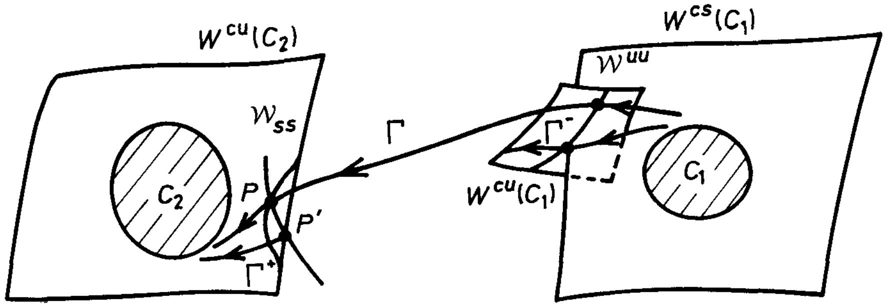

Fig. 24. An illustration to Lemma 3: if the orbit Γ^+ does not lie in $W^{cu}(C_1)$, then the manifold $W^{cu}(C_1)$ can be modified so as to include the orbit Γ, and it is impossible for $\Gamma^+ \subseteq W^{cu}(C_1)$.

The necessity is almost evident. Indeed, if the set $C_1 \cup C_2 \cup \Gamma$ is properly situated, then there exist invariant manifolds W^{cs} and W^{cu} containing this set. Suppose, for instance, $\mathcal{W}^{ss}(\Gamma)$ contains an orbit $\Gamma^+ \subset C_2$. By definition, the manifold $W^{cu}(C_1 \cup C_2 \cup \Gamma)$ contains both the orbits Γ and Γ^+. If P is a point on Γ, then the strong-stable manifold $\mathcal{W}_P^{ss}$ intersects Γ^+ at some point P'. Thus, the manifold W^{cu} intersects the manifold $\mathcal{W}_P^{ss}$ at more than one point, but this is impossible (like in the example with a homoclinic butterfly in Sec. 4): as t grows, the points P and P' become arbitrarily close to each other, and the manifold W^{cu} which is transverse[9] to the manifold $\mathcal{W}_P^{ss}$ cannot intersect $\mathcal{W}_P^{ss}$ in two close points.

The sufficience is a simple geometrical fact. The sets C_1 and C_2 are properly situated and, by Theorem 5, there exist the invariant manifolds $W^{cs}(C_1)$, $W^{cu}(C_1)$ and $W^{cs}(C_2)$, $W^{cu}(C_2)$. The intersection of the invariant manifolds $\mathcal{W}^{ss}(\Gamma)$ and $\mathcal{W}^{cu}(C_2)$ is an orbit which we denote as Γ^+. The intersection of $\mathcal{W}^{uu}(\Gamma)$ and $\mathcal{W}^{cs}(C_1)$ is an orbit Γ^-. Since $\mathcal{W}^{ss}(\Gamma)$ do not intersect C_2, the distance between points on Γ and Γ^+ decays faster than the distance to the set C_2; i.e., the distance between Γ and $W^{cu}(C_2)$ decays faster than the distance between Γ and C_2. This allows one to modify slightly the manifold $W^{cu}(C_2)$ in a small neighborhood of Γ^+ so that the new manifold would include the piece of the orbit Γ which lies near C_2 (see Fig. 24). The orbit Γ spends only a finite time outside small neighborhoods of the sets C_1 and C_2, and the modified manifold $W^{cu}(C_2)$ can easily be sewed together with the manifold $W^{cu}(C_1)$ which contains, by definition, the piece of Γ that lies in the small neighborhood of C_1, so that a manifold $\mathcal{M}^{cu}$ (non-invariant) is obtained containing the orbit Γ as well as the sets C_1 and C_2, and tangent to N^{cs}. Analogously, a manifold $\mathcal{M}^{cs}$ is constructed, and the intersection $\mathcal{M}^{cs} \cap \mathcal{M}^{cu}$ gives the required manifold $\mathcal{M}^c$, containing $\Gamma \cup C_1 \cup C_2$ and tangent to N^c.

Theorem 3 of Sec. 4 can now be proved as follows. Note that each recurrent orbit L_i in the finite contour C is a properly situated (p, q)-pseudohyperbolic set[10] with p and q from inequalities (23) and (24). By Lemma 3, if the non-coincidence condition (see Sec. 4) is fulfilled for all first-level orbits $\Gamma \in C$ (as well as the transversality conditions), then the union of all first- and zero-level orbits is a properly situated (p, q)-pseudohyperbolic set also. Next, we obtain that if the non-coincidence condition is fulfilled for all second-level orbits, then the union of all second-, first- and zero-level orbits is also a properly situated (p, q)-pseudohyperbolic set and so on. Since the contour contains only a finite number of levels, we get that the fulfillment of the non-coincidence conditions is sufficient (and necessary as well) for the contour to be properly situated. By virtue of Theorem 5, this gives Theorem 3.

5. Concluding Remarks: Hyperchaotic Contours

In this section, we pay special attention to an example of a contour for which the Lyapunov dimension can be made arbitrarily high. We show that this leads to quite non-trivial dynamics which, presumably, may be typical for high-dimensional systems.

[9] It is tangent to the space $\mathcal{N}^{cu}$ which is transverse, by definition, to the tangent $\mathcal{N}^{ss}$ to $\mathcal{W}^{ss}$.

[10] Provided that the trichotomy property holds.

Let the dimension of the phase space be even and equal $2s$ for some integer $s > 0$.[11] Consider a contour C (Fig. 14) containing two periodic orbits L_1 and L_2 and two heteroclinic orbits $\Gamma_1 \subseteq W_1^u \cap W_2^s$ and $\Gamma_2 \subseteq W_2^u \cap W_1^s$ such that W_1^u has a tangency with W_2^s along the orbit Γ_1 and W_2^u intersects W_2^s transversely at the points of Γ_2. Suppose that multipliers of L_1 are

$$\gamma > 1\,, \quad \lambda_1 e^{\pm i\omega_1}, \ldots, \lambda_{s-1} e^{\pm i\omega_{s-1}}$$

where

$$1 > \lambda_1 > \cdots > \lambda_{s-1} > 0$$

and multipliers of L_2 are

$$\alpha > 1\,, \quad \delta_1,\, \delta_2 e^{\pm i\phi_2}, \ldots, \delta_{s-1} e^{\pm i\phi_{s-1}},\, \delta_s$$

where

$$1 > \delta_1 > \delta_2 > \cdots > \delta_{s-1} > \delta_s > 0$$

Since only one of the multipliers lies outside the unit circle for each orbit L_i, the unstable manifolds of L_i are one-dimensional.

Suppose also that the product of multipliers of L_1 is less than unity and the product of multipliers of L_2 is greater than unity:

$$\gamma\lambda_1^2 \cdots \lambda_{s-1}^2 < 1$$

$$\alpha\delta_1\delta_2^2 \cdots \delta_{s-1}^2 \delta_s > 1$$

Systems with such kinds of contours form codimension one surfaces in the space of dynamical systems; i.e., such contours can occur in general one-parameter families of dynamical systems. Moreover, the presence of such a contour is, in a sense, a persistent phenomenon. The last is connected with the well-known persistence of homoclinic tangencies (structurally unstable Poincaré homoclinic orbits) which was discovered by Newhouse. It is proved in Newhouse [1979] for two-dimensional diffeomorphisms and in Gonchenko *et al.* [1993a] for a general multi-dimensional case (see also Palis & Viana [1992]) that in the space of dynamical systems, in any neighborhood of any system having a saddle periodic orbit with a homoclinic orbit along which the stable and unstable manifolds are tangent, there exist open regions (*Newhouse regions*) where systems are dense for which the periodic orbit has new orbits of a homoclinic tangency. Absolutely analogously, the following assertion is established:

> *In the space of dynamical systems, in any neighborhood of the system with the contour under consideration, there exist open sets (the Newhouse regions) where systems having a contour of such kind (i.e., systems for which $W^u(L_2)$ intersects $W^s(L_1)$ and $W^u(L_1)$ has a tangency with $W^u(L_2)$) are dense.*

The critical dimension for such a contour equals the dimension of the phase space. Indeed, to satisfy the exponential trichotomy property, the spectra of characteristic exponents of L_1 and L_2 must admit a decomposition onto three parts. For such decomposition, the center part for the orbit L_1 would contain the following characteristic exponents (see Sec. 3): $\ln\gamma > 0$, zero (trivial) characteristic exponent and a number of pairs of complex conjugate characteristic exponents $\ln\alpha_j \pm i\omega_j$. Therefore, the dimension of the center eigenspace of L_1 is even. Analogously, for the orbit L_2, the center part of the spectrum might contain $\ln\alpha > 0$, zero, $\ln\delta_1 < 0$ and a number of pairs of complex conjugate characteristic exponents $\ln\delta_j \pm i\phi_j$. In this case the dimension of the center eigenspace of L_2 would be odd, but this is a contradiction to the trichotomy property according to which the center eigenspaces must have equal dimensions for all orbits L_i. The only possibility to make the dimension of the center eigenspace of L_2 even is to include all characteristic exponents in the center part of the spectrum (the left and right parts of the spectrum are empty). Thus, for the contours under consideration, the minimal dimension of the center subspace $\mathcal{N}^c$ equals the dimension of the phase space; i.e., $d_c = 2s$ here.

Since the product of multipliers of L_1 is less than unity and the product of multipliers of L_2 is greater than unity, the divergence in L_1 is negative and positive in L_2. Thus, the Lyapunov dimension d_L equals the critical dimension and it is equal to the dimension of the phase space. The realization conjecture can be proved in this case; i.e., the following result takes place (we postpone the proof for a forthcoming paper):

> *A periodic orbit with $(2s-1)$ non-trivial multipliers equal to unity can be born in an arbitrarily small neighborhood of the contour by a small perturbation of the system.*

[11] The odd-dimensional case can be considered in an analogous way.

This "realization theorem" implies, in particular, that structurally stable periodic orbits with arbitrary dimensions of the unstable manifold (from 1 to $2s$) can be born at the bifurcations of the contour C under consideration (the case $\dim W^u = 1$ corresponds to an attractive periodic orbit, *a sink*, the case $2 \leq \dim W^u \leq 2s - 1$ to different types of saddle periodic orbits and the case $\dim W^u = 2s$ corresponds to a completely unstable periodic orbit, *a source*). Moreover, the following assertion is valid:

In a small neighborhood of the system with the contour C, in the Newhouse regions there exists a dense set of systems any of which possesses, simultaneously for each $j = 1, \ldots, 2s$, an infinite number of periodic orbits with $\dim W^u = j$.

The proof is analogous to the well-known Newhouse proof of density of systems with infinitely many sinks in open regions [Newhouse, 1974]. By definition, in an arbitrarily small neighborhood of any system in the Newhouse region there exists a system with a contour of the kind under consideration. According to the realization theorem, near such a system there exists a system having a periodic orbit with $(2s-1)$ non-trivial multipliers equal to unity. By a small perturbation, any given number of the unit multipliers can be moved outside the unit circle and the others can be moved inside. Hence, the system can be perturbed so that a structurally stable periodic orbit $\mathcal{L}_1$ with $\dim W^u = j_1$ arise for an arbitrary prescribed j_1. This orbit retains for all close systems, and among them there exists a system with a new contour of the given kind (since we are still in the Newhouse region). By a small perturbation of the last system one can achieve, in addition to the orbit $\mathcal{L}_1$, a birth of one more structurally stable periodic orbit $\mathcal{L}_2$ with $\dim W^u = j_2$ for an arbitrary j_2. Repeating the procedure, for an arbitrary sequence $\{j_i\}$ of integers $j_i \in \{1, \ldots, 2s\}$, a system can be found in arbitrary closeness to the initial system which has a sequence of periodic orbits $\mathcal{L}_i$ such that $\dim W^u(\mathcal{L}_i) = j_i$.

In this proof, each periodic orbit is born in a small neighborhood of some contour containing the orbits L_1 and L_2. The size of these neighborhoods can be taken to be smaller and smaller with each step of the inductive procedure, so that the closure of any subsequence of the constructed sequence of the periodic orbits $\mathcal{L}_i$ contains the orbits L_1 and L_2. In particular, we can take this subsequence such that $\dim W^u(\mathcal{L}_i) = 1$ or such that $\dim W^u(\mathcal{L}_i) = 2s$. Thus, we arrive at the following corollary:

In a small neighborhood of the system with the contour C, in the Newhouse regions there exists a dense set of systems for which the closure of sinks has a non-empty intersection with the closure of sources.

It is not clear how to introduce correctly the notion of *attractor* for such kinds of systems, but the previous statement shows that the attractor here may happen to contain sources as well as the repeller may happen to contain sinks.

These results have a direct relation to the problem of *hyperchaos*. Usually, those attractors are called hyperchaotic for which more than one positive Lyapunov exponent is found. As we see, for the contour under consideration, the number of positive Lyapunov exponents may vary for different orbits if the system belongs to a Newhouse region. It is not clear, therefore, in what sense the number of positive Lyapunov exponents can be considered as a characteristic of such a system as a whole. For instance, if the system has an absorbing domain containing the contour, then all the periodic orbits appearing in a small neighborhood will belong to the maximal attractor whose dimension, estimated from below by the maximal dimension of the unstable manifolds of the periodic orbits, is therefore equal to the dimension $2s$ of the phase space. This holds true though all the orbits lying in the neighborhood of the contour spend most of the time near the "basic" orbits L_1 and L_2 the dimension of the unstable manifolds of which is very low (it is equal to 2). Nevertheless, if one made calculations by the Kaplan–Yorke formula [Kaplan & Yorke, 1979] with characteristic exponents of L_2, this would give the proper value $2s$ for the dimension of the maximal attractor and not only an estimate from above as usual.

Analogous results can be established, evidently, in a neighborhood of any contour which is (1) *persistent*, for which (2) *the realization conjecture is valid* and (3) $d_L > 1$. For instance, this is true for multi-dimensional systems possessing orbits of homoclinic tangency as it was shown in Gonchenko *et al.* [1993b, 1995]. The example that we consider here provides additional arguments for the point of view proposed in Gonchenko *et al.* [1995] that:

Coexistence of orbits with the different numbers of positive Lyapunov exponents is a general property of systems with high-dimensional attractors.

Note that, by definition, the quantity d_L which we call the Lyapunov dimension of the problem is none other than the integral part of the maximum of the Lyapunov dimensions calculated at the orbits L_i by the Kaplan–Yorke formula for the restriction of the linearized flow onto the center subspace N^c. If the realization conjecture is valid for some contour, then in its small neighborhood there may appear orbits with d_L-dimensional unstable manifolds. This means that, in the general case, the Kaplan–Yorke formula (modified so that taken into account the factorization along strong-stable and strong-unstable foliations) seem to give a proper estimate for the dimension of the maximal attractor (compare with Douady & Oesterle [1980]; Il'yashenko [1982]; Babin & Vishik [1983]).

Another important consequence of validity of the realization conjecture would be that if the Lyapunov dimension d_L is sufficiently high for some contour, then the study of the bifurcations in its neighborhood is a multi-parameter problem: according to the realization conjecture, periodic orbits having $(d_L - 1)$ non-trivial multipliers on the unit circle may appear in a neighborhood of the contour and the study of the bifurcations of these orbits requires at least $(d_L - 1)$ independent control parameters (in spite of the fact that the contours under consideration may occur, for instance, in general one-parameter families of dynamical systems). Therefore, if d_L is large, the detailed bifurcation analysis is, evidently, impossible. For such kinds of contours the determination of the dimension of the problem (which includes not only the calculation of the quantities d_c and d_L but requires also a proof of some kinds of realization theorems) is not just a preliminary step like in the local bifurcation theory, but it is, presumably, a final step of the investigation.

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