

AVERAGING OVER A NON-ERGODIC SUBSYSTEM*

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One of the most basic results of the averaging theory was discovered and proven by Anosov [1]:

If, in a slow-fast system, the fast subsystem preserves a smooth invariant measure μ and is ergodic for almost all values of the frozen slow variables, then the evolution of the slow variables is close to that given by the averaged system (averaged over the measure μ in the space of fast variables) for any finite time interval (which can be taken as long as we want) and for all initial conditions except for a set of a small measure, provided the separation between the slow and fast scales is sufficiently large.

It is a very general theorem, which has almost no assumptions - it uses only smoothness and ergodicity of the measure μ . The natural application is given by slow-fast Hamiltonian systems:

$$\dot{x} = \varepsilon \Omega_x^{-1} \partial_x H(x, y, \varepsilon), \quad \dot{y} = \Omega_y^{-1} \partial_y H(x, y, \varepsilon),$$

where Ω_x and Ω_y are the matrices defining the standard symplectic form in the x - and y -spaces respectively, H is the Hamiltonian function (its value is preserved by the system), and ε is a small parameter, so the x variables are slow and the y variables are fast. This system preserves the standard volume form, so the Anosov theorem is applied if the y -subsystem at $\varepsilon = 0$ is ergodic with respect to the Liouville measure $\mu_x^L(dy) = \frac{\delta(E - H(x, y, 0))dy}{\int \delta(E - H(x, y, 0))dy}$ at the given value of $H(x, y, 0) = E$ for almost every value of x (here δ stays for the delta-function). Under the ergodicity condition, Anosov theorem implies that given any $\gamma > 0$, $\nu > 0$ and $T > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ we have

$$\|x(t) - \hat{x}(t)\| < \gamma \quad \text{on the interval } 0 \leq \varepsilon t \leq T$$

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for all initial conditions except, may be, for a set of measure less than ν , where $\hat{x}(t)$ is the solution of the averaged system

$$\dot{x} = \varepsilon \int \Omega_x^{-1} \partial_x H(x, y, 0) \mu_x^L(dy).$$

It is a simple computation to show that this system can be written in the following form:

$$\dot{x} = -\varepsilon T(x) \Omega_x^{-1} \partial_x S(x)$$

where the effective Hamiltonian $S(x) = \ln \int_{H(x,y,0) \leq E} dy$ can be identified with the Gibbs volume entropy of the y -subsystem, and the scalar time-reparameterisation factor $T(x) = (\partial_E S)^{-1}$ can be viewed as the temperature [2]. Obviously, the averaged system preserves $S(x)$. Therefore, Anosov theorem actually establishes that the fundamental physical fact of the entropy preservation at adiabatic (i.e. sufficiently slow) changes of system parameters follows from the ergodicity of the system.

In this talk we address the question of what happens if the fast system is not ergodic. We need to consider this question because the ergodicity does not seem to be the prevalent feature of the Hamiltonian dynamics. We discuss a theory which is developing in joint works with V. Gelfreich, T. Pereira, V. Rom-Kedar, and K. Shah [3–9] and suggest that in the non-ergodic case the behaviour of the slow variables is approximated by a random process, and not a single, deterministic averaged system. Namely, we propose the following conjecture (corroborated by an extensive set of numerical experiments).

Conjecture. *For fixed tolerance parameters γ and ν , there exists a finite set of smooth non-negative functions $\rho_k(x, y)$, $\sum_k \rho_k \equiv 1$, such that for any $T > 0$ and all sufficiently small ε the evolution of the slow variables is approximated on the time interval $[0, T/\varepsilon]$ by the solutions of the system*

$$\dot{x} = \varepsilon \int \Omega_x^{-1} \partial_x H(x, y, 0) \rho_k(t)(x, y) \mu_x^L(dy).$$

Here the index k of the “approximate ergodic component” ρ_k , over which we perform the averaging at each given moment of time, is a random function obtained as a realisation of the Markov process with the x -dependent transition probabilities

$$\Pr[k(t) = i \longrightarrow k(t + \Delta t) = j] = \frac{\int \rho_j(x(t + \Delta t), y) \rho_i(x(t), y) \mu_{x(t)}^L(dy)}{\int \rho_i(x(t), y) \mu_{x(t)}^L(dy)}.$$

The conjectured Markov property is crucial here, as it implies that a typical process of this type must equilibrate at an exponential rate. This means an exponential convergence of any absolutely continuous initial measure on the given energy level $H = E$ to a unique stationary one, i.e., to the Liouville measure. Moreover, the equilibration process can be attributed to the increase of the Gibbs volume entropy. It is not clear whether a typical random process described above corresponds to a typical slow-fast Hamiltonian system. However, a success in establishing the validity of this conjecture would offer a way of explaining the equilibration in adiabatically evolving systems of statistical mechanics as an effect of ergodicity violation.

References

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