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NORMAL FORMS AND LORENZ ATTRACTORS

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Normal forms for eleven cases of bifurcations of codimension-3 are considered, basically, in systems with a symmetry, which can be reduced to one of the two three-dimensional systems. The first system is the well-known Lorenz model in a special notation, the second is the Shimizu-Morioka model. In contrast with two-dimensional normal forms which admit, in principle, a complete theoretical study, in three-dimensional systems such analysis is practically impossible, except for particular parameter values when a system is close to an integrable system. Therefore, the main method of the investigation is qualitatively-numerical. In that sense, a description of principal bifurcations which lead to the appearance of the Lorenz attractor is given for the models above, and the boundaries of the regions of the existence of this attractor are selected.

We pay special attention to bifurcation points corresponding to a formation of a homoclinic figure-8 of a saddle with zero saddle value and that of a homoclinic figure-8 with zero separatrix value. In L. P. Shil'nikov [1981], it was established that these points belong to the boundary of the existence of the Lorenz attractor. In the present paper, the bifurcation diagrams near such points for the symmetric case are given and a new criterion of existence of the nonorientable Lorenz is also suggested.

1. Introduction

It is well known that a local bifurcation analysis is based upon a consideration of a normal form on the center manifold. An advantage of the normal-form method is that the normal-form system is determined by the character of the bifurcation rather than the specific features of the equations under consideration. It is also important to note that the dimension of the space of solutions of the original equations is not correlated with the dimension of the normal-form system which depends only on the number of characteristic exponents lying on the imaginary axis.

The local bifurcations of codimension less than 3 generate only one or two-dimensional normal forms which have been well studied to date (see Afraimovich *et al.* [1989], Guckenheimer & Holmes [1986] for details). The essential distinction of multi-dimensional normal forms is the possibility of chaotic behavior. It was shown in Arneodo *et al.* [1985] that in the case of a bifurcation of an equilibrium with three zero eigenvalues and a complete Jordan block there can arise spiral chaos associated with a homoclinic loop to a saddle-focus. Spiral chaos in a normal form was also found in concrete PDE's describing a convection in a rotating layer of salt fluid [Arneodo *et al.*, 1984]. In this list the work of Vladimirov & Volkov [1991] should be mentioned where one of normal forms for bifurcations of a zero-intensity state of the LSA[1] model [Lugiato *et al.*, 1978; Abraham *et al.*, 1988] was established to be the well-known Shimizu–Morioka model [Shimizu & Morioka, 1976] which has a strange attractor of the Lorenz type [A. L. Shil'nikov, 1986]. We emphasize that the approach based on the reduction to the center

[1]Laser with a saturable absorber

manifold is very promising for the rigorous proof of chaos in multiparameter problems, particularly in magneto and hydrodynamics.

Normal forms which can be reduced to the Lorenz model in some canonical notation, are here considered. They are associated with codimension-3 bifurcations of equilibrium states and periodic motions in systems with a symmetry. We shall consider three-parameter families of such systems which are assumed to have an equilibrium state with either three zero eigenvalues or two zero and a pair of pure imaginary eigenvalues. The often roots of the characteristic equation are supposed to lie to the left of the imaginary axis, which allows the problem to be reduced onto a center manifold. Also periodic orbits which have three multipliers equal to either $+1$ or -1, or a pair of complex-conjugate multipliers on the unit circle together with two multipliers equal to either $+1$ or -1 are considered.

We shall show that normal forms for such bifurcations can be reduced by rescaling the phase and time variables to the system

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= x(1-z) - Bx^3 - \lambda y\,, \\ \dot{z} &= -\alpha(z - x^2)\,. \end{aligned} \tag{1}$$

Here α and λ are rescaled bifurcation parameters and may take arbitrary values. Parameter B is determined only from the coefficients of a Taylor expansion at the bifurcation point and its value remains unchanged when the small parameters vary.

System (1) is also remarkable in that the Lorenz model [E. Lorenz, 1963]

$$\begin{aligned} \dot{x} &= -\sigma(x - y)\,, \\ \dot{y} &= rx - y - xz\,, \\ \dot{z} &= -bz + xy\,, \end{aligned}$$

is reduced to it when $r > 1$. The connection between parameters of the two systems is

$$\begin{aligned} \alpha &= b/\sqrt{\sigma(r-1)}\,, \\ \lambda &= (1+\sigma)/\sqrt{\sigma(r-1)}\,, \\ B &= b/(2\sigma - b)\,. \end{aligned} \tag{2}$$

It follows from (2) that the region of the Lorenzian parameters is bounded by the plane $\alpha = 0$ and the surface $\frac{\lambda}{\alpha} = \frac{1}{2}\left(\frac{1}{B} + 1\right)$ which tends to $\alpha = 0$ as $B \to 0$.

We note also that the particular case of system (1) at $B = 0$ is the Shimizu–Morioka model

$$\begin{aligned} x &= y\,, \\ y &= x(1-z) - \lambda y\,, \\ z &= -\alpha z + x^2\,. \end{aligned} \tag{3}$$

To verify this, one can make the transformations $x \to x/\sqrt{\alpha}$, $y \to y\sqrt{\alpha}$.

We shall show that for each $B > -1/3$ in the sector $\alpha > 0$, $\lambda > 0$ there is a region V_{LA} of existence of the Lorenz attractor. The idea of the proof is to find the bifurcation curve $\mathcal{P}$ in the the parameter space (α, λ, B) which corresponds to formation of a homoclinic butterfly with unit saddle index or, equivalently, with zero saddle value. In accordance with L. P. Shil'nikov [1981], it guarantees the existence of the Lorenz attractor under some additional conditions.

Figure 1 represents the result of our numerical reconstruction of this curve. Note that the curve $\mathcal{P}$ is the line of intersection of the bifurcation surface $H1$ corresponding to the existence of homoclinic loops and the surface $\mathcal{M}$ corresponding to the unit saddle index. The existence of the bifurcation set $H1$ in the parameter space of system (1) was proven in the paper by Belykh [1984] by the method of comparison systems. The proof in this reference can be easily revised in order to confirm the numerically established fact of the existence of the intersection

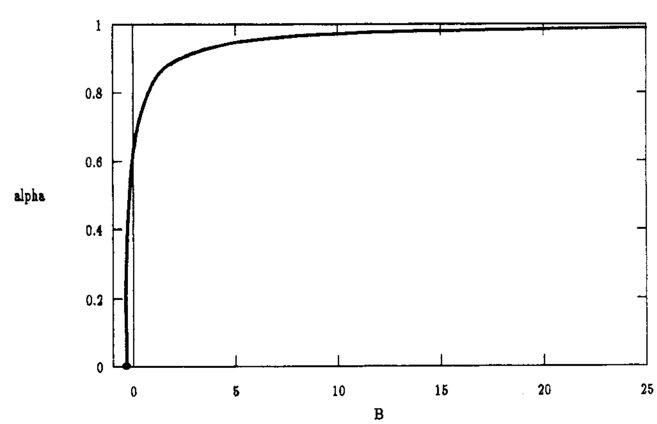


Fig. 1. The numerical reconstruction of the bifurcation curve $\mathcal{P}$ corresponding to a homoclinic butterfly with unit saddle index; here $\lambda = (1 - \alpha^2)/\alpha$.

of this set with the surface $\mathcal{M}$ in the parameter space.

It follows from the work of A. L. Shil'nikov [1986] that near the curve $\mathcal{P}$ the size of the region V_{LA} is exponentially narrow. Therefore, in order that the existence of the Lorenz attractor is not a fact of the pure mathematics, we continue numerically the boundaries of the region V_{LA} and show that the region of the existence of the Lorenz attractor is sufficiently large and is playing an essential role in organizing the global bifurcation portrait.

2. The List of Normal Forms

We shall give the normal forms for eleven cases of bifurcations of equilibrium states and periodical orbits. The procedure of reduction to the normal form is quite regular involving a step-by-step elimination of the nonresonant terms (see Arnold [1983], Guckenheimer & Holmes [1986], and Wiggins [1990]). Therefore, we omit the details of the calculation.

1. The bifurcation of an equilibrium state with three zero characteristic exponents in the case of the symmetry $(x, y, z) \leftrightarrow (-x, -y, z)$, where x, y, z denotes coordinates on the center manifold in a neighborhood of the equilibrium state; y, z are projections on the eigenvectors and x on the adjoined vector. The standard normalizing transformations reduce the system to the form

$$\begin{aligned}
\dot{x} &= y\,, \\
\dot{y} &= x(\bar{\mu} - az(1 + g(x, y, z)) \\
&\quad - a_1(x^2 + y^2)(1 + \ldots)) \\
&\quad - y(\bar{\lambda} + a_2 z(1 + \ldots) \\
&\quad + a_3(x^2 + y^2)(1 + \ldots))\,, \\
\dot{z} &= -\bar{\alpha} + z^2(1 + \ldots) \\
&\quad + b(x^2 + y^2)(1 + \ldots)\,,
\end{aligned} \tag{4}$$

where $\bar{\mu}$, $\bar{\lambda}$, $\bar{\alpha}$ are small parameters, the letter g and the dots denote the terms which vanish at $(x = 0, y = 0, z = 0)$.

2. If, in addition to the conditions of the first case, the system is invariant with respect to the involution $(x, y, z) \leftrightarrow (x, y, -z)$, then the normal form is as follows:

$$\begin{aligned}
\dot{x} &= y\,, \\
\dot{y} &= x(\bar{\mu} - az^2(1 + g(x, y, z^2)) \\
&\quad - b(x^2 + y^2)(1 + \ldots)) \\
&\quad - y(\bar{\lambda} + a_1 z^2(1 + \ldots) \\
&\quad + b_1(x^2 + y^2)(1 + \ldots))\,, \\
\dot{z} &= z(\bar{\alpha} - cz^2(1 + \ldots) \\
&\quad + d(x^2 + y^2)(1 + \ldots))\,.
\end{aligned} \tag{5}$$

3. An equilibrium state with two zero and a pair of pure imaginary characteristic exponents.

Denote by x, y and u coordinates on the center manifold, where x, y correspond to the zero characteristic roots and $u = ze^{i\varphi}$ to the pair of pure imaginary roots. If the system is invariant with respect to the involution $(x, y, u) \leftrightarrow (-x, -y, u)$, the normal form is given by

$$\begin{aligned}
\dot{x} &= y\,, \\
\dot{y} &= x(\bar{\mu} - az^2(1 + \ldots) \\
&\quad - b(x^2 + y^2)(1 + \ldots)) \\
&\quad - y(\bar{\lambda} + a_1 z^2(1 + \ldots) \\
&\quad + b_1(x^2 + y^2)(1 + \ldots))\,, \\
\dot{z} &= z(\bar{\alpha} - cz^2(1 + \ldots) \\
&\quad + d(x^2 + y^2)(1 + \ldots))\,, \\
\dot{\varphi} &= \omega - c_1 z^2(1 + \ldots) \\
&\quad + d_1(x^2 + y^2)(1 + \ldots)\,,
\end{aligned} \tag{6}$$

where the dots denote the terms which vanish at $(x = 0, y = 0, z = 0)$ and depend periodically on φ. Omitting the terms of the order higher than 3 in system (6) yields the truncated normal form

$$\begin{aligned}
\dot{x} &= y\,, \\
\dot{y} &= x(\bar{\mu} - az^2 - b(x^2 + y^2)) \\
&\quad - y(\bar{\lambda} + a_1 z^2 + b_1(x^2 + y^2))\,, \\
\dot{z} &= z(\bar{\alpha} - cz^2 + d(x^2 + y^2))\,, \\
\dot{\varphi} &= w - c_1 z^2 + d_1(x^2 + y^2)\,.
\end{aligned} \tag{7}$$

Note that, in complete analogy with the bifurcation of the equilibrium state with one zero and a pair of the pure imaginary eigenvalues [Gavrilov, 1978; Guckenheimer & Holmes, 1986], the variable φ does not enter the first three equations; therefore, they can be considered independently.

4. The bifurcation of a periodic orbit with three $(+1)$-multipliers.

On the center manifold near the periodic orbit, we introduce the coordinates (x, y, z, ψ), where ψ is the angle and (x, y, z) are the normal coordinates. Assume the original system to be symmetric with respect to the involution $(x, y) \leftrightarrow (-x, -y)$. Then the normal form truncated beyond the second-order terms has the form

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= x(\bar{\mu} - az) - y(\bar{\lambda} + a_2 z)\,, \\ \dot{z} &= -\bar{\alpha} + z^2 + b(x^2 + y^2)\,, \\ \dot{\psi} &= 1\,, \end{aligned} \tag{8}$$

(the period of the cycle is supposed to be equal one).

5. System (8) is also the truncated normal form in the case of the periodic orbit with one $(+1)$-multiplier and a pair of multipliers equal to -1, with no assumption concerning symmetry.

6, 7. In cases 4 and 5, the additional symmetry $z \leftrightarrow -z$ leads to the following truncated normal form:

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= x(\bar{\mu} - az^2 - b(x^2 + y^2)) \\ &\quad - y(\bar{\lambda} + a_1 z^2 + b_1(x^2 + y^2))\,, \\ \dot{z} &= z(\bar{\alpha} - cz^2 + d(x^2 + y^2))\,, \\ \dot{\psi} &= 1\,. \end{aligned} \tag{9}$$

System (9) is also the normal form truncated through third-order terms for the two following bifurcations:

8. When there are three (-1)-multipliers and the Jordan block is not complete; i.e., the associated linear part of the Poincare map has the form

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

It is also assumed here that the system is invariant with respect to the involution, either $(x, y) \leftrightarrow (-x, -y)$ or $z \leftrightarrow -z$.

9. When two multipliers are equal to $(+1)$, and the third one to -1:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and the system is invariant with respect to the involution $(x, y) \leftrightarrow (-x, -y)$.

The normal form

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= x(\bar{\mu} - az^2 - b(x^2 + y^2)) \\ &\quad - y(\bar{\lambda} + a_1 z^2 + b_1(x^2 + y^2))\,, \\ \dot{z} &= z(\bar{\alpha} - cz^2 + d(x^2 + y^2))\,, \\ \dot{\varphi} &= \omega - c_1 z^2 + d_1(x^2 + y^2)\,, \\ \dot{\psi} &= 1\,, \end{aligned} \tag{10}$$

appears in the following two cases:

10. In the bifurcation of a periodic orbit with a pair of multipliers on the unit circle $e^{\pm i\omega}$, $0 < \omega < \pi$, $\omega \neq \pi/2$, $\omega \neq 2\pi/3$ (the condition of absence of the strong resonances) and a pair of multipliers equal $+1$, with the symmetry $(x, y) \leftrightarrow (-x, -y)$.

11. In the bifurcation of a periodic orbit with a pair of multipliers on the unit circle without strong resonances and another pair of the multipliers equal to -1 (symmetry is not required).

The first three equations of system (8) do not depend on ψ. If we omit the last equation, then the system is reduced to system (4). Similarly, if we omit the last equation in systems (7) and (9), or the two last equations in system (11), we then obtain system (5). Thus, all of the enumerated normal forms are reduced to either system (4) (cases 1, 4, 5) or to system (5) (cases 2, 3, 6–11).

Let us consider system (4) for $ab > 0$. Let $\tau^2 = \mu + a\sqrt{\bar{\alpha}}(1 + g(0, 0, -\sqrt{\bar{\alpha}})) > 0$, $\bar{\alpha} > 0$. The scaling of the time $t \to s/\tau$, and of the space variables $x \to x\sqrt{\frac{\tau^3}{ab}}$, $y \to y\tau\sqrt{\frac{\tau^3}{ab}}$, $z \to -\sqrt{\bar{\alpha}} + \frac{\tau^2}{a}z$ and of the parameters $\bar{\lambda} = \lambda\tau$, $\bar{\alpha} = (\alpha\tau/2)^2$ gives the system

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= x(1 - z) - \lambda y + O(\tau)\,, \\ \dot{z} &= -\alpha z + x^2 + O(\tau)\,, \end{aligned}$$

where parameters α, λ are not already small. If we omit the terms of the order τ, we then obtain the Shimizu–Morioka model (3).

We shall consider system (5) for $c > 0$, $ad > 0$. In the parameter region $\tau^2 = \bar{\mu} - a\bar{\alpha}/c(1 + g(0, 0, \bar{\alpha}/c)) > 0$ and $\bar{\alpha} > 0$, let us make the scaling $t \to s/\tau$, $x \to x\tau\sqrt{\frac{c}{ad}}$, $y \to y\tau^2\sqrt{\frac{c}{ad}}$, $z \to \sqrt{\frac{\bar{\alpha}}{c} + \frac{\tau^2}{a}z}$, $\bar{\lambda} = \lambda\tau$, $\bar{\alpha} = \alpha\tau/2$. By denoting $B = \frac{bc}{ad}$

and omitting the terms of the order τ, we have the Lorenz system in the form (1).

3. Proving the Lorenz Attractor

We shall show that for each $B \in (-1/3, \infty)$ a region V_{LA} of the existence of the Lorenz attractor exists in the sector $\alpha > 0$, $\lambda > 0$. To do this, we shall point out in the parameter space (α, λ, B) the bifurcation curve of codimension-2 which corresponds to the homoclinic butterfly with unit saddle index.

The equilibrium $O(0, 0, 0)$ is a saddle for positive values of parameters α and λ. Its unstable manifold W_0^u is one-dimensional and consists of O itself and a pair of orbits (separatrices) Γ_1 and Γ_2 ; the stable manifold W_0^S is two-dimensional. The separatrices Γ_1 and Γ_2 are symmetrical to each other under the involution $(x, y, z) \leftrightarrow (-x, -y, z)$. The symmetry axis $(x = y = 0)$ lies in the stable manifold W_0^S. In the case where a separatrix comes back to the saddle we shall say that a homoclinic loop is formed; by virtue of the symmetry both separatrices form loops synchronously. A pair of such loops is called a homoclinic figure-8; or a homoclinic butterfly (Fig. 2), if Γ_1 and Γ_2 come back tangentially to each other and to the z-axis. We shall distinguish homoclinic loops by the number of their circuits around the equilibrium states O_1 and O_2 with the coordinates $(\pm 1/\sqrt{(B+1)}, 0, 1/(B+1))$.

In general, a homoclinic butterfly can be of one of the two following types: stable and unstable. It is well known that stability is determined by the saddle index $\gamma = |\lambda_2|/\lambda_3$, where λ_i's are the characteristic exponents $\lambda_1 < \lambda_2 < 0 < \lambda_3$ of the saddle. The butterfly is stable for $\gamma > 1$ and unstable for $\gamma < 1$. The formation of the unstable homoclinic butterfly is well known to be the first of the two bifurcations [Afraimovich *et al.*, 1977; Kaplan & Yorke, 1979] which lead to the Lorenz attractor arising.

We denote the two-dimensional surface in the space of parameters (α, λ, B) corresponding to the unstable single-circuit homoclinic butterfly as $H1^+$. Typically, from one side of $H1^+$ the separatrices Γ_1 and Γ_2 tend to O_1 and O_2, respectively [Fig. 3(a)]; from the other side Γ_1 tends already to O_2, and Γ_2 to O_1 [Fig. 3(b)]. This switching is accompanied with the homoclinic explosion [Afraimovich *et al.*, 1977]: in the phase space a hyperbolic set Ω is born which is topologically equivalent to the suspension over the Bernoulli shift of two symbols and

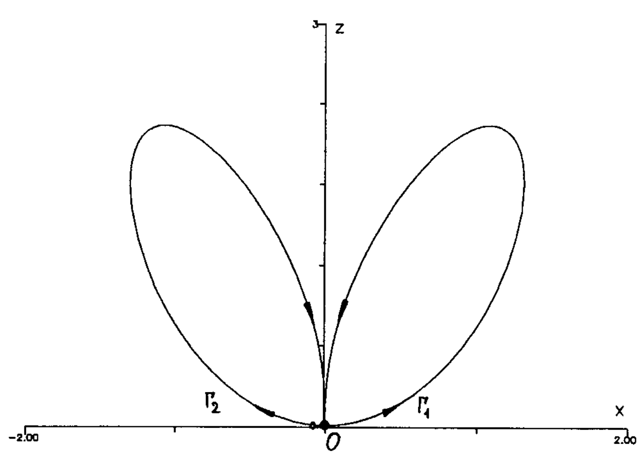


Fig. 2. A single homoclinic butterfly.

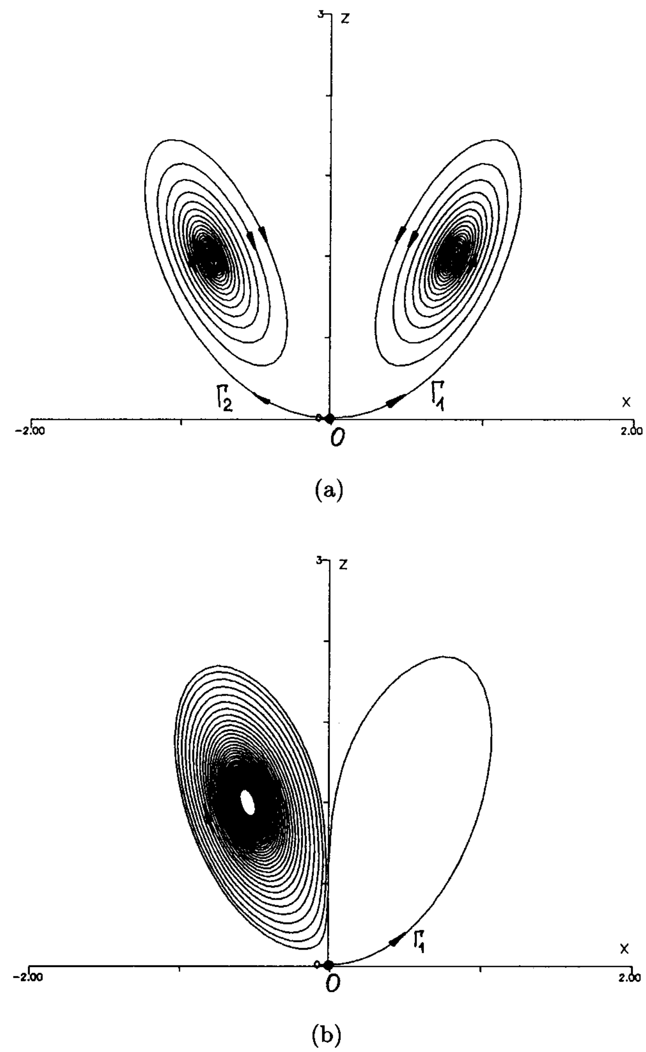


Fig. 3. The switching of the separatrices while crossing the surface H_1^- of a single unstable homoclinic butterfly. (a) above H_1^-: the separatrix Γ_1 tends to O_1, Γ_2 tends to O_2; (b) below H_1^-: Γ_1 tends to O_2, Γ_2 tends to O_1.

contains a countable set of the saddle periodic orbits and a continuum of the nonclosed Poisson-stable orbits. Near the bifurcation parameter values, the set Ω lies entirely in a small neighborhood of the homoclinic butterfly which has just split. Its orbits, in correspondence with sequences of scrolls around O_1 and O_2, are coded by infinite sequences of 1 and 2; furthermore orbits with any possible codings exist in Ω. The codings $\{\ldots 111 \ldots\}$ and $\{\ldots 222 \ldots\}$ correspond to the single-circuit periodic orbits C_1 and C_2 respectively, which are symmetric to each other.

The set Ω is nonattractive, since the two-dimensional unstable manifolds of its orbits intersect transversely the stable manifold W_0^S of the saddle O and, consequently, the orbits close to Ω escape along with the separatrices Γ_1 and Γ_2 to the attractors O_2 and O_1. The absorbing domain for Ω is formed on the bifurcation surface $LA1$, where Γ_1 and Γ_2 lie on the two-dimensional stable manifolds of the saddle periodic orbits C_2 and C_1, respectively (Fig. 4); at the moment of crossing $LA1$ the set Ω becomes the Lorenz attractor (Fig. 5).

The rigorous mathematical investigation of this bifurcation chain and also that of the structure of the Lorenz attractor has been carried out by Afraimovich *et al.* [1982], where conditions were pointed out which the system (more precisely, the Poincare map) should satisfy in order that the Lorenz attractor would exist. We shall reproduce a number of the statements from Afraimovich *et al.* [1982] which we will need in what follows.

Let us construct the cross-section through the equilibrium states O_1 and O_2 (in our case, this is $\{z = 1/(B+1),\ |x| < 1/(B+1)^{1/2}\}$. Suppose that there exists N such that any orbit originating from the rectangle $|y| \leq N$ on the cross-section comes back inside the rectangle [Fig. 6(a)]. The Poincare map T along the orbits of the system is smooth everywhere except for the discontinuous line Π_0 : $x = h_0(y)$ which is the trace of the stable manifold W_0^S of the cross-section. In order for the formulas below to be less awkward, we assume that $h_0(y) = 0$ (it can be riched by the coordinate transformation $x \to x - h_0(y)$ on the cross-section). Then, near Π_0 the map T can be written in the form [Afraimovich & L. P. Shil'nikov, 1983]

$$\begin{aligned} \bar{x} &= (x^* + A|x|^\gamma + O(x^{2\gamma}, yx^\nu))\operatorname{sgn}(x)\,, \\ \bar{y} &= (y^* + B|x|^\gamma + O(x^{2\gamma}, yx^\nu))\operatorname{sgn}(x)\,, \end{aligned} \tag{11}$$

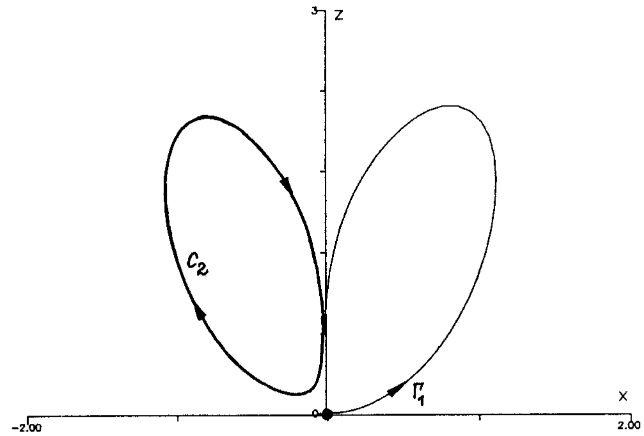


Fig. 4. A moment of the emergence of the Lorenz attractor: the separatrix $\Gamma_1(\Gamma_2)$ lies on the stable manifold of the saddle periodic orbit $C_2(C_1)$.

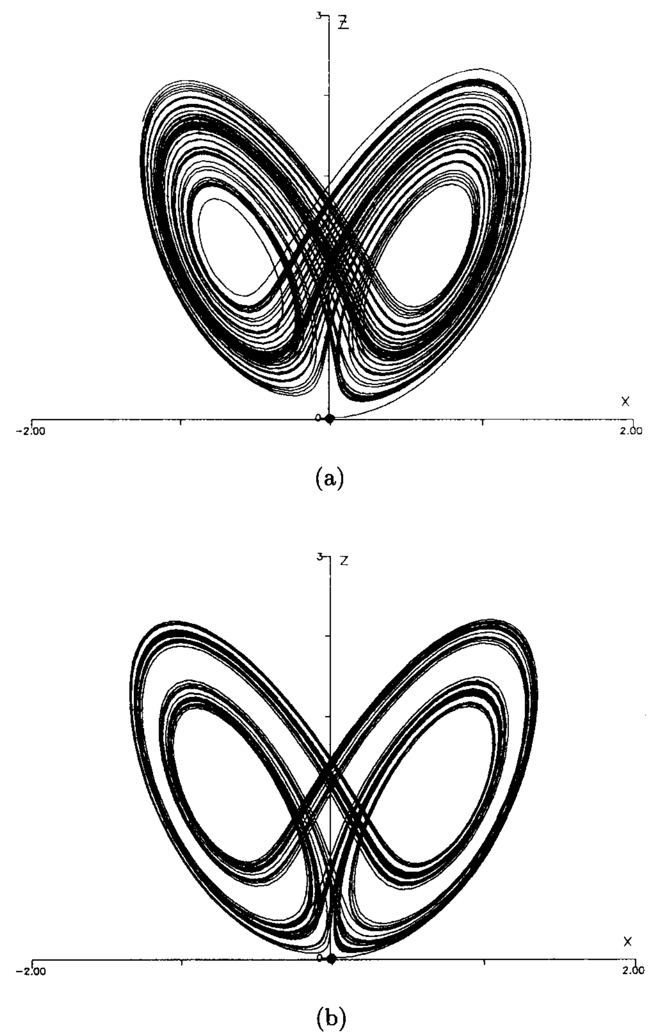


Fig. 5. Two kinds of the Lorenz attractor: (a) standard, (b) with a hole — a lacuna containing a saddle symmetric periodic motion C_{12}^* inside.

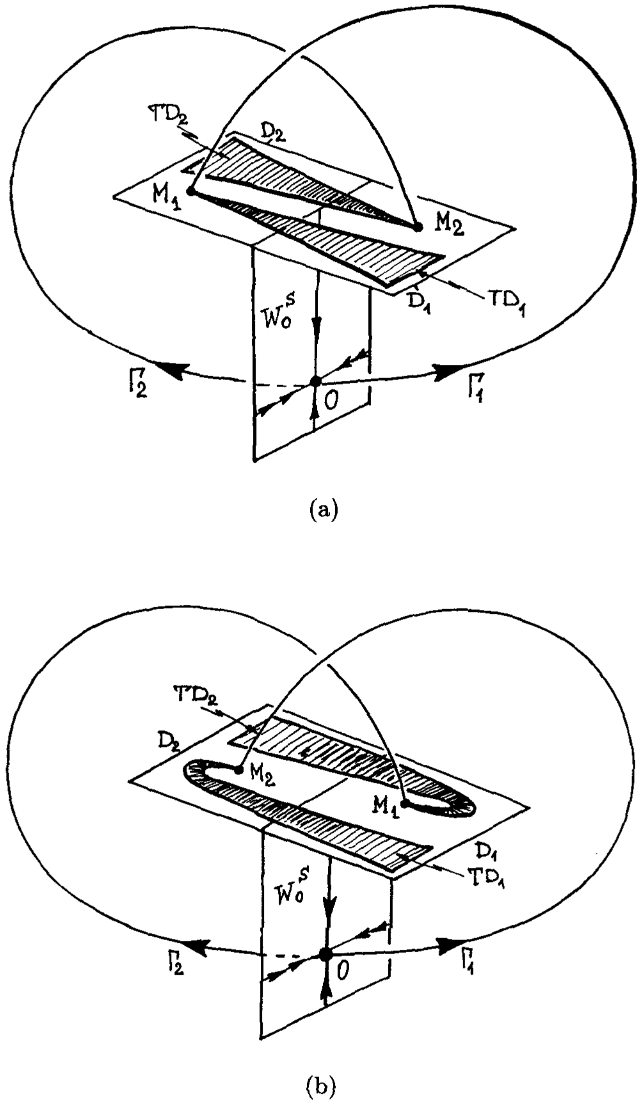


Fig. 6. A qualitative representation of the Poincare map T for parameters values where a separatrix value A is positive (a) and negative (b).

where A is called the separatrix value and $\gamma = |\lambda_2|/\lambda_3$ is the saddle index, ν is equal to $|\lambda_1|/\lambda_3$ (here $\lambda_1 < \lambda_2 < O < \lambda_3$ are the characteristic exponents of the saddle O), (x^*, y^*) are the coordinates of the point M_1 of the intersection of the separatrix Γ_1 with the cross-section. It is easy to check formula (11) assuming the system to be linear in a neighborhood of the saddle O. The general nonlinear case is treated by using the technique has been developed by L. P. Shil'nikov [1968] and Turaev [1991].

Write the map T as follows:

$$\bar{x} = f(x, y), \quad \bar{y} = g(x, y). \tag{12}$$

If γ is less than one and A is not equal to zero, then near Π_0 the inequalities

$$\begin{gathered} \|(f'_x)^{-1}\| < 1, \quad \|g'_y\| < 1, \\ \|g'_x(f'_x)^{-1}\| \cdot \|f'_y\| < (1 - \|(f'_x)^{-1}\|)(1 - \|g'_y\|) \end{gathered} \tag{13}$$

(where $\|\cdot\| = \max_{(x,y)} |\cdot|$) are valid.

It follows from Afraimovich *et al.* [1982] *that if* there exists the curvilinear rectangle

$$\begin{gathered} D : |y| \le N, \quad h_1(y) < x < h_2(y) \\ (h_1(y) < h_0(y) < h_2(y)), \end{gathered}$$

such that $TD \in D$, and inequalities (13) hold everywhere on D, then the system has the Lorenz attractor[2]. Geometrically, conditions (13) mean the contraction along the y-direction and the expansion along the x-direction under the map T. Moreover, these conditions guarantee [Afraimovich *et al.*, 1982] the existence of the stable invariant foliation which leaves the map T is contracting being restricted in. Each stable leaf is of the form $x = h^s(y)$ (the surfaces $x = h_0(y)$, $x = h_1(y)$, $x = h_2(y)$ are included in the foliation). Besides that, an invariant system of unstable leaves of the form $y = h^u(x)$ exists, each of these leaves is transversal to the stable foliation.

The structure of the Lorenz attractor is given by the following theorem:

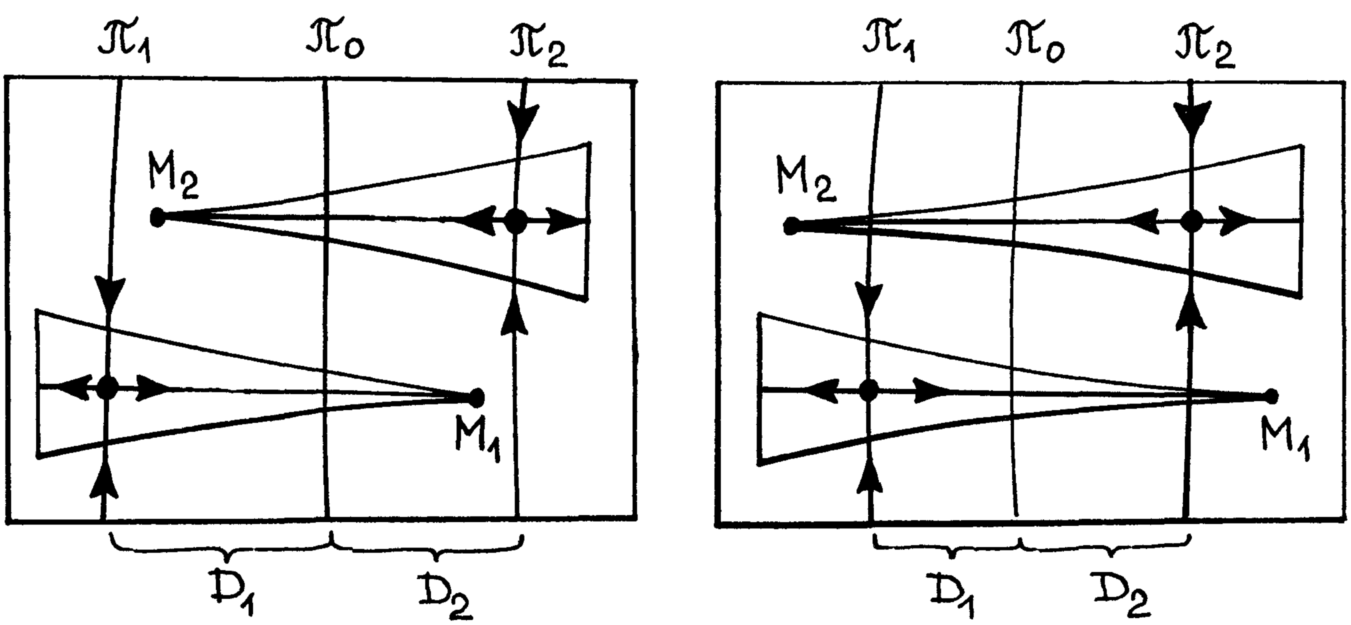
Theorem 1. [*Afraimovich et al.*, 1982]. *Under conditions above the system has the two-dimensional limit set* Λ *(the Lorenz attractor) such that*

(1) *the separatrices* Γ_1, Γ_2 *and the saddle* O *belong to* Λ*;*
(2) *saddle periodic trajectories are dense in* Λ*;*
(3) Λ *is the limit of a sequence of invariant sets each of which is equivalent to the suspension over the subshift of finite type with nonzero topological entropy;*
(4) Λ *is the structurally unstable set: under small perturbations the birth and the disappearance of the saddle periodic trajectories through the bifurcations of homoclinic butterflies happen in* Λ.

[2]Inequalities (13) coincide with conditions a), b), d) in V. S. Afraimovich *et al.* [1982] up to the replacement x onto y and f onto g. We have omitted the condition c) from this reference because it follows from a), b) and d).

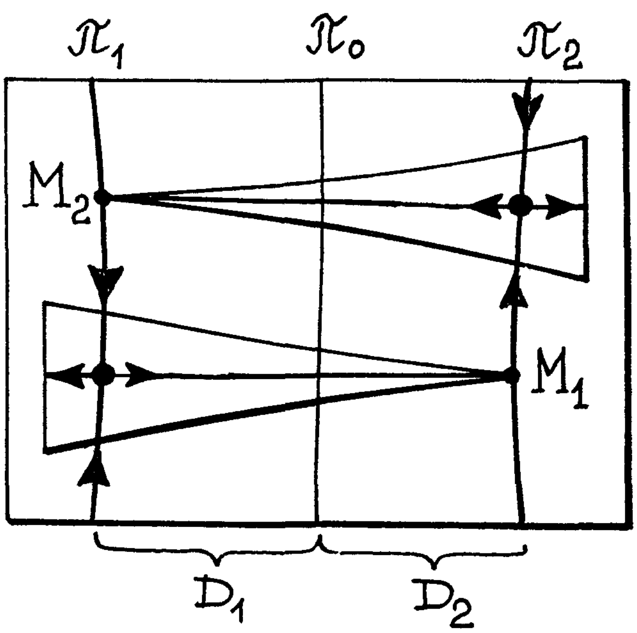
Generally speaking, orbits of some points of D can be non-asymptotical to the Lorenz attractor (if the condition of the complete dilation is not satisfied [Afraimovich *et al.*, 1982; Afraimovich & L. P. Shil'nikov, 1983]). Such orbits, if any, form a one-dimensional invariant set $\Sigma = \cup_{i=0}^{N} \Sigma_i$ where Σ_i is either a saddle periodic orbit or a nontrivial hyperbolic set equivalent to the suspension over the subshift of finite type. Each component Σ_i lies in a *lacuna* — "a hole" within the Lorenz attractor. The crucial role in the evolution of the Lorenz attractor is played by the lacuna which contains the symmetric figure-8 saddle periodic orbit C_{12}^* with the coding $\{\ldots 1212121212 \ldots\}$ [see Fig. 5(b)].

Depending on the sign of the separatrix value A the Lorenz attractor may be of the two types: orientable ($A > 0$) and nonorientable ($A < 0$) (one more type can occur in nonsymmetrical systems,



(a)

(b)



(c)

Fig. 7. The Poincare maps satisfying to conditions (13). (a) There is the Lorenz attractor within the absorbing domain $D_1 \cup D_2$ which is bounded by the stable manifolds Π_1 and Π_2 of the saddle fixed points corresponding to the single periodic orbits C_1 and C_2. (b) The region D between Π_1 and Π_2 is not taken onto itself under the Poincare map. Most of trajectories escape along with the separatrices Γ_1 and Γ_2 to the attractors O_2 and O_1. (c) This is the moment of formation of the absorbing domain and, therefore, of the Lorenz attractor. The separatrices are lying on the stable manifolds of the saddle periodic orbits C_2 and C_1. The phase portraits associated with the maps (a), (b) and (c) are shown in Figs. 5(a), 3(b) and 4 respectively.

namely the so-called semi-orientable Lorenz attractor which we shall not consider here). In the orientable case, the curves $x = h_1(y)$ and $x = h_2(y)$ bounding the region D [Fig. 7(a)] are, typically, the traces Π_1 and Π_2 of the stable manifolds of the single-circuit cycles C_1 and C_2, which are born from the homoclinic butterfly when $\gamma < 1$.

This bifurcation takes place, if $x^* = 0$ in formula (11). It is easily seen that for small $x^* < 0$ the following estimates

$$\text{distance}(\Pi_i, \Pi_0) \sim (|x^*|/A)^{1/\gamma}$$

and

$$\text{distance}(M_i, \Pi_0) \sim |x^*|$$

are valid, where M_i are the points of the first intersections of the separatrices Γ_i with the cross-section. If $A > 0$ and $\gamma < 1$, then

$$|x^*| \gg (|x^*|/A)^{1/\gamma}\,. \tag{14}$$

Hence, despite conditions (13) holding, the Lorenz attractor is not born after this bifurcation since the region D is not taken onto itself under the map T [see Fig. 7(b)].

In order for the Lorenz attractor to be born, it is necessary that the points M_i lie inside the region D; i.e. inequality (14) should be violated. This can be achieved, if either value A or $(\gamma - 1)$ is close to zero. The precise statement, which enumerates the main cases of the homoclinic bifurcations leading to the appearance of the Lorenz attractor, is given by L. P. Shil'nikov [1981]. We formulate here only the consequence from that theorem (see also Robinson [1989] and Rychlic [1989]) which we apply to system (1).

Theorem 2. *Let a system have a homoclinic butterfly and either* (1) $\gamma = 1$ *and* $0 < A < 1$ *or* (2) $A = 0$ *and* $1/2 < \gamma < 1$, $\nu > 1$. *Then in the parameter plane* $(x^*, 1-\gamma)$ *in case* (1) *and* (x^*, A) *in case* (2) *there exists an open set* V_{LA} *adjoined to the point* $P(0, 0)$ *such that for parameter values from* V_{LA} *the system has an orientable Lorenz attractor.*

In both the cases of Theorem 2 the region V_{LA} of the existence of the Lorenz attractor is bounded by two curves $LA1$ and $LA2$ which originate from the point $P(0, 0)$ (Fig. 8). The sequence of the bifurcations while moving from $LA1$ to $LA2$ is described in Afraimovich *et al.* [1982] and A. L. Shil'nikov [1993]

Theorem 3. *On the parameter plane* (x^*, ε) [$\varepsilon = 1-\gamma$ *in case* (1) *and* $\varepsilon = A$ *in case* (2)] *the following six bifurcation curves comes from the point* $P(0, 0)$ (*Fig. 8*):

(1) *the curve* $LA1$ *on which the separatrix* Γ_1 *lies on* W^s_{c2} *and, symmetrically,* Γ_2 *lies on* W^s_{c1} [*see Fig. 4; on the cross-section the points* M_1 *and* M_2 *lie on* Π_2 *and* Π_1*, respectively, see Fig. 7(c)*]. *This is the moment of forming the absorbing region* D*, and the hyperbolic set* Ω*, being born from the homoclinic butterfly, becomes attractive. On crossing* $LA1$ *the set* Ω *transforms into the Lorenz attractor;*
(2) *the curve* LC *which corresponds to the appearance of the simple lacuna containing the symmetric figure-8 saddle periodic orbit* C^*_{12}*. Formation of the lacuna occurs when* Γ_1 *and* Γ_2 *lie on the two-dimensional stable manifold of* C^*_{12} [*Fig. 9(a)*];
(3) *the curve* PF *on which the cycle* C^*_{12} *undergoes the pitch-fork bifurcation: a pair of asymmetrical saddle periodic orbits* C_{12} *and* C_{21} *bifurcates from it and the cycle* C^*_{12} *becomes stable. The basins of the Lorenz attractor and now stable cycle* C^*_{12} *are separated by the two-dimensional stable manifolds of the cycles* C_{12} *and* C_{21}.
(4) *the curve* $LA2$ *on which the separatrices* Γ_1 *and* Γ_2 *lie on the stable manifolds of* C_{21} *and* C_{12}*, respectively* [*Fig. 9(b)*]. *At this moment the Lorenz attractor is terminated and a nonattractive hyperbolic set remains on its place;*

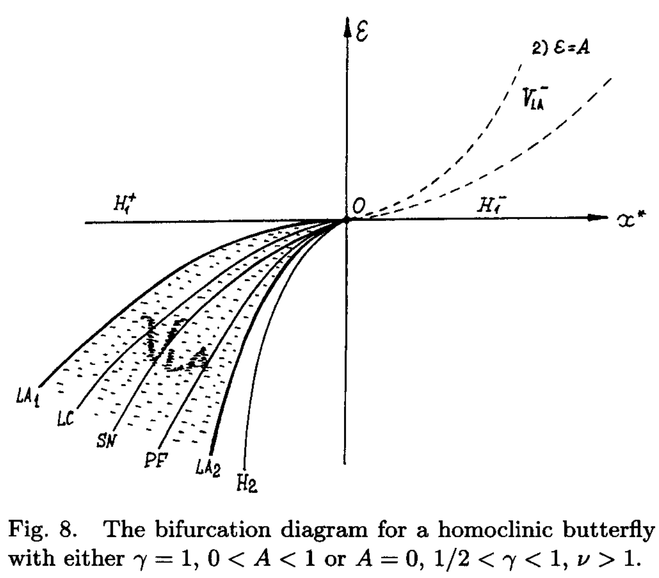
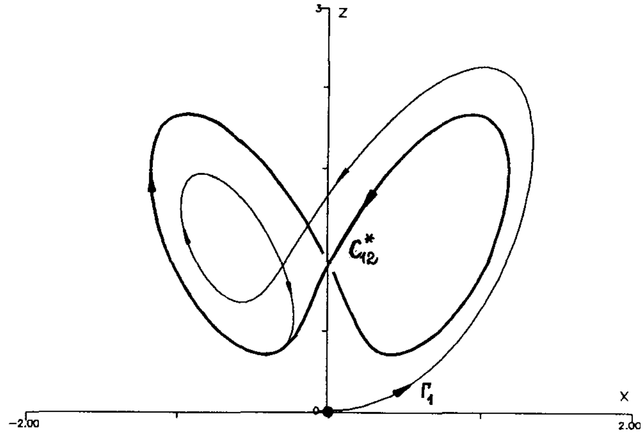
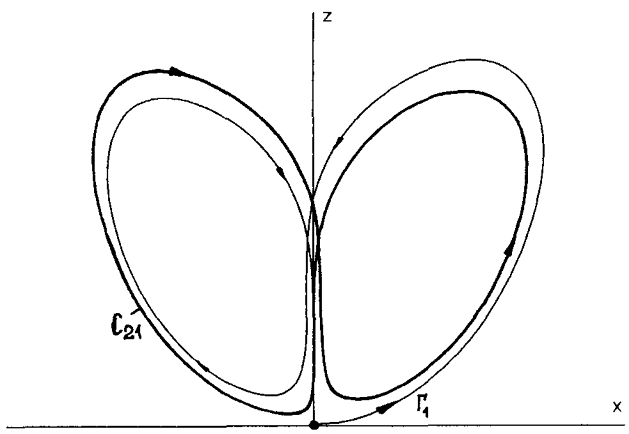


Fig. 8. The bifurcation diagram for a homoclinic butterfly with either $\gamma = 1$, $0 < A < 1$ or $A = 0$, $1/2 < \gamma < 1$, $\nu > 1$.



(a)



(b)

Fig. 9. The separatrix $\Gamma_1(\Gamma_2)$ lies on the stable manifold of the saddle figure-8 cycle: (a) symmetric, which is accompanied by the emergence of a simple lacuna; (b) asymmetric — this bifurcation destroys the Lorenz attractor.

(5) *the curve $H2$ on which the remaining hyperbolic set inclinates into the double-circuit homoclinic butterfly (Fig. 10);*

(6) *the curve SN corresponding to the saddle-node bifurcation of the cycles C_1 and C_2. These cycles coalesce with the stable single-circuit cycles C_1^* and C_2^*, respectively, and annihilate. In the case of the bifurcation $\gamma = 1$ these cycles exists everywhere in a neighborhood of the point P in the sector bounded by the curve SN and by the right branch of the curve $H1$ which corresponds to a stable homoclinic butterfly ($\gamma > 1$).*

Theorem 3 gives the complete description of bifurcations while splitting the symmetric homoclinic butterfly with $\gamma = 1$. In the case $A = 0$ the bifurcation patience is more complex. Thus, in addition to the result of L. P. Shil'nikov [1981], the following theorem can be established by the methods of Afraimovich *et al.* [1982], A. L. Shil'nikov [1990] and Turaev [1991]:

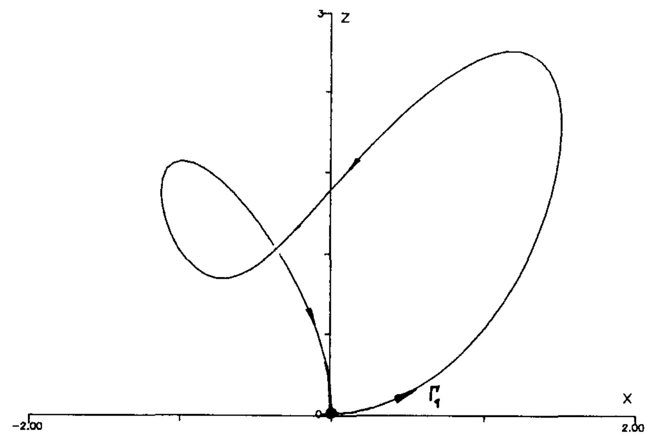


Fig. 10. A double-circuit homoclinic loop.

Theorem 4. *Let a system have a homoclinic butterfly with zero separatrix value and $0 < \gamma < 1$. Then the region V_{LA}^- of the existence of the nonorientable Lorenz attractor adjoins to the point $P(0, 0)$ on the parameter plane (x^*, A) (see Fig. 8)*

The attractor is called nonorientable because within it there exists a dense set of saddle periodic orbits with negative multipliers, whose invariant manifolds are homeomorphic to a Möbius strip [Afraimovich *et al.*, 1982]. The Poincare map T in this region is schematically shown in Fig. 11(a). The images of the right and the left half of the region D have the distinctive "hook"-shape [Afraimovich & L. P. Shil'nikov, 1983]. The Lorenz attractor is situated between the traces Π_1^* and Π_2^* of the stable manifold of the figure-8 cycle C_{12}^*.

In order to prove the existence of the Lorenz attractor in model (1) we use the first case of Theorem 2. Note that, concerning the local bifurcations, parameters α, λ and B of the model play different roles. Here α and λ are the rescaled small parameters and their values can be arbitrary. The parameter B is determined through coefficients of a Taylor expansion at the moment of the bifurcation and its value remains unchanged while changing the small parameters. Therefore, we shall show the existence of the Lorenz attractor on the plane (α, λ) for fixed values B.

At $B = 0$ the point of codimension-2 with the coordinates ($\alpha = 0.606$, $\lambda = 1.045$) which

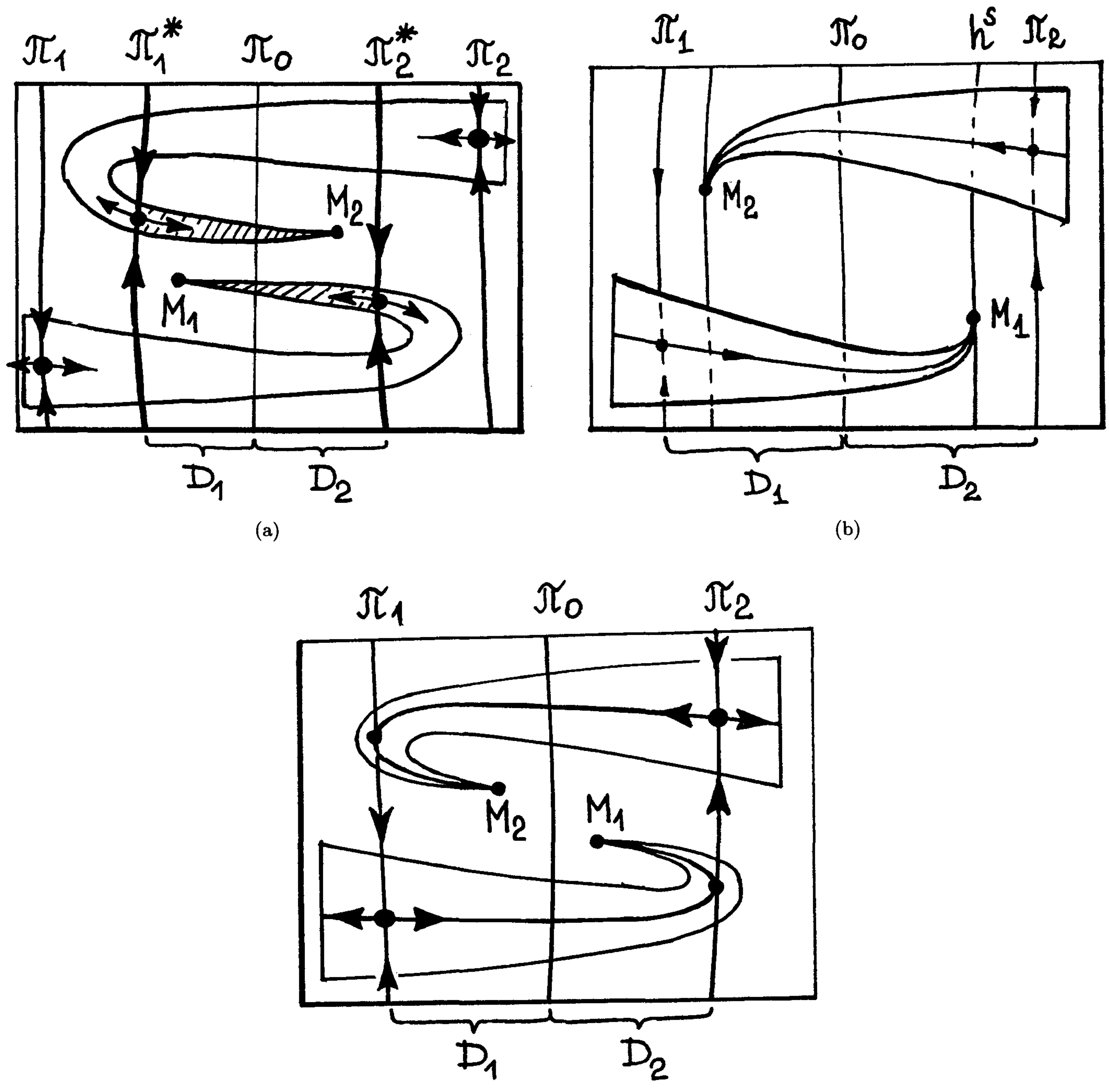


Fig. 11. The Poincare map corresponding to (a) a non-orientable Lorenz attractor, (b) a transition from a Lorenz attractor to a quasi-attractor, (c) a heteroclinic contour including the single saddle periodic cycles C_1 and C_2.

corresponds to the single-circuit homoclinic butterfly with $\gamma = 1$ was found by A. L. Shil'nikov [1986]. We have continued numerically the curve $\mathcal{P}$ associated with this bifurcation from the point ($B = 0$, $\alpha = 0.606$, $\lambda = 1.045$) in the space of the three parameters. This curve lies on the surface $\mathcal{M}$: $\alpha\lambda = 1 - \alpha^2$ defined by the condition $\gamma = 1$ and has the form $\alpha = \eta(B)$, where η is a function monotonic on the interval $B \in (-1/3, \infty)$, $\eta(-1/3) = 0$, $\eta(\infty) = 1$.

The graph of the function η is shown in Fig. 1. The end points ($\alpha = 0$, $\lambda = \infty$, $B = -1/3$) and ($\alpha = 1$, $\lambda = 0$, $B = 1$) of the obtained curve correspond to the cases where system (1) is solved exactly. Near these points the existence of the given curve can be analytically shown, in analogy with the work of Robinson [1989] and Rychlic [1989]. Note that without the assumption of the closeness to the integrable cases, the existence of the bifurcation set ($H1$ in our notations) corresponding to the formation of single-circuit homoclinic loops has been proved for system (1) by Belykh [1984] at $B > 0$. Revising the method he used, it may be shown that this bifurcation set actually intersects the surface $\mathcal{M}$ thereby confirming the results of the numerical calculation. We have also checked numerically that everywhere on this curve the separatrix value A is positive and less than unity; i.e. the conditions of Theorem 2 are fulfilled.

Thus, Theorem 2 allows us to state that a region of the existence of the Lorenz attractor of model (1) exists on the plane (α, λ) for any $B \in (-1/3, \infty)$. The same takes place for any system close enough since the Lorenz attractor cannot disappear under small perturbations [Guckenheimer & Williams, 1979; Afraimovich *et al.*, 1982]. We establish, therefore, that the sector of existence of the Lorenz attractor in a small neighborhood of the origin in the phase space adjoins to the point (0, 0, 0) in the space of the small parameters $(\bar{\mu}, \bar{\alpha}, \bar{\lambda})$ at $ab > 0$ for the first case of our normal form list and at $c > 0$, $ad > 0$, $3bc + ad > 0$ for the second case. The remaining cases are essentially more complicated because of the presence of the angle variables. We only note that any model (finite-parameter family of differential equations) in which bifurcations corresponding to these cases occur cannot be "good" in the sense of Gonchenko *et al.* [1990].

4. Global Bifurcation Analysis

It follows from A. L. Shil'nikov [1986] that near the points $P_B(\alpha = \eta(B),\ \lambda = \alpha^{-1} - \alpha)$ corresponding to the homoclinic butterfly with unit saddle index, the width of the region V_{LA} is of the order $e^{-1/(1-\gamma)}$; i.e. it is extremely narrow. Therefore, in order to give real content to the statement of the existence of the Lorenz attractor, it is necessary to continue the curves $LA1$ and $LA2$ out of the small neighborhood of the point P_B and to investigate the global structure of the boundary of the region V_{LA}.

Figures 12–14 show the typical bifurcation diagrams on the parameter plane (α, λ) for three values of B, respectively: $B = 1$, $B = 0$, $B = -0.1075$.

1. $B = 1$. (Fig. 12) The curve H1 of the single-circuit homoclinic butterfly intersects the line $\gamma = 1$ at the point $P(\alpha = 0.830,\ \lambda = 0.374)$ from which, in correspondence with the theory above, the curves $LA1$, LC, SN, PF, $LA2$, $H2$ originate.

On the curve $H1$, to the right of P, the cycle C^*_{12} [Fig. 15(a)] sticks into the stable ($\gamma > 1$) homoclinic butterfly and a pair of stable single-circuit periodic orbits C^*_1 and C^*_2 [Fig. 15(b)] is born. They collapse into the equilibria O_1 and O_2 on the curve AH: $(\alpha + \lambda)(1 + \alpha\lambda) = 2\alpha$ corresponding to the Andronov–Hopf bifurcation. This bifurcation is supercritical on the branch AH^- of this curve and subcritical on AH^+. The point $Q(\alpha = 0.551,\ \lambda = 0.366)$, at which the first Lyapunov value vanishes, is the limit point of the curve SN originating from the point P and corresponding to a pair of nonrough single-circuit periodic orbits of the saddle-node type. The region of existence of the cycles C^*_1 and C^*_2 is bounded by SN and the curves $H1^+$ and AH^-. The saddle periodic orbits C_1 and C_2, being born from the homoclinic loops on the branch $H1^+$, either coalesce with C^*_1 and C^*_2 on SN or collapse into O_1 and O_2 on the branch AH^+.

The curve $LA1$ is terminated by the point $(\alpha = 0,\ \lambda = 0)$, $LA2$ by the point $R_1(\alpha = 0.3247,\ \lambda = 0.2679)$ on $H2$, where the separatrix value A vanishes for double-circuit loops. It should be noted that the curves $LA1$ and $LA2$ do not belong entirely to the boundary of the existence of the Lorenz attractor. The third boundary curve AZ which links the points $R_2(\alpha = 0.247,\ \lambda = 0.252)$ on $LA1$ and $R_3(\alpha = 0.3218,\ \lambda = 0.2671)$ on $LA2$, corresponds to the vanishing of the separatrix value A. This curve is analogous to that found in Bykov & A. L. Shil'nikov [1992] for the original Lorenz model.

Geometrically, the vanishing of the value A is accompanied with a contact of leaves of the stable and the unstable foliation at the points M_1 and M_2 corresponding to the first intersection of the separatrix Γ_1 and Γ_2 with the cross-section [Fig. 11(b)]. Below the curve AZ, the distinctive hooks appear for the Poincare map, like in Fig. 6(b); we do not give the precise formulations here.

Whereas the curves $LA1$ and $LA2$ separate the regions of the simple and the Lorenzian dynamics, the curve AZ plays an essentially different role. Below AZ the nontrivial hyperbolic sets with an infinite number of the saddle periodic orbits preserve as before, but the formation of the hooks implies the homoclinic tangencies of the stable and unstable manifolds of these trajectories. For instance, the curves corresponding to the homoclinic tangencies of the invariant manifolds of the single-circuit periodic orbits [C_1 and C_2; see Fig. 11(c)], and the

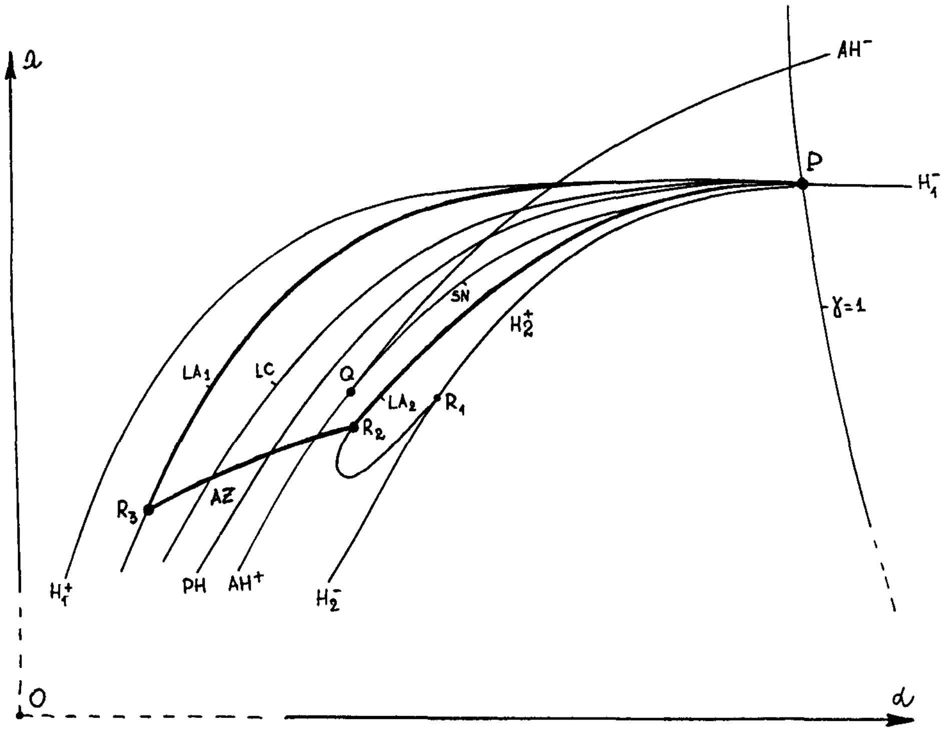


Fig. 12. The (α, λ) bifurcation diagram for $B = 1$.

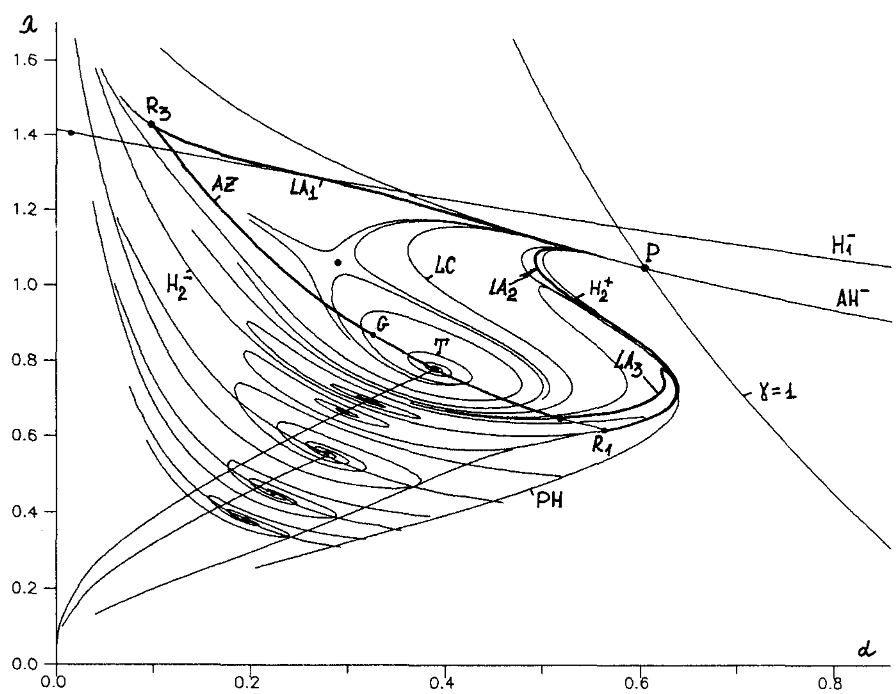


Fig. 13. The (α, λ) bifurcation diagram for $B = 0$.

double-circuit saddle periodic trajectories (C_{12} and C_{21}) start, respectively, from the end points R_2 and R_3 of AZ.

The presence of homoclinic tangencies implies the plethora of different dynamical phenomena, namely an appearance of a large and even infinite number of co-existing stable periodic trajectories with the narrow and judge basins [Gavrilov & L. P. Shil'nikov, 1973; Newhouse, 1979], non-rough periodic trajectories [Gonchenko & L. P. Shil'nikov,

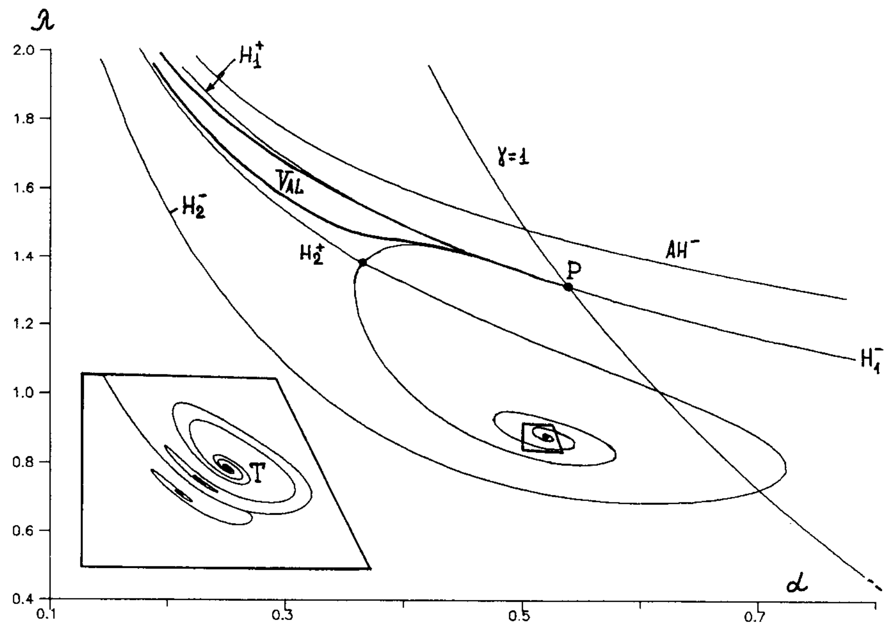
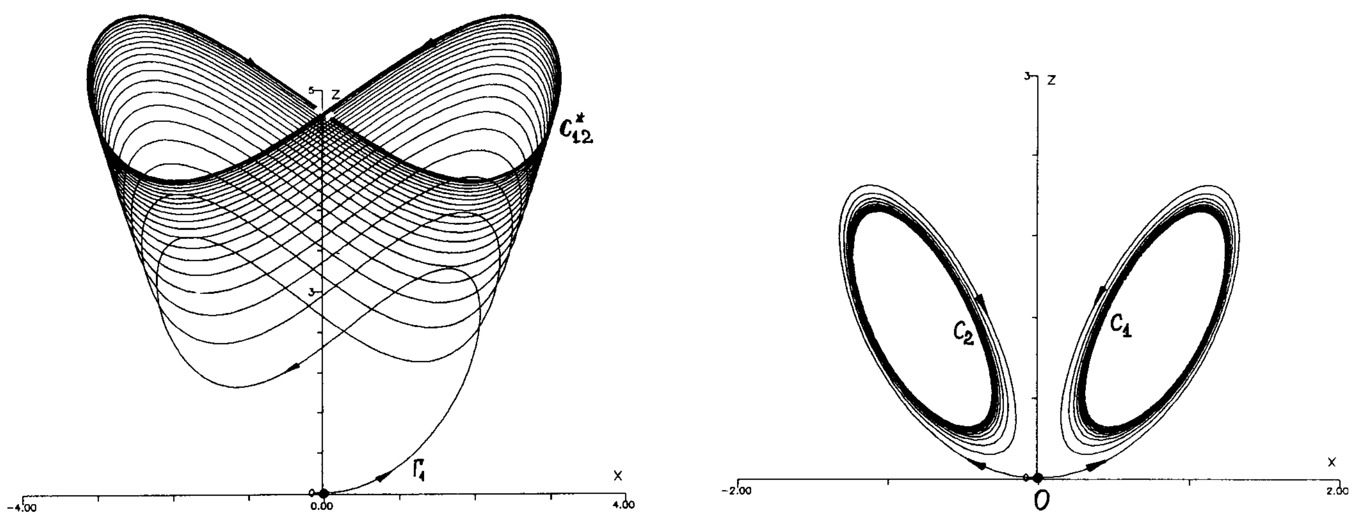


Fig. 14. The (α, λ) bifurcation diagram for $B = -0.1075$.



(a) (b)

Fig. 15. The transition through the curve H_1^- corresponding to a stable homoclinic butterfly: (a) below H_1^-, the stable symmetric figure-8 cycle C_{12}^* is the unique stable limit set, (b) above H_1^-, the cycle C_{12}^* is broken into the two stable single cycles C_1 and C_2.

1986], period-doubling cascades, Henon-like attractors, and so on.

We note also that on the curve AZ the points are dense where the system has homoclinic loops with zero separatrix value. It was mentioned above that to each such points there adjoins (from below) the sector which corresponds to the existence of the non-orientable Lorenz attractor. These homoclinic loops have a very large "number of scrolls" hence the non-orientable Lorenz attractors appearing are also multi-circuit, in the sense that the Poincare return times of the trajectories of such attractors are very long. Furthermore, the basins of these attractors are extremely thin.

The attractive sets containing an infinite number of saddle periodic trajectories together with "weak" stable periodic orbits (or with other attractors) and homoclinic tangencies are called quasiattractors [Afraimovich & L. P. Shil'nikov, 1983] or "wild" attractors. Thus, the curve AZ separates the regions of the Lorenzian and wild dynamics on the parameter plane.

In contrast with the Lorenz attractor, the structure of quasiattractors is not clear. Moreover, it was shown in Gonchenko *et al.* [1993] that a complete description of their structure cannot be obtained due to uncontrolled bifurcations which densely occur in the "wild" regions.

2. $B = 0$ (the Shimizu–Morioka system). The bifurcation diagram (Fig. 13) looks like the corresponding diagram for the case $B = 1$. Note, however, that at $B = 1$ the curves $H1$, $H2$ and AH finish at the point ($\alpha = 0$, $\lambda = 0$); but at $B = 0$, AH finishes at ($\alpha = 0$, $\lambda = \sqrt{2}$), and $H1$ and $H2$ finish at ($\alpha = 0$, $\lambda = 2.154\ldots$).

The essential difference of the Shimizu–Morioka system from the case $B = 1$ is that we have $\gamma < 1/2$ everywhere on the curve AZ at $B = 1$, whereas at $B = 0$ the index γ can be less than $1/2$ [to the left of the point $G(\alpha = 0.33,\ \lambda = 0.87)$] or greater than $1/2$ (to the right of the point G). The branch of the curve AZ from G to the point $R_3(\alpha = 0.12,\ \lambda = 1.45)$ belongs entirely to the boundary of the region of existence of the Lorenz attractor and the passage through AZ is the same as described above for the case $B = 1$.

To the right of the point G, the curve AZ is not already the boundary of the region of the existence of the Lorenz attractor. On AZ there is a countable set of the codimension-2 points R_i corresponding to the homoclinic butterflies with zero separatrix value. In contrast with the branch determined by the condition $\gamma < 1/2$, these points are not dense on AZ. To each such point the region of the existence of the orientable Lorenz attractor adjoins by a narrow sector (see previous section). When approaching AZ there appear the lacunas in the Lorenz attractor with multi-circuit periodic trajectories inside, and the destruction of the Lorenz attractor occurs when the separatrices lie on the stable invariant manifolds of such cycles.

Figure 16 shows this bifurcation on the boundary component $LA3$ for the asymmetrical saddle four-circuit cycle. The curve $LA3$ begins with the point $R_1(\alpha = 0.57,\ \lambda = 0.66)$ where the separatrix value A vanishes for the double-circuit homoclinic loop.

The second essential feature of this bifurcation diagram is the presence of the point $T(\alpha = 0.38,\ \lambda = 0.79)$ on the boundary of the region of existence of the Lorenz attractor. This point is also of codimension-2 and corresponds to the formation of the heteroclinic contour including all three equilibrium states: the saddle O and the saddle-foci O_1 and O_2 (Fig. 17). In accordance with Bykov, [1980, 1993], the existence of such points implies that there is a countable set of points T_i corresponding to more complicated contours with same properties. A countable set of bifurcation curves of homoclinic butterflies spirals to each such a point. Besides this, lines of homoclinic and heteroclinic orbits of the saddle-foci O_1 and O_2 come from these points and finish at ($\alpha = 0$, $\lambda = 0$).

The complete bifurcation analysis of the Shimizu–Morioka system is given in A. L. Shil'nikov [1993].

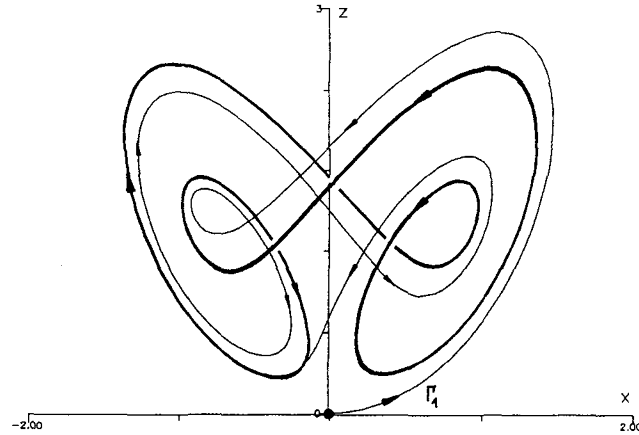


Fig. 16. Disappearance of the Lorenz attractor on the boundary $LA3$: the separatrices tend to the asymmetrical saddle four-circuit cycles.

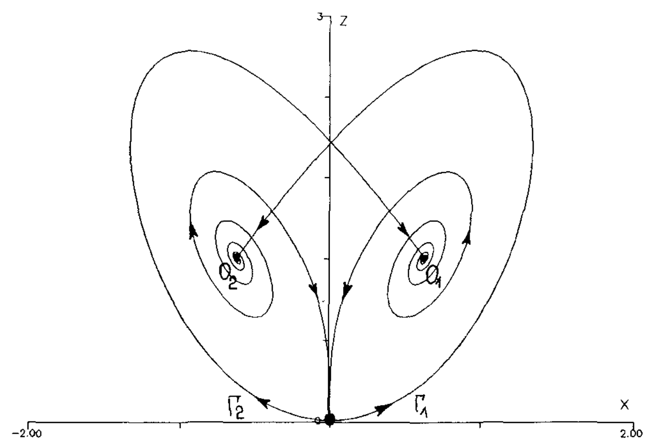


Fig. 17. A heteroclinic contour containing all equilibrium states.

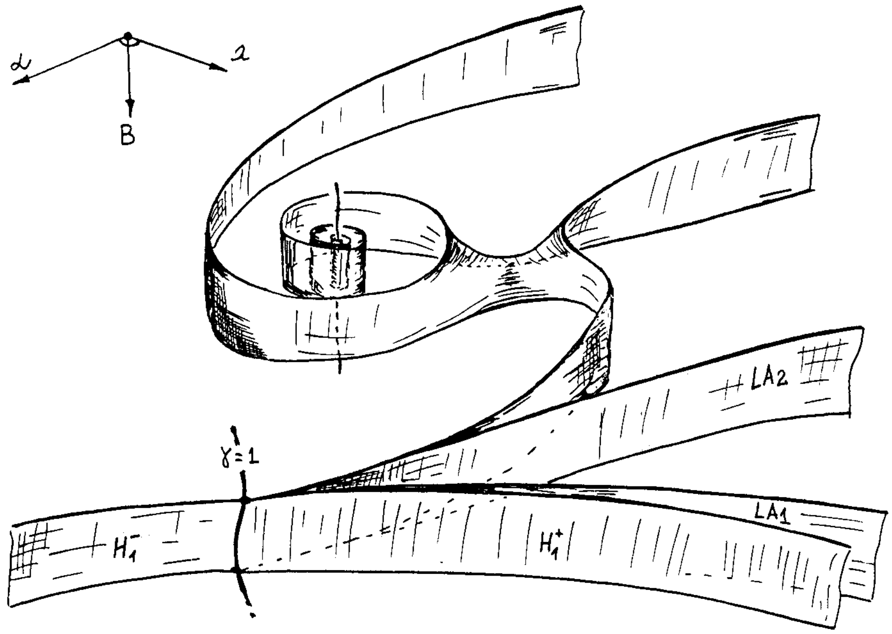


Fig. 18. A fragment of the bifurcation diagram in the parameter space (α, λ, B).

3. As the parameter B decreases, the structure of the boundary of the region of existence of the Lorenz attractor is simplified. We shall not describe all reconstructions of the boundary, but only point out the final moment $B = -0.1075\ldots$

The associated bifurcation diagram is shown in Fig. 14. At this value B there occurs the overlinkage of the bifurcation curves corresponding to the double-circuit homoclinic butterflies. This is stipulated with the fact that the bifurcation surface of such loops has the saddle shape in the three-parameter space (α, λ, B) (see Fig. 18). The upper branch $H2^+$ of the intersection of this surface with the plane $B = -0.1075\ldots$ lies entirely in the region where the separatrix value A is positive. Since the curves $LA1$ and $LA2$ starting from the point $P(\alpha = 0.542,\ \lambda = 1.387)$ are situated between this branch and the curve $H1$ of the single-circuit homoclinic butterfly, the value A is also positive everywhere on them.

Thus, the region of the existence of the Lorenz attractor adjoined to P is bounded only by these curves and goes up to ($\alpha = 0$, $\lambda = +\infty$). When B is decreasing till $-1/3$, the point P tends to ($\alpha = 0$, $\lambda = +\infty$), and the region of the existence of the Lorenz attractor moves away to infinity. We note that the down branch of the curve $H2$ intersects the line $\gamma = 1$ thereby causing the appearance of the new regions of the existence of the Lorenz attractor near these intersections.

Acknowledgments

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