Arch. Rational Mech. Anal. 145 (1998) 129–159. © Springer-Verlag 1998

Normal Forms, Resonances, and Meandering Tip Motions near Relative Equilibria of Euclidean Group Actions

Bernold Fiedler & Dmitry Turaev

Communicated by M. Golubitsky

Abstract

The equivariant dynamics near relative equilibria to actions of noncompact, finite-dimensional Lie groups G can be described by a skew-product flow on a center manifold: $\dot{g} = g\boldsymbol{a}(v)$, $\dot{v} = \varphi(v)$ with $g \in G$, with v in a slice transverse to the group action, and $\boldsymbol{a}(v)$ in the Lie algebra of G. We present a normal form theory near relative equilibria $\varphi(v{=}0) = 0$, in this general case. For the specific case of the Euclidean groups $SE(N)$, the skew product takes the form

$$\dot{R} = R\boldsymbol{r}(v), \qquad \dot{S} = R\boldsymbol{s}(v), \qquad \dot{v} = \varphi(v)$$

with $\boldsymbol{r}(v) \in SO(N)$, $\boldsymbol{s}(v) \in \mathbb{R}^N$. We give a precise meaning to the intuitive idea of tip motion of a meandering spiral: it corresponds to the dynamics of $S(t)$. This clarifies the notion of meander radii and drift resonance in the plane $N = 2$. For illustration, we discuss the unbounded tip motions associated with a weak focus in v, on the verge of Hopf bifurcation, in the case of resonant Hopf and rotation frequencies of the spiral, and study resonant relative Hopf bifurcation. We also encounter random Brownian tip motions for trajectories $v(t) \to \Gamma$, which become homoclinic for $t \to +\infty$. We conclude with some comments on the homoclinic tip shifts and drift resonance velocities in the Bogdanov-Takens bifurcation, which turn out to be small beyond any finite order.

1. Introduction

Going beyond rigidly rotating spirals, meandering and drifting spiral wave patterns have been observed in Belousov-Zhabotinsky media [UNUM93, JSW89, BE93] and in low-pressure CO-oxidation on platinum monocrystals [NvORE93]. Mathematically speaking, the wave patterns are described by concentration vectors $u = u(t, x)$ depending on time t and location $x \in \mathbb{R}^2$. The partial differential equations, which model the dynamics of the solutions $u(t, x)$, are equivariant with respect to the standard affine action of the planar Euclidean group $E(2)$.

The *Euclidean group* $E(N)$, $N = 2, 3, \dots$, is a semidirect product $E(N) = O(N) \times \mathbb{R}^N$ of the orthogonal group $O(N)$ with the Abelian translation group $\mathbb{R}^N$. The composition for $(R, S), (R', S') \in O(N) \times \mathbb{R}^N$ is defined by

$$(R, S) \circ (R', S') := (RR', S + RS'); \tag{1.1}$$

this rule is compatible with the standard affine representation

$$(R, S)x := Rx + S \tag{1.2}$$

on $x \in \mathbb{R}^N$. *Equivariance* of our dynamical system means that $u(t, \cdot)$ is a solution if and only if $(R, S)u(t, \cdot)$ is a solution for any (R, S). Here the linear representation of (R, S) in the state space X of solution x-profiles $u(t, \cdot)$ is given by

$$((R, S)u(t, \cdot))\,(x) := u(t, (R, S)^{-1}x). \tag{1.3}$$

The inverse $(R, S)^{-1}$ is given explicitly by

$$(R, S)^{-1} = (R^{-1}, -R^{-1}S). \tag{1.4}$$

A *spiral wave* $u(t, \cdot)$ is a special time-periodic solution, for which the time orbit is contained in a single group orbit. After a fixed shift of x-coordinates, it can be written as

$$u(t, \cdot) = (R(t), 0)\, u(0, \cdot). \tag{1.5}$$

The rotations $R(t) \in SO(N)$ are given as a periodic one-parameter subgroup

$$R(t) = \exp(\boldsymbol{r}_0 t) \tag{1.6}$$

generated by $\boldsymbol{r}_0$ in the Lie algebra $so(N)$ of anti-symmetric matrices. In the terminology of [Ran82, Ren82, Fie88], and others, non-stationary spiral waves are called *rotating waves*; see also Section 3. The term "spiral" arises from the applied context, where the concentration patterns $u(t, \cdot)$ largely follow Archimedian spirals. Quite analogously, a *meandering wave* $u(t, \cdot)$ is a special solution of the form

$$u(t, \cdot) = (R(t), S(t))\, v(t, \cdot), \tag{1.7}$$

where now $v(t, \cdot)$ is a nonstationary time-periodic solution and the shifts $S(t)$ remain bounded. If the shifts $S(t)$ are unbounded, we call the solution $u(t, \cdot)$ *drifting.*

Numerically, meandering and drifting one-armed spirals have been observed in planar ($N = 2$) models by BARKLEY [Bar94]. Emphasizing the lack of a theoretical framework, based on Euclidean $E(2)$ equivariance, he also presented an ad-hoc heuristic ordinary differential equation model exhibiting meandering and drifting solutions.

The first mathematically rigorous analysis of these phenomena has recently been achieved by WULFF; see [Wul96]. Her result is based on a careful Lyapunov-Schmidt reduction in a scale of Banach spaces. This resolves the difficulties of non-differentiability and, in some cases, *non-continuity* of the group action (1.3) on the infinite-dimensional Banach space $u(t,\cdot) \in X$. For technically related earlier results, restricted to compact group actions, see [Ren82, Ran82]. For recent progress concerning meandering and drifting multi-armed spirals with nontrivial isotropy of u_0 see [GLM97, AM97]. The method of center bundles used there is similar in spirit to a previous approach to bifurcation from relative equilibria of *compact* group actions due to KRUPA; see [Kru90].

It has recently been shown, for the first time, that a center-manifold reduction to a finite-dimensional globally group-invariant and locally time-invariant C^{k+1} manifold $M \subseteq X$ can also be achieved in an $E(2)$-equivariant context, if the nonlinearity of the differential equation governing the dynamics of the spiral waves is smooth; see [SSW97a, SSW97b]. The reduction is based on the assumption that the linearization at the spiral wave does not exhibit continuous spectrum near the imaginary axis. Most notably, the group action becomes differentiable on M, even though it is possibly not continuous on X. For differentiable noncompact group actions on Banach spaces, the first center-manifold reduction near relative equilibria is due to [Mie91]. His reduction is carefully designed to preserve symplectic Hamiltonian structure. From this abstract viewpoint, early observations of what we now call drifting spirals go back as far as [Lov92], in the context of buckling of elasticae in continuum mechanics.

In [FSSW96] we have therefore considered a finite-dimensional, typically noncompact Riemannian manifold $\mathscr{M}$ with a differentiable proper action of a possibly non-compact finite-dimensional Lie group G. We recall that a (local) flow on $\mathscr{M}$ is G-equivariant, provided that $g \cdot u(t)$ is a solution whenever $u(t) \in \mathscr{M}$ is a solution and g is fixed in G. We call $u_0 \in \mathscr{M}$ a *relative equilibrium* if the solution curve $u(t)$ through $u_0 = u(0)$ lies entirely in the group orbit $G \cdot u_0$ of u_0. Equivalently, the manifold $G \cdot u_0$ is flow-invariant. We also recall that $G \cdot u_0$ is diffeomorphic to G/H, where H is the isotropy of u_0:

$$H := \{h \in G | hu_0 = u_0\}. \tag{1.8}$$

It was shown in [FSSW96] that G-equivariant flows in a tubular neighborhood U of a relative equilibrium $G \cdot u_0,\ u_0 \in \mathscr{M}$, with compact isotropy H of u_0, can be represented by a skew product flow

$$\dot{g} = g\boldsymbol{a}(v), \qquad \dot{v} = \varphi(v). \tag{1.9}$$

Here g is in the Lie group G and $\boldsymbol{a}$ is in the associated Lie algebra alg (G). The vector v is in a "linear" slice V, transverse to the group action. The slice V is called a *Palais slice*, and (g, v) are *Palais coordinates* near the relative equilibrium. Note the nonuniqueness of these coordinates due to a certain arbitrariness in our choice of the Palais slice V.

The induced local flow (1.9) on $G \times V$ is equivariant under the action of $(g_0, h) \in G \times H$ on $(g, v) \in G \times V$, given by

$$(g_0, h) \cdot (g, v) \ = \ (g_0 g h^{-1}, hv). \tag{1.10}$$

Specifically, $\{\mathrm{id}\} \times H$-equivariance of the flow on $G \times V$ implies that

$$\begin{aligned} \varphi(hv) &= h\varphi(v), \\ \boldsymbol{a}(hv) &= h\boldsymbol{a}(v)h^{-1} \end{aligned} \tag{1.11}$$

for all h and v. The original flow on U is equivalent to the induced flow on $\{\mathrm{id}\} \times H$-orbits in $G \times V$.

The special case $\mathscr{M} = T^*G$, the cotangent bundle of G, with trivial isotropy $H =$ id arises in Hamiltonian dynamics. Aimed at applications in elasticity, $G = SE(2)$ or $SE(3)$, a skew product formulation like (1.9) was derived in [Mie91].

In the following we fix any arbitrarily large, finite smoothness C^κ for the functions $\boldsymbol{a} : V \to \mathrm{alg}\,(G)$ and $\varphi : V \to \mathbb{R}^n$. Note that

$$\varphi(0) = 0 \tag{1.12}$$

because u_0, corresponding to $v = 0$, is a relative equilibrium.

The main goal of the present paper is a *normal form method* which further simplifies system (1.9) and applies in bifurcation situations where

$$\boldsymbol{a} = \boldsymbol{a}(\lambda, v), \qquad \varphi = \varphi(\lambda, v). \tag{1.13}$$

depend on one or several additional real parameters λ. As usual in bifurcation theory, we first consider the parameter-independent case, suppressing λ notationally.

We may assume the driving $\dot{v}$-equation in the skew product (1.8) to be in normal form already. To preserve equivariance with respect to the (orthogonal) action of the compact isotropy H on V, we favor the approach of [ETB+87]; see [Van89]. Of course, we assume in particular that the linearization matrix

$$M = \varphi'(0) \tag{1.14}$$

at the relative equilibrium $v = 0$ is already in (complex) linear Jordan normal form:

$$\begin{aligned} (Mv)_{pq} &= \mu_p v_{pq} + v_{p,q+1} &&\text{for } 1 < q < d_p, \\ (Mv)_{pq} &= \mu_p v_{pq} &&\text{for } q = d_p. \end{aligned} \tag{1.15}$$

Here $1 \leqq p \leqq m$ enumerates the Jordan blocks of size d_p and with (not necessarily distinct) eigenvalues μ_p. The v_{pq} enumerate the components $v_1, \dots, v_N$ of v. Equivariance with respect to H is preserved here.

Our normal form method aims at eliminating as many monomials

$$\boldsymbol{a_k} v^{\boldsymbol{k}} = \boldsymbol{a_k} \prod_{p,q} v_{pq}^{k_{pq}} \tag{1.16}$$

as possible from the Taylor expansion of the coupling term $\boldsymbol{a}(v)$ of the Lie algebra skew product in the $\dot{g}$-equation

$$\dot{g} = g\boldsymbol{a}(v) \tag{1.17}$$

of (1.9). To this end, we consider the adjoint action

$$\begin{aligned} \mathrm{ad}\,(\boldsymbol{a}_0): \quad & \mathrm{alg}\,(G) \to \mathrm{alg}\,(G) \\ & \gamma \mapsto [\boldsymbol{a}_0, \gamma] := \boldsymbol{a}_0\gamma - \gamma\boldsymbol{a}_0 \end{aligned} \tag{1.18}$$

of $\boldsymbol{a}_0 := \boldsymbol{a}(v = 0)$ on the Lie algebra. Let η_i denote the distinct eigenvalues of $\mathrm{ad}(\boldsymbol{a}_0)$ and decompose spectrally

$$\begin{aligned} \mathrm{alg}\,(G) &= \bigoplus_i \mathscr{A}_i \\ \boldsymbol{a}_{\boldsymbol{k}} &= \sum_i \boldsymbol{a}_{\boldsymbol{k}}^{(i)}. \end{aligned} \tag{1.19}$$

Definition 1.1. We call a term $\boldsymbol{a}_{\boldsymbol{k}}^{(i)} v^{\boldsymbol{k}}$ in the Taylor expansion of $\boldsymbol{a}(v)$ *resonant* if the nonnegative integer $\boldsymbol{k}$-components k_{pq}, $1 \leqq p \leqq m$, $1 \leqq q \leqq d_p$ satisfy the *resonance condition*

$$0 = \eta_i + (\boldsymbol{k}, \mu) \tag{1.20}$$

with the eigenvalues η_i of $\mathrm{ad}(\boldsymbol{a}_0)$. Here $(\boldsymbol{k}, \mu) = \sum_{p,q} k_{pq}\mu_p$.

Theorem 1.2. *With the above assumptions and notations, for any finite order κ of differentiability, there exists a transformation*

$$g \mapsto g\ g_0(v) \tag{1.21}$$

on $g \in G$ which preserves the skew product structure (1.9) *and equivariance* (1.11), *but eliminates all nonresonant terms $\boldsymbol{a}_{\boldsymbol{k}}^{(i)} v^{\boldsymbol{k}}$ for*

$$1 \leqq |\boldsymbol{k}| = \sum_{p,q} \boldsymbol{k}_{p,q} \leqq \kappa. \tag{1.22}$$

In the parameter-dependent case $\boldsymbol{a} = \boldsymbol{a}(\lambda, v)$, $\varphi = \varphi(\lambda, v)$ the same statements remain valid, with $\boldsymbol{a}_0 := \boldsymbol{a}(\lambda = 0, v = 0)$, and for suitable transformations $g_0 = g_0(\lambda, v)$ and resonant terms $\boldsymbol{a}_{\boldsymbol{k}}^{(i)}(\lambda)v^{\boldsymbol{k}}$.

It is worthwhile to briefly interpret our normal-form transformations (1.21) geometrically in terms of the skew product (1.9). The Palais coordinates (g, v) are nonunique, because the Palais slice V just needs to be chosen locally transversely to the group orbits. Geometrically, the normal-form transformation only corresponds to a correction in the choice of the Palais slice V. Algebraically, however, this correction may significantly simplify the form of the bifurcation equations in Palais coordinates (g, v), as we will see below.

We give an outline of the paper. In Section 2, we prove Theorem 1.2. The special case $G = SE(N)$ is addressed in Section 3. The skew product (1.9) takes the form

$$\dot{R} = R\boldsymbol{r}(v), \qquad \dot{S} = R\boldsymbol{s}(v), \qquad \dot{v} = \varphi(v) \tag{1.23}$$

with $\boldsymbol{r}(v) \in so(N)$, $\boldsymbol{s}(v) \in \mathbb{R}^N$, in that case. For $\mathrm{spec}\, M = \{0\}$ and N even, we obtain $\dot{S} = 0$ for the drift components, to any finite order and even in absence of further nontrivial isotropy H; see Lemma 3.2. For odd N, a drift along an axis can occur, but again the axis itself remains fixed to any finite order.

We caution our readers that these results hold in normal form, only. For proper interpretation, they have to be translated back to the original coordinates (g, v) by inversion of the normal-form transformation (1.21). This point is important since, for example, the changes of coordinates that are used to transform to normal form can mix the R and S variables. For example, the equation $\dot{S} = 0$, for even N, amounts to a "slaving principle"

$$S = S(R, v) = -RS_0(v) + \text{const} \tag{1.24}$$

for the drift variable S, in original coordinates, which holds to any finite order in v.

For a first illustration of this effect consider the trivial case of a saddle-node bifurcation of relative equilibria to $G = SE(2)$. Typically, relative equilibria are rigidly rotating waves, for example, of spiral profiles. The associated simple zero eigenvalue, in the v-equation, produces an equation $\dot{S} = 0$ for the drift component, in normal form. In original coordinates, however, the bifurcating spirals generically exhibit nonstationary drift components, according to our slaving principle (1.24). Indeed, the rotation angle $R \in SO(2)$ generically rotates at constant nonzero speed, accounting for a synchronous circular motion of the drift component S. Of course, a simple time-independent transformation in $SE(2)$ eliminates this circular motion of S. Our normal-form transformation performs this elimination, automatically, to any finite order. For further details and examples concerning normal forms involving multiple zero eigenvalues we refer to Sections 3 and 8 below.

In Section 4, we begin a detailed analysis of the case $G = SE(2)$ which has motivated our introduction, and has driven much of the previous theoretical and experimental work. We give a precise meaning to the intuitive idea of tip motion of a meandering spiral. Section 5 applies these notions to epicyclic meandering of tips in the plane. We clarify the notion of meander radii and address drift resonance. Section 6 investigates Hopf bifurcation under $m : 1$ drift resonance of Hopf and rotation frequencies of the spiral. In Section 7, we discuss the unbounded tip motions associated with a weak focus in v, on the verge of Hopf bifurcation, in this case of resonant frequencies. In Section 8, we compute tip shifts associated with homoclinic and heteroclinic orbits $v(t) \in \Gamma$. We also discuss random Brownian tip motions for trajectories $v(t) \to \Gamma$, which become homoclinic for $t \to +\infty$. We conclude with some comments on the homoclinic tip shifts and drift resonance velocities in the Bogdanov-Takens bifurcation, which turn out to be small beyond any finite order.

2. Normal Forms of Skew Products

In this section, we prove Theorem 1.2. We recall that transformations

$$g_{\text{new}} = g\, g_0(v) \tag{2.1}$$

on the Lie group G do not change the general form

$$\dot{g} = g\boldsymbol{a}(v), \qquad \dot{v} = \varphi(v) \tag{2.2}$$

of the skew product (1.9). Indeed, the new $\boldsymbol{a}(v)$, after transformation (2.1) is given explicitly by

$$\boldsymbol{a}_{\mathrm{new}}(v) = g_0(v)^{-1}\boldsymbol{a}(v)g_0(v) + g_0(v)^{-1}g_0'(v)\varphi(v). \tag{2.3}$$

It is our goal to successively eliminate nonresonant terms $\boldsymbol{a}^{(i)}v^{\boldsymbol{k}} \in \mathscr{A}_i$ with $|\boldsymbol{k}| = \sum_{p,q} k_{p,q} \geqq 1$, for which

$$\eta_i + (\boldsymbol{k}, \mu) = \eta_i + \sum_{p,q} k_{p,q}\mu_p \neq 0. \tag{2.4}$$

Here $\mu_p \in \operatorname{spec} M$ indicate the eigenvalues of the Jordan blocks $1 \leqq p \leqq m$ of $M = \varphi'(0)$, and η_i is the eigenvalue of $A = \operatorname{ad} \boldsymbol{a}(0)$ on the generalized eigenspace $\mathscr{A}_i$ in alg (G). Specifically, we proceed by induction on multiindices $\boldsymbol{k}$ with $|\boldsymbol{k}| \geqq 1$. In fact, we will define a total order $\operatorname{ord}(\boldsymbol{k})$ on $\boldsymbol{k}$, i. e., on the monomials $v^{\boldsymbol{k}}$. We then show how nonresonant terms $\boldsymbol{a}^{(i)}v^{\boldsymbol{k}}$ with $|\boldsymbol{k}| \geqq 1$ can be eliminated, successively with respect to $\operatorname{ord}(\boldsymbol{k})$, by specific choices

$$g_0(v) = \exp(\gamma v^{\boldsymbol{k}}), \tag{2.5}$$

with $\gamma = \gamma_{\boldsymbol{k}} \in \mathscr{A}_i$. We conclude the proof by showing how equivariance with respect to the isotropy H of the relative equilibrium u_0 is preserved.

We now define the *total order* $\operatorname{ord}(\boldsymbol{k})$ of $\boldsymbol{k}$, i. e., of the monomials $v^{\boldsymbol{k}}$. If $|\boldsymbol{k}| > |\boldsymbol{k}'|$, then $\operatorname{ord}(\boldsymbol{k}) > \operatorname{ord}(\boldsymbol{k}')$. If $|\boldsymbol{k}| = |\boldsymbol{k}'|$, consider $|\boldsymbol{k}|_p := \sum_q k_{pq}$ and the smallest p such that $|\boldsymbol{k}|_p \neq |\boldsymbol{k}'|_p$. Then define $\operatorname{ord}(\boldsymbol{k}) > \operatorname{ord}(\boldsymbol{k}')$, if $|\boldsymbol{k}|_p > |\boldsymbol{k}'|_p$. Finally, if $|\boldsymbol{k}|_p = |\boldsymbol{k}'|_p$, for all p, then choose the smallest p, and subsequently the smallest $q = q_p$, such that $k_{pq} \neq k'_{pq}$. Define $\operatorname{ord}(\boldsymbol{k}) > \operatorname{ord}(\boldsymbol{k}')$, if $k_{pq} < k'_{pq}$, in that case. Note the reversal of inequalities, in this last definition.

This order is well adapted to compute the effect of the transformation $g \mapsto g\, g_0(v)$, with $g_0(v) := \exp(\gamma v^{\boldsymbol{k}})$, on $\boldsymbol{a}_{\mathrm{new}}(v)$ given by (2.3). Indeed, let us expand

$$\begin{aligned} \boldsymbol{a}(v) &= \textstyle\sum_{|\boldsymbol{k}'| \leqq \kappa} \boldsymbol{a}_{\boldsymbol{k}'} v^{\boldsymbol{k}'} + \cdots, \\ \varphi(v) &= Mv + \cdots, \\ g_0(v) &= \mathrm{id} + \gamma v^{\boldsymbol{k}} + \cdots, \\ g_0(v)^{-1} &= \mathrm{id} - \gamma v^{\boldsymbol{k}} + \cdots. \end{aligned} \tag{2.6}$$

Then our transformation $g_0(v)$ does not change lower-order coefficients of $\boldsymbol{a}$. Indeed,

$$(\boldsymbol{a}_{\mathrm{new}})_{\boldsymbol{k}'} = \boldsymbol{a}_{\boldsymbol{k}'} \tag{2.7}$$

for $|\boldsymbol{k}'| < |\boldsymbol{k}|$. Even on the level $|\boldsymbol{k}'| = |\boldsymbol{k}|$, the order $\operatorname{ord}(\boldsymbol{k})$ has been defined such that (2.7) still holds, for $\operatorname{ord}(\boldsymbol{k}') < \operatorname{ord}(\boldsymbol{k})$, by expansions (2.6) and the Jordan normal form (1.15) for M. Indeed, for $|\boldsymbol{k}| \geqq 1$ and up to terms of order higher than $\operatorname{ord}(\boldsymbol{k})$, we compute

$$\begin{aligned} g_0^{-1}g_0'(v)\varphi(v) &= (\mathrm{id} - \gamma v^{\boldsymbol{k}})\gamma(v^{\boldsymbol{k}})'Mv + \cdots \\ &= (\boldsymbol{k}, \mu)\gamma v^{\boldsymbol{k}} + \cdots. \end{aligned} \tag{2.8}$$

For $\boldsymbol{k}' = \boldsymbol{k}$, we compute the correction

$$(\boldsymbol{a}_{\text{new}})_{\boldsymbol{k}} = \boldsymbol{a}_{\boldsymbol{k}} + (\text{ad } \boldsymbol{a}_0)\gamma + (\boldsymbol{k}, \mu)\gamma \tag{2.9}$$

again by (1.15), (2.6), (2.8), with the abbreviation

$$(\text{ad } \boldsymbol{a}_0)\gamma = \boldsymbol{a}_0\gamma - \gamma\boldsymbol{a}_0. \tag{2.10}$$

Again by definition of $\text{ord}(\boldsymbol{k})$, the remaining corrections in the Taylor expansion of $\boldsymbol{a}(v)$ are of order higher than $\text{ord}(\boldsymbol{k})$.

We now choose $\gamma = \gamma_{\boldsymbol{k}}$ such that $(\boldsymbol{a}_{\text{new}})_{\boldsymbol{k}} = 0$, for as many $\boldsymbol{k}$ as possible. More precisely, we decompose $\boldsymbol{a}_{\boldsymbol{k}} = \sum_i \boldsymbol{a}_{\boldsymbol{k}}^{(i)}$ according to the spectral decomposition of $\text{alg}\,(G)$ with respect to the eigenvalues η_i of ad $\boldsymbol{a}_0$. We now use the crucial assumption that $\boldsymbol{a}_{\boldsymbol{k}}^{(i)} v^{\boldsymbol{k}}$ is nonresonant; see Definition 1.1 and (2.4). Since $(\boldsymbol{k}, \mu)$ is not resonant to the eigenvalue η_i of ad $\boldsymbol{a}_0$, we can invert on $\mathscr{A}_i$ and define

$$\gamma = \gamma_{\boldsymbol{k}}^{(i)} := -\Big(((\boldsymbol{k}, \mu) + \text{ad } \boldsymbol{a}_0)|_{\mathscr{A}_i}\Big)^{-1} \boldsymbol{a}_{\boldsymbol{k}}^{(i)}. \tag{2.11}$$

This immediately implies that

$$\boldsymbol{a}_{\text{new},\boldsymbol{k}}^{(i)} = 0, \tag{2.12}$$

eliminating all nonresonant terms.

In the parameter-dependent case $\varphi{=}\varphi(\lambda, v)$, $\boldsymbol{a}{=}\boldsymbol{a}(\lambda, v)$, we subsume the artificial equation $\dot{\lambda} = 0$ in the $\dot{v}$-equation: $v_{\text{new}} := (\lambda, v)$. The above procedure then puts $\dot{g} = g\boldsymbol{a}(v_{\text{new}})$ into normal form, as before. Note that the additional λ-terms do not contribute to the resonance conditions (1.20), (2.4). Moreover, our normal-form transformation of g does not alter the parameter foliation $\dot{\lambda} = 0$. Therefore, our results remain valid in the parameter-dependent case.

It remains to address equivariance of $\boldsymbol{a}(v)$ with respect to the compact isotropy H of the relative equilibrium u_0, i. e., $v = 0$. To ensure H-equivariance as in (1.11),

$$\boldsymbol{a}_{\text{new},\boldsymbol{k}}(hv) = h\boldsymbol{a}_{\boldsymbol{k}}(v)h^{-1}, \tag{2.13}$$

we would like to simply integrate with respect to Haar measure dh over H:

$$\tilde{\boldsymbol{a}}_{\text{new}}(v) := \int_H h\boldsymbol{a}_{\text{new}}(h^{-1}v)h^{-1}dh. \tag{2.14}$$

Here $\boldsymbol{a}_{\text{new}}$ is determined as in (2.9) above, and $\tilde{\boldsymbol{a}}_{\text{new}}$ is our candidate for the transformed $\boldsymbol{a}$. By H-invariance of Haar measure, $\tilde{\boldsymbol{a}}_{\text{new}}$ now is indeed H-equivariant as required in (2.13). It remains to be shown, however, that $\tilde{\boldsymbol{a}}_{\text{new}}$ can be obtained by a transformation $g \mapsto g\ g_0(v)$ and, moreover, still consist of only resonant terms.

For this double purpose, we consider slightly more general transformations

$$g_0(v) := \exp(\gamma(v)) \tag{2.15}$$

where $\gamma(v)$ is now a polynomial in v, homogeneous of degree $\kappa = |\boldsymbol{k}|$, with values in alg (G). We introduce some notation: Let $\mathrm{alg}^{\kappa}(G)$ denote the space of these polynomials. With terms of lower degree left unchanged, the effect of transformation (2.15) on terms $\boldsymbol{a}_\kappa$ of degree κ in the Taylor expansion is

$$\boldsymbol{a}_{\mathrm{new},\kappa}(v) = \boldsymbol{a}_\kappa(v) + (\mathrm{ad}\, \boldsymbol{a}_0)\gamma(v) + \gamma'(v)Mv, \tag{2.16}$$

similar to (2.9) above. In particular, the range of $\boldsymbol{a}_{\mathrm{new},\kappa}$ which is obtained by transformations (2.15) with $\gamma(\cdot) \in \mathrm{alg}^{\kappa}(G)$, is an affine linear subspace $\boldsymbol{a}_\kappa(v) + \mathscr{A}^\kappa$ through $\boldsymbol{a}_\kappa(v)$ in $\mathrm{alg}^{\kappa}(G)$. The subspace $\mathscr{A}^\kappa$ is the range of the linear map

$$\begin{aligned} A: \quad & \mathrm{alg}^{\kappa}(G) \to \mathrm{alg}^{\kappa}(G) \\ & \gamma(v) \mapsto (\mathrm{ad}\, \boldsymbol{a}_0)\gamma(v) + \gamma'(v)Mv \end{aligned} \tag{2.17}$$

We consider the linear H-action

$$(h\gamma)(v) := h\gamma(h^{-1}v)h^{-1}, \tag{2.18}$$

on $\gamma \in \mathrm{alg}^{\kappa}(G)$.

We now reach our double purpose. To prove that averaging (2.14) produces $\tilde{\boldsymbol{a}}_{\mathrm{new}}(v)$, which can be realized by a transformation (2.15), we only have to show that the affine subspace $\boldsymbol{a}_\kappa(v) + \mathscr{A}^\kappa$ is invariant under the H-action (2.18). Indeed

$$(h\boldsymbol{a}_\kappa)(v) = h\boldsymbol{a}_\kappa(h^{-1}v)h^{-1} = \boldsymbol{a}_\kappa(v), \tag{2.19}$$

by equivariance (1.11) of the original Lie algebra term $\boldsymbol{a}(v)$. Moreover, the linear map A, with range $\mathscr{A}^\kappa$, is H-equivariant by (2.17), (2.18):

$$\begin{aligned} (h(A\gamma))(v) &= h((\mathrm{ad}\, \boldsymbol{a}_0)\gamma(h^{-1}v) + \gamma'(h^{-1}v)Mh^{-1}v)h^{-1} \\ &= (\mathrm{ad}\, \boldsymbol{a}_0)h\gamma(h^{-1}v)h^{-1} + h(\gamma'(h^{-1}v)h^{-1}Mv)h^{-1} \\ &= (\mathrm{ad}\, \boldsymbol{a}_0)(h\gamma)(v) + (h\gamma)'(v)Mv \\ &= (A(h\gamma))(v). \end{aligned} \tag{2.20}$$

Therefore $\mathscr{A}^\kappa = \mathrm{range}\, A$ is also H-invariant. In particular, averaging (2.14) remains in the subspace which can be reached by transformations (2.15).

Finally, we have to prove that averaging (2.14) lets us remain in the subspace of resonant terms. This subspace is precisely the kernel of the semisimple part A^s of A, given by

$$(A^s\gamma)(v) = (\mathrm{ad}\, \boldsymbol{a}_0)^s\gamma(v) + \gamma'(v)M^s v. \tag{2.21}$$

Here M^s denotes the unique semisimple (here: diagonal) part of $M = \varphi'(0)$. But A is H-equivariant, by (2.20), and therefore A^s is also H-equivariant, and so is $\ker A^s$, i. e., the subspace of resonant terms. This completes the proof of normal-form Theorem 1.2. □

Obviously, more refined results are possible, which exploit the possible nilpotency of both ad $\boldsymbol{a}_0$ and M. Specifically, only terms in an H-invariant complement to range A need to remain, in normal form. Because

$$\text{codim range } A = \dim\ker A \leqq \dim\ker A^s, \tag{2.22}$$

these can be fewer terms than those of ker A^s which are called resonant in Definition 1.1. For the applications which we have in mind in the present paper, however, the (semi-)simpler normal-form Theorem 1.2 proves sufficient.

3. Relative Equilibria of $SE(N)$ with Zero Eigenvalues

In this section, we apply normal-form Theorem 1.2 to relative equilibria $v = 0$ of the special Euclidian group $SE(N)$; see (1.1)–(1.4) for notation. Here and below, for simplicity of presentation, we only consider the case of trivial isotropy $H = \{\text{id}\}$. Nontrivial isotropy can eliminate further terms in our normal forms, of course, according to Theorem 1.2, because the normal form preserves any additional H-equivariance.

In coordinates (R, S) on $SE(N)$, and $(\boldsymbol{r}, \boldsymbol{s})$ on $se(N)$, the skew product (1.9) takes the form

$$\dot{R} = R\boldsymbol{r}(v), \qquad \dot{S} = R\boldsymbol{s}(v), \qquad \dot{v} = \varphi(v). \tag{3.1}$$

In Lemma 3.1, we compute and interprete the eigenvalues η_i of $\boldsymbol{a}_0$, which are used in Definition 1.1 of resonance. In Lemma 3.2, we compute and discuss the normal form of (1.9), (3.1), given by Theorem 1.2, in the case of an equilibrium $v = 0$ with only zero eigenvalues μ_p of $M = \varphi'(0)$. We conclude this section with a discussion of additional formal integrals, appearing in normal form, and of their proper interpretation as integrals beyond finite order in the context of the original system.

To compute the spectrum of ad $\boldsymbol{a}_0$, we first recall the commutator on elements $(\boldsymbol{r}, \boldsymbol{s})$ of the Lie algebra $se(N) = so(N) \times \mathbb{R}^N$ to be given by

$$[(\boldsymbol{r}_0, \boldsymbol{s}_0), (\boldsymbol{r}, \boldsymbol{s})] = ([\boldsymbol{r}_0, \boldsymbol{r}], \boldsymbol{r}_0\boldsymbol{s} - \boldsymbol{r}\boldsymbol{s}_0); \tag{3.2}$$

see for example [FSSW96, (4.3)]. To compute spec(ad $\boldsymbol{a}_0$), let $l = [N/2]$ and $\boldsymbol{a}_0 = (\boldsymbol{r}_0, \boldsymbol{s}_0)$. Note that the infinitesimal rotation matrix $\boldsymbol{r}_0$ is skew symmetric, and hence is (orthogonally) diagonalizable with purely imaginary spectrum. Typically

$$\text{spec } \boldsymbol{r}_0 = \begin{cases} \{\pm i\omega_j;\ j = 1, \ldots, l\} & \text{for } N = 2l, \\ \{0\} \cup \{\pm i\omega_j;\ j = 1, \ldots, l\} & \text{for } N = 2l + 1 \end{cases} \tag{3.3}$$

with $\omega_j > 0$.

Lemma 3.1. *Assume that* (3.3) *holds. Then* spec (ad $\boldsymbol{a}_0$) *is given by*

$$\begin{aligned} \text{spec (ad } \boldsymbol{a}_0) = \{0\} &\cup \{\pm i\omega_j;\ 1 \leqq j \leqq l\} \\ &\cup \{\pm i(\omega_{j_1} \pm \omega_{j_2});\ 1 \leqq j_i, j_2 \leqq l\}. \end{aligned} \tag{3.4}$$

Proof. We first observe that $(\boldsymbol{r}, \boldsymbol{s}) \in \{0\} \times \mathbb{C}^N$ form an invariant subspace. In fact

$$(\text{ad}\ \boldsymbol{a}_0)(0, \boldsymbol{s}) = (0, \boldsymbol{r}_0 \boldsymbol{s}) \tag{3.5}$$

embeds spec $\boldsymbol{r}_0$ into spec (ad $\boldsymbol{a}_0$). Also by (3.5) and (3.2), it is therefore sufficient to study

$$\text{spec}\ (\text{ad}_{so(N)}\ \boldsymbol{r}_0) \tag{3.6}$$

in order to determine spec (ad $\boldsymbol{a}_0$).

We describe $\boldsymbol{r}_0$ in block-diagonal form by complex eigenvectors. Let e_j denote the eigenvectors in $\mathbb{C}^N$ of the eigenvalues $i\omega_j$ of $\boldsymbol{r}_0 \in so(N)$, for $1 \leqq j \leqq l$. The complex conjugates $\bar{e}_j$ belong to $-i\omega_j$. For odd $N = 2l+1$, include a real nonzero eigenvector $e_0 \in \ker \boldsymbol{r}_0$ of $\omega_0 = 0$. Then the complex matrix $e_{j_1} e_{j_2}^T \in gl(N)$ is an eigen"vector" of $[\boldsymbol{r}_0, \cdot]$ with eigenvalue

$$i(\omega_{j_1} + \omega_{j_2}) \tag{3.7}$$

for $0 \leqq j_1 \leqq j_2 \leqq l$. Indeed,

$$\begin{aligned} [\boldsymbol{r}_0, e_{j_1} e_{j_2}^T] &= \boldsymbol{r}_0 e_{j_1} e_{j_2}^T - e_{j_1} e_{j_2}^T \boldsymbol{r}_0 \\ &= i(\omega_{j_1} + i\omega_{j_2}) e_{j_1} e_{j_2}^T. \end{aligned} \tag{3.8}$$

The complex conjugate transpose $\bar{e}_{j_2} \bar{e}_{j_1}^T$ is another eigen"vector" corresponding to the same eigenvalue. In particular, their skew-Hermitian sum $e_{j_1} e_{j_2}^T - \bar{e}_{j_2} \bar{e}_{j_1}^T$ is an eigen"vector" of ad $\boldsymbol{r}_0$. Note that the real and imaginary parts of this matrix are each skew symmetric, and hence in $so(N)$, by block diagonalization of $\boldsymbol{r}_0$, for $0 \leqq j_1 < j_2 \leqq l$. This accounts for eigenvalues $\pm i(\omega_{j_1} + \omega_{j_2})$ of ad $\boldsymbol{r}_0$. Similarly, $e_{j_1} \bar{e}_{j_2}^T$ produce eigenvalues $\pm i(\omega_{j_1} - \omega_{j_2})$, this time for $0 \leqq j_1 \leqq j_2 \leqq l$, excepting the case $j_1 = j_2 = 0$. Note that the matrix for $j_1 = j_2$ has zero real part and zero diagonal. Counting dimensions right, we therefore have spanned $so(N)$. This proves the lemma. □

We now proceed to compute the normal form of an $SE(N)$-equivariant skew product (1.9) near a relative equilibrium $v = 0$. By Theorem 1.2, nonresonant terms in $\boldsymbol{a}(v)$ can be eliminated, up to any finite order in v. If $\boldsymbol{a}(v)$ is analytic and contains only resonant terms, at all orders, we call the skew product (1.9) a *formal normal form.*

An actual smooth transformation may reduce the system to its formal normal form only up to terms of some prescribed order κ. Therefore, the original vector field (3.2), after normal-form transformation (1.21), differs from its formal normal form of Theorem 1.2 by some terms of order $\mathscr{O}(\|v\|^\kappa)$, where κ may be taken arbitrarily large. The influence of such terms becomes significant on times of order ε^{-k}, where ε is the size of the small neighborhood of the origin in the slice V in which the system is defined.

Therefore, the pictures given by formal normal forms are approximately valid on finite time intervals $|t| < t(\varepsilon)$, with error terms as just described, for $t(\varepsilon)$ tending to infinity as $\varepsilon \to 0$, proportionally to any finite power of ε^{-1}. This simple observation is completely analogous to the usual normal-form theory for vector fields; see for example [Van89]. In short, we say that formal normal forms are valid *beyond finite order*.

Let us, for a moment, consider the $\dot{v}$-equation of system (1.9) separately. It describes the flow of v in a small neighborhood of the equilibrium $v = 0$. If some number (say, n_c) of the eigenvalues μ_j of the linearization matrix $M = \varphi'(0)$ lie on the imaginary axis, then it is well known that there exists a smooth invariant n_c-dimensional *center manifold* W^c which we denote schematically as

$$\Psi(v) = 0 \tag{3.9}$$

where $\Psi : \mathbb{R}^n \to \mathbb{R}^{n-n_c}$. The center manifold contains the relative equilibrium $v = 0$ and it is tangent, at the equilibrium, to the generalized eigenspace of M which corresponds to the purely imaginary and zero eigenvalues.

For the complete system (1.9), equation (3.9) still defines an invariant manifold $\mathcal{M}^c = W^c \times SE(N)$ in $SE(N) \times \mathbb{R}^n$. This is no longer a local manifold, since the "global" factor $SE(N)$ is not compact. Nevertheless, $\mathcal{M}^c$ still contains all solutions which remain in a sufficiently small neighborhood of the relative equilibrium $\{0\} \times SE(N)$, for all real times. Therefore, without loss of generality, we may restrict any bifurcation considerations to $\mathcal{M}^c$. On $\mathcal{M}^c$, all eigenvalues μ_p lie on the imaginary axis. Specifically, we now consider the simplest case where all μ_p are zero.

Lemma 3.2. *Let $v = 0$ be a relative equilibrum of* (1.9) *such that all eigenvalues μ_p of the linearization $M = \varphi'(0)$ at $v = 0$ are zero. Furthermore assume that the eigenvalues $\pm i\omega_j$ of $\boldsymbol{r}_0 = \boldsymbol{r}(v = 0)$ to be distinct with $\omega_j > 0,\ j = 1, \dots, l,\ l = [N/2]$. Then the formal normal forms of systems* (1.9), (3.1) *are as given below.*

If $N = 2l$ is even, then

$$\dot{R} = R \begin{pmatrix} \Omega_1(v) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \Omega_l(v) \end{pmatrix}, \quad \dot{S} = 0, \quad \dot{v} = \varphi(v). \tag{3.10}$$

If $N = 2l + 1$ is odd, then

$$\dot{R} = R \begin{pmatrix} \Omega_1(v) & & \mathbf{0} & 0 \\ & \ddots & & \vdots \\ \mathbf{0} & & \Omega_l(v) & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}, \quad \dot{S} = R \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sigma_N(v) \end{pmatrix}, \quad \dot{v} = \varphi(v) \tag{3.11}$$

where

$$\Omega_j(v) = \begin{pmatrix} 0 & -\omega_j(v) \\ \omega_j(v) & 0 \end{pmatrix}.$$

Proof. Since all eigenvalues μ_p of M are zero, the term $a^{(i)}_{\boldsymbol{k}} v^{\boldsymbol{k}}$ is resonant if and only if

$$\eta_i = 0; \tag{3.12}$$

see (1.20). We therefore have to compute the generalized kernel of $\mathrm{ad}\,\boldsymbol{a}_0$, with a little help from Lemma 3.1.

We note that conjugation of $\boldsymbol{a}_0 = (\boldsymbol{r}_0, \boldsymbol{s}_0)$ by $(R_0, S_0) \in SE(N)$ does not change $\mathrm{spec}(\mathrm{ad}\,\boldsymbol{a}_0)$; it corresponds to a basis change in $\mathrm{alg}\,(G)$ and is an automorphism of the commutator. Because

$$(R_0, S_0)(\boldsymbol{r}_0, \boldsymbol{s}_0)(R_0, S_0)^{-1} = (R_0\boldsymbol{r}_0R_0^{-1}, -R_0\boldsymbol{r}_0R_0^{-1}S_0 + R_0\boldsymbol{s}_0), \tag{3.13}$$

we can always assume that

$$\boldsymbol{s}_0 \in \ker \boldsymbol{r}_0 = (\mathrm{range}\ \boldsymbol{r}_0)^{\perp} \tag{3.14}$$

after a translation $R_0 = \mathrm{id}$, $S_0 \in \mathbb{R}^N$. Also, the skew symmetric matrix $\boldsymbol{r}_0$ can be assumed to be block-diagonal, by some $R_0 \in SO(N)$ and $S_0 = 0$.

We first consider the case $N = 2l$ even. Then $\ker \boldsymbol{r}_0 = \{0\}$, by assumption, and hence $\boldsymbol{s}_0 = 0$, by (3.14). In particular, (3.2) implies that

$$(\mathrm{ad}\ \boldsymbol{a}_0)(\boldsymbol{r}, \boldsymbol{s}) = ([\boldsymbol{r}_0, \boldsymbol{r}], \boldsymbol{r}_0\boldsymbol{s}). \tag{3.15}$$

Therefore, $\mathrm{ad}\ \boldsymbol{a}_0$ is semisimple, and resonant terms belong to the kernel, rather than to the generalized kernel. The kernel is given by $\boldsymbol{s} = 0$ and $\boldsymbol{r}$ an arbitrary block-diagonal matrix of rotations $\Omega_j(v)$, one for each eigenspace of eigenvalues $\pm i\omega_j$ of $\boldsymbol{r}_0$. See the proof of Lemma 3.1 for further details.

Now we consider the case $N = 2l + 1$ odd. If $\boldsymbol{s}_0 = 0$, the previous arguments apply and yield $\sigma_N(v) \equiv 0$ in the formal normal form. If $\boldsymbol{s}_0 \neq 0$, we may assume that $\boldsymbol{s}_0 = e_0$ spans $\ker \boldsymbol{r}_0$, without loss of generality; see the proof of Lemma 3.1 for notation. Then

$$(\mathrm{ad}\ \boldsymbol{a}_0)(\boldsymbol{r}, \boldsymbol{s}) = ([\boldsymbol{r}_0, \boldsymbol{r}], \boldsymbol{r}_0\boldsymbol{s} - \boldsymbol{r}e_0). \tag{3.16}$$

The expressions for $\boldsymbol{r}(v), \boldsymbol{s}(v)$ appearing in the right-hand sides of (3.1), (3.11) span $\ker(\mathrm{ad}\ \boldsymbol{a}_0)$, as before. A generalized kernel does not appear. Indeed, $[\boldsymbol{r}_0, \cdot]$ is semisimple on $so(N)$. In particular, $\boldsymbol{r} \in \ker_{so(N)}[\boldsymbol{r}_0, \cdot]$ for $(\boldsymbol{r}, \boldsymbol{s})$ in the generalized kernel. Therefore, $\boldsymbol{r}e_0 = 0$, and $r_0\boldsymbol{s} = 0$. This implies that $(\boldsymbol{r}, \boldsymbol{s}) \in \ker(\mathrm{ad}\ \boldsymbol{a}_0)$, and the lemma is proved. □

We now discuss the evolution $(R(t), S(t))$ on the group, which is generated by the v-flow in the (formal) normal forms (3.10), (3.11) of Lemma 3.2. By $SE(N)$-equivariance, we can always take $R(0) = \mathrm{id}$, $S(0) = 0$ as an initial condition. As was pointed out earlier, our conclusions on the *formal normal flow*, that is, the flow of the formal normal form, are valid beyond finite order $\varepsilon^{-\kappa}$ in time.

We begin with the formal normal flow for $N = 2l$ even; see (3.10). Then

$$R(t) = \begin{pmatrix} R_1(t) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & R_l(t) \end{pmatrix} \tag{3.17}$$

preserves block-diagonal form for all t, with (2×2)-blocks

$$R_j(t) = \begin{pmatrix} \cos\alpha_j(t) & -\sin\alpha_j(t) \\ \sin\alpha_j(t) & \cos\alpha_j(t) \end{pmatrix}. \tag{3.18}$$

In particular, $R(t)$ possesses l fixed two-dimensional planes of rotation. The rotation phases α_j in these planes are found by direct integration of

$$\dot{\alpha}_j = \omega_j(v) \tag{3.19}$$

with initial conditions $\alpha_j(0) = 0$. Moreover, $\dot{S} = 0$ in (3.10) implies

$$S(t) \equiv 0. \tag{3.20}$$

In particular, we obtain $\frac{1}{2}N^2$ integrals of motion on the group $SE(N)$ of dimension $\frac{1}{2}N(N+1)$: the components of the shift, and the positions of the l rotation planes.

For proper interpretation of this nondrift result in terms of the original evolution $(\tilde{R}(t), \tilde{S}(t))$ on $SE(N)$, we have to invert the normal-form transformation (1.21). We should also remember the approximate nature of our (and any) formal normal form, due to error terms of order $\mathscr{O}(\|v\|^\kappa)$ in formal normal form vector fields.

Inverting the normal form transformation (1.21), we still obtain $\frac{1}{2}N^2$ first integrals for the original evolution $(\tilde{R}(t), \tilde{S}(t))$ on $SE(N)$, up to order $\mathscr{O}(\|v\|^\kappa)$. With $g_0(v) =: (R_0(v), S_0(v))$, these approximate integrals are given by (1.21) as relations

$$\begin{aligned} \tilde{R}(t)R_0(v(t)) &= R(t), \\ \tilde{R}(t)S_0(v(t)) + \tilde{S}(t) &= S(t) = 0 \end{aligned} \tag{3.21}$$

in terms of the block structure of $R(t)$ and the vanishing of $S(t)$.

Recall that these are only *formal integrals*: they are approximately conserved for only finite, though very large, time — beyond finite order κ. For example, we observe a relative slaving beyond finite order of the original drift $\tilde{S}$ in terms of original rotation $\tilde{R}$ and original shape v:

$$\tilde{S}(t) = -\tilde{R}(t)S_0(v(t)) + t\mathscr{O}(\|v\|^\kappa) = S(\tilde{R}, v) + t\mathscr{O}(\|v\|^\kappa). \tag{3.22}$$

Here $\|v\|$ refers to the sup norm over the interval from 0 to t.

In the odd-dimensional case $N = 2l + 1$, integration of the formal normal form (3.11) gives the rotation matrix $R(t)$ in block-diagonal form again:

$$R(t) = \begin{pmatrix} R_1(t) & & & \mathbf{0} \\ & \ddots & & \\ & & R_l(t) & \\ \mathbf{0} & & & 1 \end{pmatrix} \tag{3.23}$$

where $R_j(t)$ and the evolution of the rotation phases α_j are as in (3.18), (3.19). Thus, there are l fixed two-dimensional rotation planes and one axis of rotation which remain fixed under the action of $R(t)$. Also, there is no shift in the directions orthogonal to the rotation axis. At the same time, the shift $s(t)$ along the rotation axis is governed by

$$\dot{s} = \sigma_{N+1}(v). \tag{3.24}$$

Returning to the original system by transformation (1.21), we obtain $\frac{1}{2}(N^2 - 1)$ formal integrals, which define the v-dependence of the rotation axis, of the rotation planes, and of the shift in the directions orthogonal to the rotation axis. Mutatis mutandis, our interpretation of the case of even N applies.

4. Tip Motion in $SE(N)$ Systems

The partial differential equations mentioned in the introduction exhibit solution patterns $u = u(t, x)$, both experimentally and numerically, which intuitively look like spirals with a superimposed rotational and translational motion; see Fig. 4.1. In a local center manifold $\mathscr{M}^c$ near a relative equilibrium $G \cdot u_0$, $G = SE(N)$, the dynamics is described by

$$\dot{R} = R\boldsymbol{r}(v), \qquad \dot{S} = R\boldsymbol{s}(v), \qquad \dot{v} = \varphi(v); \tag{4.1}$$

see (3.1) and Section 3. In experiments, the translational motion is usually represented as the motion of some distinguished point $\boldsymbol{x}(t) \in \mathbb{R}^N$ on the solution profile; for $N = 2$ see Fig. 4.1. Examples for $\boldsymbol{x}(t)$ are the "tip" (b) of maximal curvature of a distinguished, sharp reaction front, the inflection point (d) of the front, or the "core" (a) of the spiral. In the present section, we try to clarify both ambiguity and benefit of these concepts. We give a precise, albeit general, meaning to the concept of a "tip" $\boldsymbol{x}(t)$; see Definition 4.1. In Proposition 4.2 we then prove that the tip motion $\boldsymbol{x}(t)$ satisfies a differential equation which coincides with $\dot{S} = R\boldsymbol{s}(v)$ from (4.1), except for an explicit correction term of the vector field $\boldsymbol{s}(v) \in \mathbb{R}^N$. In discussing the dynamical behavior of systems (4.1), for general $\boldsymbol{s}(v)$, it is therefore justified to call $S(t)$ itself a "tip motion".

Definition 4.1. Let u_0 be a relative equilibrium to $G = SE(N)$ with differentiable, G-invariant local center manifold $\mathscr{M}^c$. By "the" *tip* of $u \in \mathscr{M}^c$, we mean the value $\boldsymbol{x}(u)$ at u of a differentiable, G-equivariant function

$$\boldsymbol{x} : \mathscr{M}^c \to \mathbb{R}^N \tag{4.2}$$

Here G-equivariance is required with respect to the given action of G on $\mathscr{M}^c$ and the affine representation (1.2) of $G = SE(N)$ on $\mathbb{R}^N$.

Similarly, let H denote the (compact) isotropy of u_0. Then a *tip function* x^* on the Palais slice V is a differentiable, H-equivariant function

$$x^* : V \to \mathbb{R}^N. \tag{4.3}$$

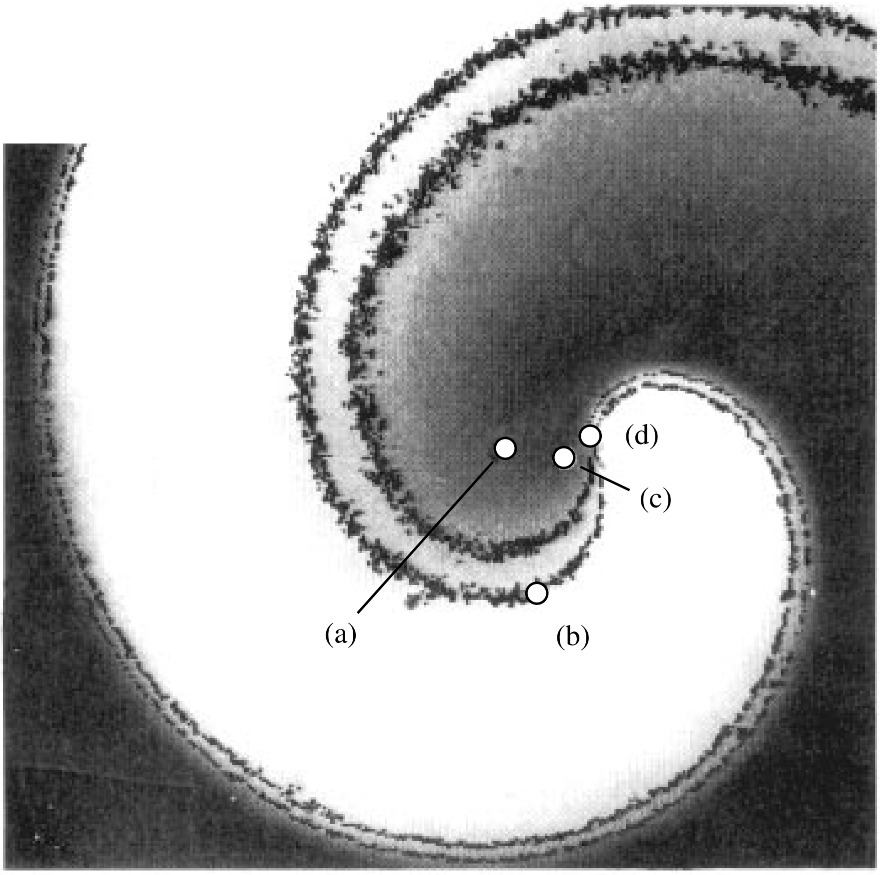

Fig. 4.1. A spiral wave pattern in the Belousov-Zhabotinsky reaction, by courtesy of [MZ94].

Differentiability is required to derive a differential equation for $\boldsymbol{x}$ in Proposition 4.2 below. We restrict our definition to $\mathscr{M}^c$, because typical G-actions are not even continuous on the underlying function spaces for u.

Our abstract definition of the tip $\boldsymbol{x}$ on $\mathscr{M}^c$ is motivated as follows. Let $u = u(x)$ be a function on $\mathscr{M}^c$. Then a "distinguished" point $\boldsymbol{x}(u)$ of the profile $u(\cdot)$ can be chosen as a "core" (a), a "tip" (b), a "rotation center" (c), an "inflection point" (d), etc., by a lot of different extraction rules. A common feature of all rules should be equivariance: a tip, core, etc., of $(gu)(x) := u(g^{-1}x)$ should sit at position $g\boldsymbol{x}$ if the position for u is denoted by $\boldsymbol{x} \in \mathbb{R}^N$. In other words, the function $\boldsymbol{x} = \boldsymbol{x}(u(\cdot))$ in (4.2) determines some specific extraction rule for whatever "distinguished" point is of interest. This rule ought to be compatible with equivariance under $G = SE(N)$.

This motivates our definition of $\boldsymbol{x}$. To clarify the relation between tips $\boldsymbol{x}$ and tip functions x^*, we represent the center manifold $\mathscr{M}^c$ by differentiable coordinates

$$\Psi : G \times V \to \mathscr{M}^c \tag{4.4}$$

which is $(G \times H)$-equivariant, that is,

$$\Psi((g_0, h) \cdot (g, v)) = g_0 \Psi(g, v), \tag{4.5}$$

under the action (1.10) of $G \times H$ on $G \times V$. We recall from [FSSW96] that $G \times V \to \mathcal{M}^c$ is a differentiable, H-principal fiber bundle with fibers given by the H-orbits

$$(\mathrm{id}, h) \cdot (g, v) = (gh^{-1}, hv). \tag{4.6}$$

Now consider the composition

$$\tilde{x}^* := \boldsymbol{x} \circ \Psi : \quad G \times V \to \mathbb{R}^N \tag{4.7}$$

for any given tip $\boldsymbol{x}$ on $\mathcal{M}^c$. Then $(G \times H)$-equivariance (4.5) implies that

$$\begin{aligned} \tilde{x}^*((g_0, h) \cdot (g, v)) &= \boldsymbol{x}(\Psi((g_0, h) \cdot (g, v))) \\ &= \boldsymbol{x}(g_0 \Psi(g, v)) = g_0 \boldsymbol{x}(\Psi(g, v)) \\ &= g_0 \tilde{x}^*(g, v). \end{aligned} \tag{4.8}$$

With $h = \mathrm{id}$, $g = \mathrm{id}$, this implies that

$$\tilde{x}^*(g_0, v) = g_0 \tilde{x}^*(\mathrm{id}, v). \tag{4.9}$$

Putting $g_0 = h$, $g = \mathrm{id}$ in (4.8) and using (1.10), we obtain

$$\tilde{x}^*(\mathrm{id}, hv) = h \tilde{x}^*(\mathrm{id}, v). \tag{4.10}$$

In other words, the tip $\boldsymbol{x}$ on the center manifold $\mathcal{M}^c$ defines a tip function $x^*(v) := \tilde{x}^*(\mathrm{id}, v)$ on the slice V. The converse is also true, because

$$\boldsymbol{x}(\Psi(g, v)) := g x^*(v) \tag{4.11}$$

is a well-defined tip on $\mathcal{M}^c$, by H-equivariance of the tip function x^* on V.

We now derive the differential equation for motion of the tip $\boldsymbol{x}$ on $\mathcal{M}^c$.

Proposition 4.2. *Let $u = u(t, x)$ denote a trajectory of the G-equivariant flow on the center manifold $\mathcal{M}^c$, and define the tip motion as*

$$\boldsymbol{x}(t) := \boldsymbol{x}(u(t, \cdot)), \tag{4.12}$$

for some tip $\boldsymbol{x}$ on $\mathcal{M}^c$ as in Definition 4.1. Then $\boldsymbol{x}(t)$ satisfies the differential equation

$$\dot{\boldsymbol{x}} = R(t) \tilde{\boldsymbol{s}}(v(t)) \tag{4.13}$$

where $R(t)$, $v(t)$ are solutions of (4.1) *and $\tilde{\boldsymbol{s}}$ is given explicitly by*

$$\tilde{\boldsymbol{s}}(v) = \boldsymbol{s}(v) + \boldsymbol{r}(v) x^*(v) + D x^*(v) \varphi(v). \tag{4.14}$$

Here $x^*(v) = \tilde{x}^*(\mathrm{id}, v)$ is the tip function on the slice V associated with $\boldsymbol{x}(u)$ by (4.7), (4.10) *above.*

Proof. Following (4.4)–(4.10), we consider the dynamics of

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{x}(u(t,\cdot)) = (\boldsymbol{x} \circ \Psi)(g(t), v(t)) \\ &= \tilde{\boldsymbol{x}}^*(g(t), v(t)) = g(t)x^*(v(t)) \\ &= R(t)x^*(v(t)) + S(t). \end{aligned} \tag{4.15}$$

By [FSSW96], the $(G \times H)$-equivariant flow (3.1), (4.1) on the total space $G \times V$ projects to the G-equivariant flow on the base $\mathscr{M}^c$, in the H-principal bundle of $G \times V$ over $\mathscr{M}^c$. Differentiation of (4.15) with respect to t by the chain rule, and (3.1), (4.1) therefore imply (4.13) and (4.14). This proves the proposition. □

We remark that flows depending on parameters λ can be treated in exactly the same way, with $\boldsymbol{r} = \boldsymbol{r}(\lambda, v)$, $\boldsymbol{s} = \boldsymbol{s}(\lambda, v)$, $\varphi = \varphi(\lambda, v)$ in (4.1), and $\boldsymbol{x} = \boldsymbol{x}(\lambda, u)$, $x^* = x^*(\lambda, v)$ in (4.2), (4.3).

We now return to system (4.1), having identified $S(t)$ as equivalent to tip motion $\boldsymbol{x}(t)$. To be more specific, we discuss the planar case $G = SE(2)$. We introduce complex notation $z(t) \in \mathbb{C}$, for $S(t) \in \mathbb{R}^2$, and $\exp(i\alpha(t)) \in \mathbb{C}$, for $R(t) \in SO(2)$. With this notation, system (4.1) with parameters becomes

$$\dot{\alpha} = \omega(\lambda, v), \qquad \dot{z} = e^{i\alpha}\sigma(\lambda, v), \qquad \dot{v} = \varphi(\lambda, v). \tag{4.16}$$

Here v belongs to some neighborhood v of the origin in $\mathbb{R}^n$, and $\lambda \in \mathbb{R}^k$ are some small parameters; ω and φ are real-valued functions, and σ is complex.

We consider the case $\omega(0, 0) \neq 0$. Since we are in a small neighborhood of the origin, we may assume $\omega(\lambda, v) \neq 0$ everywhere. Thus, we may rescale time so that system (4.16) takes the form

$$\dot{\alpha} = \omega_0, \qquad \dot{z} = e^{i\alpha}(\omega_0/\omega)\sigma, \qquad \dot{v} = (\omega_0/\omega)\varphi. \tag{4.17}$$

This system has the same form as (4.16). Therefore we may just assume that $\omega = \omega_0 \equiv \text{const}$, in (4.16). The choice of $\omega_0 \neq 0$ is, of course, completely arbitrary.

For $\omega \equiv \omega_0$ the integration of the equations for α and the tip shift z is simple:

$$\alpha(t) = \alpha_0 + \omega_0 t, \tag{4.18}$$

$$z(t) = z_0 + e^{i\alpha_0} \int_0^t e^{i\omega_0 s}\sigma(v(s))\, ds \tag{4.19}$$

where $v(t)$ is picked up from the last equation of (4.16). The evolution of v is independent of α and z, and, in principle, an arbitrary dynamics in $V \subseteq \mathbb{R}^n$ is possible. Again, we suppress λ.

According to (4.19), the increment of $z(t)$ measures the Fourier harmonic of the input $\sigma(v(t))$ at frequency ω_0. In the following, we investigate various types of dynamics of the tip motion in the z-plane, in response to various possible types of time behavior of the v-variables (stationary states, periodic motions, homoclinics).

Consider an equilibrium state: $v(t) = v_0 \equiv \text{const}$, $\varphi(v_0) = 0$. Then $\sigma(v) = \sigma_0 \equiv \text{const}$ and we obtain the dynamics

$$z(t) = z_0 - ie^{i\alpha_0}\frac{\sigma_0}{\omega_0}\left(e^{i\omega_0 t} - 1\right). \tag{4.20}$$

Thus, z moves along the circle with center $(z_0 + ie^{i\alpha_0}\frac{\sigma_0}{\omega_0})$ and with radius $|\frac{\sigma_0}{\omega_0}|$. Note that the coordinate transformation

$$z_{\text{new}} = z + ie^{i\alpha}\frac{\sigma(v)}{\omega_0} \tag{4.21}$$

preserves the form of system (4.16), transforming $\sigma(v)$ into

$$\sigma_{\text{new}}(v) = i\frac{\sigma'(v)}{\omega_0}\varphi(v). \tag{4.22}$$

In particular, $\dot{z}_{\text{new}}$ vanishes at all relative equilibria with $\varphi(v) = 0$, in the new coordinates. Therefore $z_{\text{new}}(t) = \text{const}$ for these solutions. Thus, after our coordinate transformation, the value of the tip position z coincides with the center of rotation of the rotating wave; see Fig. 4.1, point (c).

Of course, any coordinate transformation which preserves

$$\sigma(v) = 0 \ \text{ at } \ \varphi(v) = 0 \tag{4.23}$$

is allowed. Therefore the tip of the wave is still not defined uniquely outside of the relative equilibria — and cannot be.

5. Drift Resonance of Relative Periodics

Let $G = SE(2)$. In this section, we consider *relative periodics*, that is, the case of a periodic orbit in the v-variables of (4.1), (4.16). Then $\sigma(v(t))$ is a periodic function, say with minimal period $T > 0$. In degenerate cases, this period can be an integer fraction of the minimal period of $v(t)$ itself. We exclude the case where $\sigma(v(t))$ is stationary, but $v(t)$ is not. We derive a general, at most quasiperiodic expression for the tip motion of relative periodics, in (5.3). We then discuss tip motions in the case of drift resonance. We are mainly setting up notation in this section, for our readers' convenience, not claiming much originality; see also [FSSW96, GLM97].

We denote the Fourier expansion for σ by

$$\sigma(v(t)) = \sum_{k=-\infty}^{k=+\infty} \sigma_k e^{ik\omega_1 t}, \tag{5.1}$$

$\omega_1 = 2\pi/T > 0$. With time scaled to $\dot{\alpha} = \omega_0$, as in (4.17), the tip motion of the relative periodic is given by

$$z(t) = z_0 + e^{i\alpha_0}\sum_{k=-\infty}^{k=+\infty} \sigma_k \int_0^t e^{i(k\omega_1+\omega_0)s}\,ds. \tag{5.2}$$

If $\omega_0/\omega_1 \notin \mathbb{Z}$, then there are no resonant terms, and

$$z(t) = C + e^{i\omega_0 t} Q(t) \tag{5.3}$$

where Q is a complex T-periodic function and $C \in \mathbb{C}$ is a constant. By (5.3), the epicycle motion of $z(t)$ is confined to the closed annulus bounded by the two circles with center at $z = C$ and radii

$$\sigma^- := \min_{0 \leqq t \leqq T} |Q(t)|, \qquad \sigma^+ := \max_{0 \leqq t \leqq T} |Q(t)|. \tag{5.4}$$

If ω_0 and ω_1 are incommensurate, then the tip motion $z(t)$ is quasiperiodic and dense in the annulus. If $\omega_0/\omega_1 \in \mathbb{Q}$, then the tip motion $z(t)$ is periodic and hence closes up in the annulus.

Note that the position of the center C depends on initial conditions z_0, α_0, and the initial phase on the periodic orbit. The radii $\sigma^\pm$ of the annulus do not depend on these initial conditions. We call the epicyclic motion (5.3) in the annulus *a meander*, C is *the center of the meander*, and $\sigma^\pm$ are *the radii of the meander*.

In the resonant case $\omega_0 = m\omega_1$ where m is some integer, formula (5.3) is not valid and we have an $m : 1$ *drift resonance* between the rotation frequency ω_0 of the spiral wave and the frequency ω_1 of its shape modulation v. Now, integral (5.2) gives the tip motion

$$z(t) = c + e^{i\alpha_0} \sigma_{-m} t + e^{i\omega_0 t} q(t) \tag{5.5}$$

where $q(t)$ is a complex periodic function and $c \in \mathbb{C}$ is a constant. From this point of view, the center of epicyclic meandering

$$C = c + e^{i\alpha_0} \sigma_{-m} t \tag{5.6}$$

moves to infinity along a straight line with the constant velocity $|\sigma_{-m}|$. To study the transition of $\omega_0/\omega_1 \approx m$ through $m : 1$ drift resonance, we single out the Fourier term σ_{-m} in (5.2), which accounts for the difference $Q - q$. For the z-motion near the drift resonance, we thus obtain

$$z(t) = c + e^{i\alpha_0} \sigma_{-m} \frac{e^{i(\omega_0 - m\omega_1)t} - 1}{i(\omega_0 - m\omega_1)} + e^{i\omega_0 t} q(t). \tag{5.7}$$

From this point of view, the epicyclic meander $e^{i\omega_0 t} q(t)$ has radii $\min |q|$ and $\max |q|$ and the center of this meander moves with velocity $|\sigma_{-m}|$ along a large circle of radius $|\sigma_{-m}|/|\omega_0 - m\omega_1|$. The center of the large circle coincides with the constant C in representation (5.3).

6. Relative Hopf Bifurcation with Drift Resonance

In this section, we consider (relative) Andronov-Hopf bifurcation in the v-variable of system (4.16). As in (4.17), we normalize $\omega(\lambda, v) = \omega_0$ and consider

$$\dot{\alpha} = \omega_0, \qquad \dot{z} = e^{i\alpha}\sigma(\lambda, v), \qquad \dot{v} = \varphi(\lambda, v) \tag{6.1}$$

with Hopf bifurcation in v. The nonresonant case is easy; see [FSSW96, GLM97]. In this section we derive and discuss the formal normal forms of the resonant cases, as provided by Theorem 1.2. We put the v-equation in formal normal form

$$\dot{v} = \tilde{\varphi}(\lambda, |v|^2)v \tag{6.2}$$

with complex notation for $v \in \mathbb{R}^2 = \mathbb{C}$. Note that $\tilde{\varphi}$ is complex-valued and

$$i\omega_1 := \tilde{\varphi}(0, 0) \neq 0. \tag{6.3}$$

We first derive the resonance conditions (1.20) in this case. We then discuss the case $\omega_0 = m\omega_1$ of $m : 1$ resonance between the rotation frequency ω_0 and the relative Hopf frequency ω_1.

To derive the resonance conditions we compute the spectrum of the adjoint ad $\boldsymbol{a}_0$. We write $\boldsymbol{a}_0 = (\boldsymbol{r}_0, \boldsymbol{s}_0)$, in the notation of Section 3. By the normalization (4.21), (4.22), we can assume $\boldsymbol{s}_0 = 0$. By Lemma 3.1,

$$\operatorname{spec}(\operatorname{ad} \boldsymbol{a}_0) = \{0, \pm i\omega_0\}. \tag{6.4}$$

By the proof of Lemma 3.1, the eigenvalue $\eta = 0$ belongs to the $so(2)$-components, here represented by the normalized equation $\dot{\alpha} = \omega_0$. The eigenvalues $\eta = \pm i\omega_0$, in contrast, belong to the $\mathbb{R}^2$-components $\boldsymbol{s}$, here represented by $\sigma(\lambda, v)$. Since $M = D_v\varphi(\lambda{=}0, v{=}0)$ possesses only the eigenvalues $\pm i\omega_1(\lambda{=}0)$, represented in complex notation by coordinates $v, \bar{v}$, respectively, the resonance condition (1.20) becomes

$$-i\omega_0 = -\eta = i\omega_1(k_1 - k_2) \tag{6.5}$$

for terms $\exp(i\alpha)v^{k_1}\bar{v}^{k_2}$ in the $\dot{z}$-equation. This implies that

$$\omega_0 = m\omega_1 \tag{6.6}$$

produces the only possible resonance

$$k_2 = k_1 + m \tag{6.7}$$

for nonzero integer m. If ω_0, ω_1 are of the same sign, then m in (6.6) is positive, and

$$\dot{z} = e^{i\alpha}\tilde{\sigma}(\lambda, |v|^2) \cdot \bar{v}^m, \tag{6.8}$$

with $\tilde{\sigma}$ complex, becomes the desired formal normal form for tip motion at $m : 1$ resonant relative Hopf bifurcation. If ω_0, ω_1 have opposite sign, then m is negative and we obtain

$$\dot{z} = e^{i\alpha}\tilde{\sigma}(\lambda, |v|^2)v^{|m|} \tag{6.9}$$

as the formal normal form. If we take $\bar{z}$ as a new tip variable, then (6.9) coincides with (6.8). For $\omega_0 \notin \omega_1\mathbb{Z}$, that is, in the absence of resonance, the formal normal form becomes

$$\dot{z} = 0. \tag{6.10}$$

Without proper interpretation, this "nondrift" result may look surprising — or even wrong. To a naive reader, it might appear to be saying that meandering spirals do not really meander — contradicting all experimental evidence and contradicting the analysis in [Wul96, FSSW96, GLM97].

Of course, the proper interpretation of "nondrift" result (6.10) is similar to our interpretation of the zero eigenvalue nondrift result (3.20) at the end of Section 3. We have to invert the normal form transformation (1.21), with $g_0(v) =: (e^{i\alpha_0(v)}, z_0(v))$, to obtain the original evolution $(e^{i\tilde{\alpha}(v)}, \tilde{z}(v))$ on $SE(2)$. For the tip motion $\tilde{z}(t)$ in the original system, the normal form transformation (3.20) implies that

$$\tilde{z}(t) = -e^{i\omega_0 t} e^{-i\alpha_0(v(t))} z_0(v(t)) + z(t). \tag{6.11}$$

Nondrift $z(t) \equiv 0$ now clearly yields a quasiperiodic motion for the original tip position $\tilde{z}(t)$, beyond finite order.

Our normal form approach in fact leaves open the possibility of perturbation terms, beyond finite order, which destroy the very quasiperiodicity of the motion. Such terms are however not present, as our direct analysis in Section 5 has shown. In particular, our normal form analysis determines the coefficients C and $Q(t)$, beyond finite order.

It is now also easy to compute the epicycle expansion (5.7) for drift resonant tip motion (6.8), along the bifurcating sheet of periodic solutions $v(t)$. We consider two parameters, $\lambda = (\lambda_1, \lambda_2)$, normalized such that $\lambda_2^{1/2}$, for $\lambda_2 > 0$, describes the amplitude $|v(t)|$. Similarly, $T = 2\pi/\omega_1$ with $\omega_1 = \omega_1(\lambda = 0) + \lambda_1$ describes the minimal period of the bifurcating periodic solutions $v(t)$. Inserting these expressions into (6.8) yields the tip motion

$$\begin{aligned} z(\lambda_1, \lambda_2, t) &= z_0 + e^{i\alpha_0} \int_0^t e^{i\omega_0 s} \tilde{\sigma}(\lambda, \lambda_2) \lambda_2^{m/2} e^{-im\omega_1 s} ds \\ &= c + e^{i\alpha_0} \tilde{\sigma}(\lambda, \lambda_2) \lambda_2^{m/2} \frac{e^{-im\lambda_1 t} - 1}{-im\lambda_1}. \end{aligned} \tag{6.12}$$

Indeed, this has the form (5.7) with

$$c = z_0, \qquad \sigma_{-m} = \tilde{\sigma}(\lambda, \lambda_2)\lambda_2^{m/2}, \qquad q(\lambda, t) \equiv 0 \tag{6.13}$$

in formal normal form. Passing to the tip shift $\tilde{z}(t)$ in the original system, by transformation (6.11), we obtain

$$q(\lambda, t) = -e^{-i\alpha_0(v(t))} z_0(v(t)). \tag{6.14}$$

Note that the motion along the large circle persists, for $\lambda \to 0$, with limiting speed

$$\sigma_{-m} \sim \lambda_2^{m/2}. \tag{6.15}$$

Again by the analysis of Section 5, perturbation terms beyond finite order which could destroy expressions (6.13), (6.14) for the coefficients in expansion (5.7) do not exist. Therefore, (6.13), (6.14) themselves hold beyond finite order in $|v(t)| = \lambda_2^{1/2}$.

7. Relative Weak Focus with Drift Resonance

A weak focus occurs at Hopf bifurcation, when third-order terms are nondegenerate. The formal normal form for a weak focus is

$$\dot{v} = \tilde{\varphi}(|v|^2)v, \tag{7.1}$$

where $\tilde{\varphi}(r^2) = i\omega_1 + \varphi_1 r^2 + \cdots,\ \varphi_1 \in \mathbb{C},\ \omega_1 \in \mathbb{R} \setminus \{0\}$. In the present section, we study $m:1$ resonances between the rotation frequency ω_0 and the (relative) Hopf frequency ω_1. We consider the tip motion $z(t)$ in the resonant cases, as given by the formal normal form

$$\dot{z} = e^{i\alpha}\tilde{\sigma}(|v|^2)\bar{v}^m, \tag{7.2}$$

with $\alpha = \alpha_0 + \omega_0 t$, nonzero ω_0, complex $\tilde{\sigma}$, and $m > 0$; see (6.8). By putting $m = 0$ in (7.2), the nonresonant cases $\omega_0 \notin \mathbb{Z}\omega_1$ can be subsumed.

Throghout this section, we ignore the correction term (6.11) which distinguishes tip motion $\tilde{z}(t)$ in the original system from tip motion $z(t)$ in formal normal form. Similarly, terms $t\mathcal{O}(\|v\|^\kappa)$ beyond finite order are omitted.

We introduce polar coordinates $v = re^{i\psi}$. Reversing time and scaling r, if necessary, we expand

$$\begin{aligned} \dot{r} &= (\operatorname{Re}\tilde{\varphi}(r^2))r = -r^3 + \cdots, \\ \dot{\psi} &= \operatorname{Im}\tilde{\varphi}(r^2) \quad = \omega_1 + \hat{\tilde{m}}\omega_2 r^2 + \cdots. \end{aligned} \tag{7.3}$$

For $t \to +\infty$, we obtain the asymptotics

$$r(t) = (2t)^{-1/2} + \mathcal{O}(t^{-1}). \tag{7.4}$$

In the *nonresonant* case, normalization (4.21), (4.22) implies $\tilde{\sigma}(0) = 0$. Therefore

$$\dot{z} = e^{i\alpha}(\sigma_1 r^2 + \cdots) \tag{7.5}$$

for some $\sigma_1 \in \mathbb{C}$. Integration gives the tip motion

$$z(t) = z_0 + e^{i\alpha_0}\int_0^t e^{i\omega_0 s}(\sigma_1 r^2(s) + \cdots)\, ds. \tag{7.6}$$

Inserting expansion (7.4), we obtain a convergent oscillatory integral in (7.6). Hence

$$z_\infty := \lim_{t \to +\infty} z(t) \tag{7.7}$$

exists and is finite.

We consider the $m:1$ *resonant* case next. Replacing v by $\bar{v}$, if necessary, we assume that $\omega_2 > 0$ in expansion (7.3). We also fix the orientation of z such that $\omega_0 = m\omega_1$ with $m > 0$, and hence with formal normal form (7.2). The tip motion is given by

$$z(t) = z_0 + e^{i\alpha_0} \int_0^t e^{i\omega_0 s} \tilde{\sigma}(r^2) r^m e^{-im\psi(s)} ds. \tag{7.8}$$

Since $\tilde{\sigma}(r^2) r^m = 0$ for $r = 0$ is satisfied automatically, we can assume

$$\tilde{\sigma}(r^2) = \tilde{\sigma}_0 + \cdots \tag{7.9}$$

with $\tilde{\sigma}_0 \neq 0$, this time. To evaluate the integral (7.8), we introduce a new time variable

$$\tau := -\omega_0 t + m\psi(t). \tag{7.10}$$

By (7.3), (7.4), we can expand $\dot{\tau}(t)$ as

$$\begin{aligned} \dot{\tau} &= -\omega_0 + m\dot{\psi} = -\omega_0 + m\omega_1 + \omega_2 r^2 + \cdots \\ &= \omega_2 r^2 + \cdots \\ &= \tfrac{1}{2}\omega_2 t^{-1} + \cdots \end{aligned} \tag{7.11}$$

with remainder terms of order r^4, t^{-2}, for $r \to 0,\ t \to +\infty$, respectively. Note that $\dot{\tau} > 0$, because $\omega_2 > 0$; hence τ can be viewed as a transformed time, indeed. Also note the asymptotics

$$\tau(t) = \frac{1}{2}\omega_2 \log t + \cdots \tag{7.12}$$

with decaying remainder of order t^{-1}. Denoting $' = \frac{d}{d\tau}$ we obtain

$$r' = \frac{\dot{r}}{\dot{\tau}} = -\omega_2^{-1} r + \cdots \tag{7.13}$$

from (7.3), (7.11). This yields the expansion

$$r = r(\tau) = r_0 \exp(-\omega_2^{-1}\tau) + \cdots \tag{7.14}$$

for $\tau \to +\infty$. The order of the remainder is $\exp(-2\omega_2^{-1}\tau)$. By substituting τ for t, s in the integral expression (7.8) of the tip motion $z(t)$, expansions (7.9)–(7.14) imply that

$$\begin{aligned} &\int_0^t e^{i\omega_0 s} \tilde{\sigma}(r^2) r^m e^{-im\psi(s)} ds \\ &\quad = \int_0^{\tau(t)} e^{-i\tau} \tilde{\sigma}(r^2) r^m \cdot (\dot{\tau})^{-1} d\tau \\ &\quad = \int_0^{\tau(t)} e^{-i\tau} (\tilde{\sigma}_0 + \ldots) \omega_2^{-1} (r^{m-2} + \cdots) d\tau \\ &\quad = \tilde{\sigma}_0 \int_0^{\tau(t)} e^{-i\tau} r_0^{m-2} \omega_2^{-1} e^{-((m-2)/\omega_2)\tau} d\tau + \cdots \end{aligned} \tag{7.15}$$

with exponentially convergent remainder terms of order

$$\mathscr{O}\left(\exp\left(-\frac{m}{\omega_2}\tau\right)\right). \tag{7.16}$$

For drift resonance order $m \geqq 3$, this implies exponential *c*onvergence to

$$z_\infty = \lim_{t\to+\infty} z(t) \tag{7.17}$$

for the tip position $z(t)$.

For $m = 2$ drift resonance, we obtain

$$z(t) = c + i\tilde{\sigma}_0\omega_2^{-1}e^{i\alpha_0}\exp(-i\tau(t)) + o(1) \tag{7.18}$$

for $t, \tau \to +\infty$ and some complex constant c. Thus $z(t)$ converges to a circular motion with logarithmic phase $-\tau(t)$ and ever decreasing angular velocity $\dot{\tau}(t) \sim 1/t$; see (7.11), (7.12).

For $m = 1$ drift resonance, finally, the tip motion $z(t)$ becomes unbounded. Indeed, up to bounded terms, we obtain

$$\begin{aligned}
z(t) &= e^{i\alpha_0}\tilde{\sigma}_0 r_0^{-1}\omega_2^{-1}\int_0^{\tau(t)} e^{(-i+1/\omega_2)\tau}d\tau + \cdots \\
&= e^{i\alpha_0}\frac{\tilde{\sigma}_0 r_0^{-1}\omega_2^{-1}}{(-i+1/\omega_2)}e^{(-i+1/\omega_2)\tau(t)} + \cdots \\
&= e^{i\alpha_0}\frac{\tilde{\sigma}_0 r_0^{-1}\omega_2^{-1}}{-i+1/\omega_2}t^{1/2}e^{-i(\omega_2/2)\log t}\cdot(1+o(1)) + \cdots .
\end{aligned}$$

In other words, the tip moves along an exponential spiral with radius growing like $t^{1/2}$, but again with logarithmic phase $-\tau(t)$ and ever decreasing angular velocity $\dot{\tau}(t) \sim 1/t$.

8. Relative Homoclinics and Heteroclinics

In this section, we consider (relative) homoclinic or heteroclinic trajectories $\Gamma \subset \mathbb{R}^n$ to relative equilibria of $SE(2)$-actions, for the v-equation. Again, we normalize time as in (4.17), such that

$$\dot{\alpha} = \omega_0, \qquad \dot{z} = e^{i\alpha}\sigma(\lambda, v), \qquad \dot{v} = \varphi(\lambda, v). \tag{8.1}$$

We first observe that trajectories Γ in the v-equation, which are homoclinic or heteroclinic to hyperbolic (relative) equilibria, give rise to bounded tip motions $z(t)$ which are typically *heteroclinic*. The associated *tip shift*

$$\mathscr{S}(\Gamma) := z_{+\infty} - z_{-\infty} \tag{8.2}$$

is finite. We then investigate the tip motion $z(t)$ for trajectories $v(t)$ limiting to a homoclinic loop Γ, in forward time, and exhibit a relation to the random behavior of trigonometric sums. We also investigate the infinite sequence of drift resonances which appears along a one-parameter family of relative periodic orbits which limits onto Γ. We conclude with some comments on these phenomena for a (relative) Bogdanov-Takens bifurcation in v.

From (4.19) we recall the tip motion

$$z(t) = z_0 + e^{i\alpha_0} \int_0^t e^{i\omega_0 s} \sigma(v(s))\, ds. \tag{8.3}$$

As in (4.21), (4.22), we further normalize $\sigma(v_0) = 0$ at any relative equilibrium $\varphi(v_0) = 0$.

Assume that the relative equilibrium v_0 is hyperbolic, that is, $\varphi'(v_0)$ does not possess purely imaginary eigenvalues. Then, for any orbit $v(t)$ which limits to the equilibrium for $t \to +\infty$, the convergence of $v(t)$ is exponential. Correspondingly, $\sigma(v(t))$ decays to zero exponentially, and the integral in the right-hand side of (8.3) converges. Thus, there exists a finite limit

$$z_\infty = z_0 + e^{i\alpha_0} \int_0^\infty e^{i\omega_0 s} \sigma(v(s))\, ds \tag{8.4}$$

for the motion of the tip along this orbit. In other words, the evolution of the wave corresponding to the given orbit $v(t)$ limits to a stationary rotating wave with tip, i. e., a center of rotation, at $z = z_\infty$.

Analogous considerations apply to $v(t)$ which tend to an equilibrium for $t \to -\infty$. In particular, for a heteroclinic orbit Γ that connects two hyperbolic equilibria there exists the *tip shift*

$$\mathscr{S}(\Gamma) = z_\infty - z_{-\infty} = \int_{-\infty}^\infty e^{i\omega_0 s} \sigma(v(s))\, ds. \tag{8.5}$$

A heteroclinic solution $v(t)$ induces a heteroclinic solution $(g(t), v(t))$ connecting two rotating waves. Their tips are a distance $\mathscr{S}(\Gamma)$ apart from each other. By now it does not matter whether Γ is heteroclinic or homoclinic: if $\mathscr{S}(\Gamma) \neq 0$, then the tip motion $z(t)$ is always heteroclinic.

Note that only the absolute value of the shift along a relative heteroclinic or homoclinic orbit $v(t)$ is defined uniquely; the direction of the shift depends on the initial phase α_0. This gives rise to an interesting phenomenon: for a v-orbit L which tends to a homoclinic orbit Γ, in v-variables, the tip motion in the z-plane may be quite irregular. Indeed, let $v \in \mathbb{R}^2$, let O be a saddle equilibrium with eigenvalues $\mu_- < 0 < \mu_+$ such that $\mu_- + \mu_+ < 0$, and let Γ be a homoclinic orbit to O. Let an orbit L tend to the homoclinic loop $\Gamma \cup O$, for $t \to +\infty$. Take a small cross section through O and let $t_0 = 0, t_1, t_2, \ldots$ be the consecutive intersection times of L with the cross section. Then

$$t_k = c_1 \nu^k + c_2 + O(e^{-c_3 k}) \tag{8.6}$$

where c_i are some fixed constants and $\nu = |\mu_-/\mu_+| > 1$; see [OS92]. For $\alpha_0 = 0$ the shift of the tip that corresponds to the k-th cycle of $v(t) \in L$ along Γ is given by the integral

$$
\begin{aligned}
\mathscr{S}_k(L) &= \int_{t_{k-1}}^{t_k} e^{i\omega_0 s}\sigma(v(s))\,ds \\
&= e^{i\omega_0(t_{k-1}+t_k)/2}\int_{-(t_k-t_{k-1})/2}^{(t_k-t_{k-1})/2} e^{i\omega_0 s}\sigma\left(v\left(s+\frac{t_{k-1}+t_k}{2}\right)\right)ds.
\end{aligned}
\tag{8.7}
$$

The latter integral converges exponentially to the tip shift integral $\mathscr{S}(\Gamma)$ of Γ; see (8.4).

Since $\frac{1}{2}(t_k+t_{k-1})$ satisfies the same asymptotics (8.6) as t_k itself, with a modified constant $\tilde{c}_1$, we obtain the cumulative tip shift

$$
\begin{aligned}
z(t_{k_0+k}) - z(t_{k_0}) &= \textstyle\sum_{j=1}^{k} \mathscr{S}_{k_0+j}(L) \\
&= e^{i\omega_0 c_2}\mathscr{S}(\Gamma)\cdot \textstyle\sum_{j=1}^{k} \exp\left(2\pi i\{\tilde{\omega}_0 \nu^{k_0+j}\}\right) \;+\; o(1)
\end{aligned}
\tag{8.8}
$$

for $k_0 \to \infty$ and any fixed $k > 0$. Here $\tilde{\omega}_0$=$(2\pi)^{-1}\tilde{c}_1\omega_0$, $\nu > 1$, and $\{\cdot\}$ denotes the fractional part. For Lebesgue almost all $\tilde{\omega}_0$, the terms in the classical trigonometric sum (8.8) have mean value zero. Indeed, the sequence $\{\tilde{\omega}_0\nu^{k_0+j}\}$, j=1, 2, 3, ... , is uniformly distributed in the unit interval, for any fixed $\nu > 1$ and Lebesgue almost all $\tilde{\omega}_0$; see [KN74, Ch. 4]. In fact, the trigonometric sum in (8.8) behaves like a Brownian random walk for almost all $\tilde{\omega}_0$. For a recent survey see [BP96]. Specifically, consider identically distributed, independent, complex, normalized Gaussian random variables $Y_1, Y_2, \ldots$. Then the rescaled sums

$$
Y^{(k)}(t) := k^{-1/2}\sum_{j=1}^{[kt]} Y_j \tag{8.9}
$$

converge to normalized Brownian motion in the complex plane, almost surely, for subsequences $k \to \infty$ which grow fast enough; see for example [Bre68, Ch. 13.4]. Moreover, for any $\nu > 1$ we have approximation of the trigonometric sums (8.8) by the Gaussian sums (8.9):

$$
\left|\sum_{j=1}^{k}\exp(2\pi i\{\tilde{\omega}_0\nu^{k_0+j}\}) - \sum_{j=1}^{k} Y_j\right| \leqq \mathscr{O}(k^{5/12+\gamma}), \tag{8.10}
$$

for any $\gamma > 0$, as $k \to \infty$; see [PS75] and the survey [BP96]. Here $\tilde{\omega_0}$ is considered as a uniformly distributed random variable in the unit interval and $Y_j = Y_j(\tilde{\omega}_0, \tilde{\omega}_1)$. In particular, the spiral tip moves away from the origin according to the loglog law

$$
\overline{\lim}_{k\to+\infty}\,(\sqrt{k\log\log k})^{-1}\left|\sum_{j=1}^{k}\exp(2\pi i\{\tilde{\omega}_0\nu^{k_0+j}\})\right| \;=\; 1 \tag{8.11}
$$

of Brownian motion, for any $v > 1$ and almost all $\tilde{\omega}_0$. For specific sample paths see Fig. 8.1. Note that $k = 10^4$ iterates usually require astronomical waiting times ν^k and, accordingly, very high-precision calculations to produce the associated "random" walks.

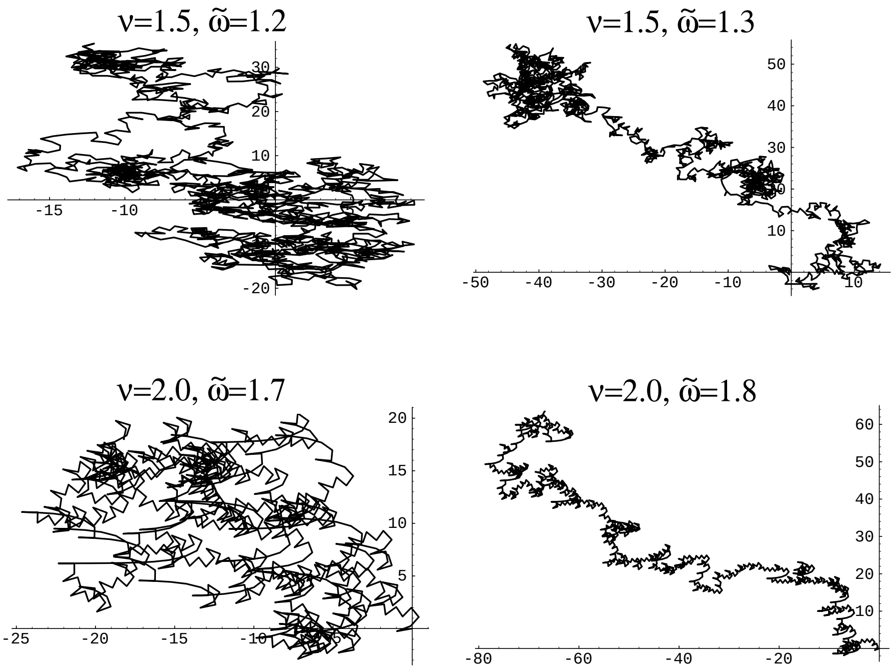

Fig. 8.1. Sample paths of Brownian tip motions.

We now consider a planar relative homoclinic Γ, with the above eigenvalue configuration, but embedded in a generic one-paramter family

$$\dot{v} = \varphi(\lambda, v). \tag{8.12}$$

Specifically, we assume that the stable and unstable separatrix of the equilibrium $v_0 = O$ cross each other with nonvanishing speed, as $\lambda \in \mathbb{R}$ increases through $\lambda_0 = 0$. Then a unique, stable, relative periodic orbit L appears, for λ on one side of $\lambda_0 = 0$, say for small $\lambda > 0$. The minimal period T of L grows to infinity, monotonically, for $\lambda \searrow 0$, and the frequency $\omega_1 = 2\pi/T$ tends to zero. Therefore, an infinite sequence of $m : 1$ drift resonances, $m \to \pm\infty$, occurs at parameters $\lambda = \lambda_m \searrow 0$ determined by

$$m\omega_1(\lambda) = \omega_0 \quad \text{at} \quad \lambda = \lambda_m. \tag{8.13}$$

To actually encounter drift resonances, we need the drift velocities $|\sigma_{-m}(\lambda)|$ to be nonzero; see (5.5). Here the Fourier coefficient σ_{-m} of $\sigma(v(\lambda_m, t))$ of the periodic orbit $v(\lambda_m, t) \in L$ is given by

$$\sigma_{-m} = \frac{1}{T_m} \int_0^{T_m} e^{i\omega_0 s} \sigma(v(\lambda_m, s)) ds. \tag{8.14}$$

For $\lambda_m \searrow 0$, the integral in fact converges to the tip shift $\mathscr{S}(\Gamma)$ of the homoclinic Γ. This follows from estimates on the essentially linear behavior of the periodic trajectories near the origin; see [OS92]. In particular, we have the estimate

$$\sigma_{-m} = \frac{1}{T_m}(\mathscr{S}(\Gamma) + o(1)), \tag{8.15}$$

for $|m| \to \infty$. In summary, a nondegenerate homoclinic orbit Γ at $\lambda = 0$ with nonzero tip shift $\mathscr{S}(\Gamma)$ is accompanied by an infinite sequence of $m : 1$ drift resonances, $|m| \to \infty$, at parameter values $\lambda_m \to 0$.

Homoclinic loops as above appear, for example, in the *Bogdanov-Takens bifurcation* corresponding to an equilibrium state with double zero eigenvalue. This is a codimension-two bifurcation with two parameters, $\lambda = (\lambda_1, \lambda_2)$. There is a curve, emanating from the origin in the parameter plane, which corresponds to homoclinic loops. Generically one can expect an infinite sequence of drift resonances to occur near this curve; see [GLM97]. However, as our normal form results of Section 3 show, for the case of zero eigenvalues μ there is a formal integral in the original variables $(e^{i\tilde{\alpha}}, \tilde{z}, v)$:

$$e^{-i\tilde{\alpha}}\tilde{z} = -S_0(\lambda, v) \tag{8.16}$$

where S_0 is a formal series. See the discussion in (3.20)–(3.22), (6.11), (6.13), (6.14). In particular, the tip motion $\tilde{z}(t)$ corresponding to a bounded motion in the v-plane ($\|v, \lambda\| \leqq \varepsilon$) must remain bounded for a long time, which grows to infinity faster than any power of ε^{-1} for $\varepsilon \to 0$. Thus, a drift velocity corresponding to a (relative) periodic orbit in the v-plane must be smaller than any power of the size of the neighborhood of the origin containing that periodic orbit. Equivalently, it must be smaller than any power of $\|\lambda\|$. Similarly, the homoclinic tip shifts $\mathscr{S}(\Gamma)$ are small beyond any finite order in $\|\lambda\|$. This means that polynomial expressions are capable of accurately reflecting neither the homoclinic tip shifts, nor the velocities of resonant drifts, near a relative Bogdanov-Takens bifurcation.

We repeat that these perhaps surprising statements are valid in the original variables $(e^{i\tilde{\alpha}}, \tilde{z}, v)$. Normal forms, i. e., appropriate choices of Palais coordinates or Palais slices, are no more than a tool in deriving such insights — after all. We hope that this tool will be useful beyond $SE(2)$ and Euclidean groups.

Acknowledgements. We are indebted to ALEXANDER MIELKE for explaining his results on skew-product descriptions near relative equilibria of differentiable group actions in Hamiltonian systems. JÖRG SCHMELING has helped us to clarify the stochastic behavior of tip shifts in Section 8. And IAN MELBOURNE has refreshed and inspired us with pointed and helpful criticism. DMITRY TURAEV has enjoyed his working visit at Freie Universität Berlin, made possible by the Schwerpunktprogramm "Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme" of the Deutsche Forschungsgemeinschaft. For combined efficient typesetting efforts, we are grateful to ANDREA BEHM and REGINA LÖHR.

References

[AM97] P. ASHWIN & I. MELBOURNE, Noncompact drift for relative equilibria and relative periodic orbits. *Nonlinearity*, **10**, 596–616 (1997).

[Bar94] D. BARKLEY, Euclidean symmetry and the dynamics of rotating spiral waves. *Phys. Rev. Lett.*, **72**, 164–167 (1994).

[BE93] M. BRAUNE & H. ENGEL, Compound rotation of spiral waves in a light-sensitive Belousov-Zhabotinsky medium. *Chem. Phys. Lett.*, **204**, 257–264 (1993).

[BP96] I. BERKE & W. PHILIPP, Trigonometric series and uniform distribution mod 1. *Studia Sci. Math. Hungar.*, **31**, 15–25 (1996).

[Bre68] L. BREIMAN, *Probability*. Addison Wesley, New York, 1968.

[ETB$^+$87] C. ELPHICK, E. TIRAPEGUI, M. BRACHET, P. COULLET, & G. IOOSS, A simple global characerization for normal forms of singular vector fields. *Physica*, **29D**, 95–127 (1987).

[Fie88] B. FIEDLER, *Global Bifurcation of Periodic Solutions with Symmetry*. Springer-Verlag, Berlin, 1988.

[FSSW96] B. FIEDLER, B. SANDSTEDE, A. SCHEEL, & C. WULFF, Bifurcation from relative equilibria of noncompact group actions: Skew products, meanders, and drifts. *Doc. Math.*, **1**, 479–505 (1996).

[GLM97] M. GOLUBITSKY, V. LEBLANC, & I. MELBOURNE, Meandering of the spiral tip — an alternative approach. *J. Nonlinear Sci.*, **7**, 557–586 (1997).

[JSW89] W. JAHNKE, W. E. SKAGGS, & A. T. WINFREE, Chemical vortex dynamics in the Belousov–Zhabotinskii reaction and in the two-variable Oregonator model. *J. Chem. Phys.*, **93**, 740–749 (1989).

[Kru90] M. KRUPA, Bifurcations of relative equilibria. *SIAM J. Math. Analysis*, **21**, 1453–1486 (1990).

[KN74] L. KUIPERS & H. NIEDERREITER, *Uniform Distribution of Sequences*. Wiley-Interscience, New York, 1974.

[Lov92] A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*. New York, 1892.

[Mie91] A. MIELKE, *Hamiltonian and Lagrangian Flows on Center Manifolds*, Springer-Verlag, Berlin, 1991.

[MZ94] S. C. MÜLLER, V. S. ZYKOV, Simple and complex spiral wave dynamics. *Phil. Trans. Royal Soc. Lond. A*, **347**, 677–685 (1994).

[NvORE93] S. NETTESHEIM, A. VON OERTZEN, H. H. ROTERMUND, & G. ERTL, Reaction diffusion patterns in the catalytic CO-oxidation on Pt(110) — front propagation and spiral waves. *J. Chem. Phys.*, **98**, 9977–9985 (1993).

[OS92] I. M. OVSYANNIKOV & L. P. SHIL'NIKOV, Systems with a homoclinic curve of a multidimensional saddle-focus and spiral chaos. *Math. USSR Sbornik*, **2**, 415–443 (1992).

[PS75] W. PHILIPP & W. STOUT, Almost sure invariance principles for partial sums of weakly dependent random variables. *Mem. Amer. Math. Soc.*, **2** (1975).

[Ran82] D. RAND, Dynamics and symmetry: Predictions for modulated waves in rotating fluids. *Arch. Rational Mech. Anal.*, **79**, 1–38 (1982).

[Ren82] M. RENARDY, Bifurcation from rotating waves. *Arch. Rational Mech. Anal.*, **79**, 49–84 (1982).

[SSW97a] B. SANDSTEDE, A. SCHEEL, & C. WULFF, Center-manifold reduction for spiral waves. *C. R. Acad. Sci., Série I*, **324**, 153–158 (1997).

[SSW97b] B. Sandstede, A. Scheel, & C. Wulff, Dynamics of spiral waves in unbounded domains using center-manifold reductions. *J. Diff. Eqs.*, **141**, 122–149 (1997).

[UNUM93] Zs. Ungvarai-Nagy, J. Ungvarai, & S. C. Müller, Complexity in spiral wave dynamics. *Chaos*, **3**, 15–19 (1993).

[Van89] A. Vanderbauwhede, Centre manifolds, normal forms and elementary bifurcations. *Dynamics Reported*, **2**, 89–169 (1989).

[Wul96] C. Wulff, *Theory of meandering and drifting spiral waves in reaction-diffusion systems*. Dissertation, Berlin, 1996.

Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2–6
D-14195 Berlin
fiedler@math.fu-berlin.de

and

Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstr. 39
D-10117 Berlin
turaev@wias-berlin.de

(Accepted March 30, 1998)