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# A New Simple Bifurcation of a Periodic Orbit of "Blue Sky Catastrophe" Type

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ABSTRACT. In this paper, we study a global bifurcation of codimension one connected with the disappearance (for positive values of a parameter $\mu$) of a saddle-node periodic orbit $L_0$ under the condition that all orbits from the locally unstable manifold $W^u$ of $L_0$ tend to $L_0$ as $t \to +\infty$. Conditions are presented which guarantee the blue sky catastrophe: the appearance of a stable periodic orbit $L_\mu$ which exists for any small positive values of $\mu$ but its length and period unboundedly increase as $\mu \to +0$.

## 1. Introduction

One of the main questions of nonlinear dynamics concerns the structure of the boundaries of stability regions of periodic orbits. It was the question which gave an initial impulse to the development of bifurcation theory, when Andronov and Leontovich [1] discovered that for two-dimensional systems of ODE's there are exactly four principal boundaries of stability of periodic orbits: on the first boundary the stable periodic orbit bifurcates from a stable equilibrium (which, in turn, loses stability), on the second boundary the periodic orbit coalesces with an unstable one and then disappears, on the third boundary the periodic orbit disappears merging into a homoclinic loop of a simple saddle-node equilibrium state and on the fourth boundary the stable periodic orbit merges into a homoclinic loop of a saddle equilibrium state with negative saddle value.

For the multidimensional case this list is extended in the following way. Obviously, there may be two types of stability boundaries: at the moment of bifurcation the periodic orbit either does or does not exist. In the first case, the intersection of the periodic orbit with a local cross-section is the fixed point of the Poincaré map, so the loss of stability corresponds to a multiplier on the unit circle. We have exactly three principal (i.e., those which are typical for generic one-parameter families) stability boundaries here: one corresponds to one multiplier of the periodic orbit equal to $(+1)$ and the remaining multipliers lying strictly inside the unit circle; this is the

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*saddle-node* bifurcation (Figure 1) analogous to the two-dimensional case; another boundary corresponds to one multiplier equal to $(-1)$, when the periodic orbit does not disappear as it crosses this stability boundary and a period-doubling bifurcation takes place; the third stability boundary corresponds to a pair of complex-conjugate multipliers; this bifurcation gives rise to the formation of an invariant torus.

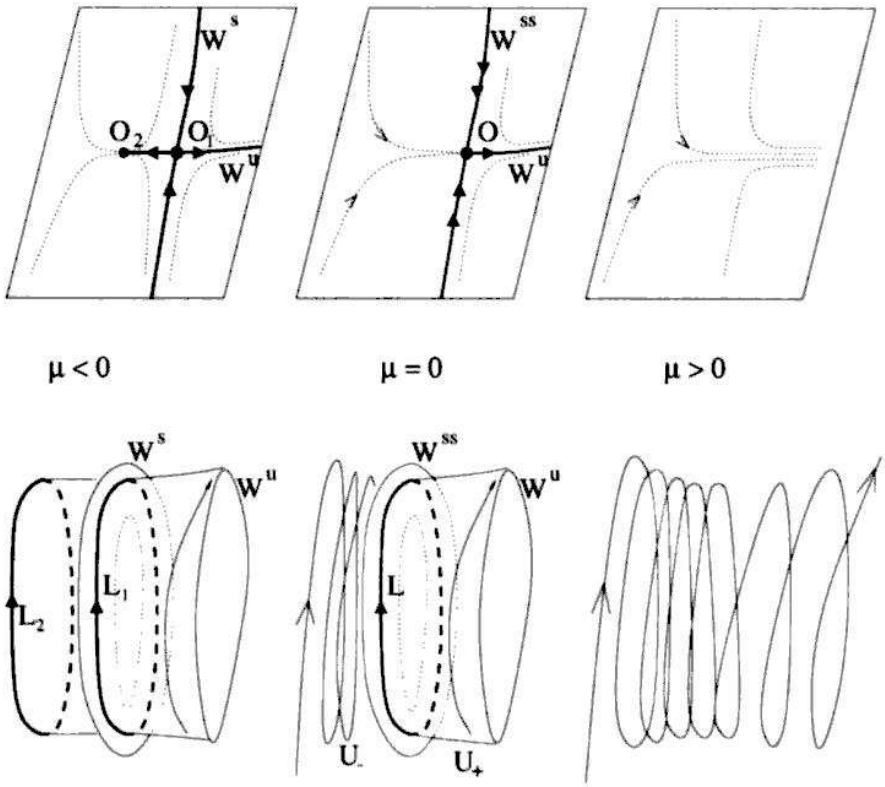


FIGURE 1. The bifurcation of a saddle-node periodic orbit. a) At $\mu < 0$ there exist stable and saddle periodic orbits. b) At $\mu = 0$ the periodic orbits unite into the saddle-node. Its strong stable manifold $W^{ss}$ divides the neighborhood into the node region $U_-$ and the saddle region $U_+$. The unstable manifold $W^u$ lies in $U_+$. c) At $\mu > 0$ the saddle-node disappears and all the orbits leave the small neighborhood.

Stability boundaries of the other type correspond, as in the two-dimensional case, to the birth of a periodic orbit from the stable equilibrium state (the Andronov-Hopf bifurcation) or to the merging of the periodic orbit into a homoclinic loop [**2**] of either a simple saddle-node equilibrium state (Figure 2) or a hyperbolic equilibrium state with one-dimensional unstable manifold and with negative saddle value.

The following question immediately arises: Can there be other types of stability boundaries of codimension one? It can be shown that the list above gives *all* the principal stability boundaries for the case in which the *length* of the periodic orbit remains bounded when approaching the bifurcation moment (although the period may tend to infinity if the orbit merges into a homoclinic loop). Thus, the search for new stability boundaries must presuppose the unbounded growth of the orbit length.

In the present paper we do find such a stability boundary, which does not have two-dimensional analogs. The boundary is an open subset of a codimension

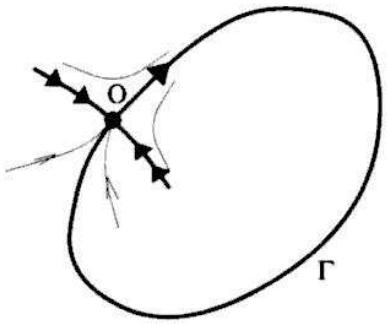


FIGURE 2. A homoclinic loop $\Gamma$ to a saddle-node equilibrium $O$.

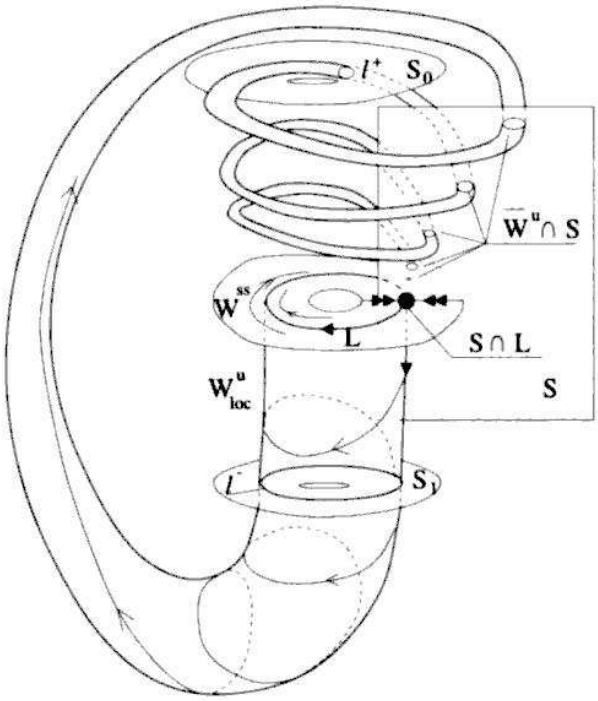


FIGURE 3. The global structure of the set $\overline{W}^u$ in the case under consideration. The image $l^+$ of the intersection $l^-$ of the local unstable manifold $W^u_{\text{loc}}$ of the saddle-node $L$ with the cross-section $S_1$ is homotopic to zero on the cross-section $S_1$. The intersection of $\overline{W}^u$ with the local cross-section $S$ in the node region is a countable set of circles which accumulate at $S \cap L$.

one bifurcational surface corresponding to the existence of a saddle-node periodic orbit. This open set is distinguished by some qualitative conditions determining the geometry of the unstable set of the saddle-node (see Figure 3) and also by some quantitative restrictions (a certain value should be less than 1, see below). We shall show under these conditions that when the saddle-node disappears, a new stable periodic orbit arises whose period and length both tend to infinity when approaching the bifurcation moment (Theorem 1).

This is one of the possible scenarios for the global bifurcation involving the disappearance of a saddle-node periodic orbit when all the orbits of its unstable set return to the saddle-node as $t \to +\infty$.

The study of this global bifurcation has a long history. Originally, this problem was raised in the twenties in connection with the study of the transition from synchronization to the amplitude modulation regime in the van der Pol equation

$$\ddot{x} + \mu(1 - x^2)\dot{x} + \omega_0^2 x = \mu A \cos \omega t.$$

Under the assumption that $\mu$ is a small parameter and that $1:1$ resonance occurs (i.e., $\omega - \omega_0 \sim \mu$), Andronov and Vitt showed that the transition from the synchronization to the amplitude modulation regime is connected with a bifurcation involving the birth of a stable limit cycle from a homoclinic loop to a saddle-node equilibrium state (as in Figure 2) in the time-averaged system. Returning to the initial equation, one can see that a similar picture occurs for the two-dimensional Poincaré map, where the saddle-node is now the fixed point of the map and the homoclinic loop is not a single orbit but forms a continuum of orbits that constitute the unstable set of the saddle-node. At that time, this kind of analysis was not carried out.

The study of this bifurcation started in [**3**] under the assumption that the dynamical system with the saddle-node is either nonautonomous and periodically depends on time, or autonomous but possesses a global cross-section (at least at the part of the phase space under consideration). Essentially, the problem was reduced to the study of a one-parameter family of $C^r$-diffeomorphisms ($r \geqslant 2$) which has, at $\mu = 0$, a saddle-node fixed point such that all orbits of the unstable set of the saddle-node return to it as the number of iterations tends to $+\infty$.

Recall that the saddle-node point has one multiplier equal to 1, the remaining multipliers lying inside the unit circle. Near the fixed point, the diffeomorphism (the Poincaré map) has the form

$$\begin{cases} \bar{y} = Ay + H(y, z), \\ \bar{z} = z + G(y, z) \end{cases} \tag{1}$$

where $z \in \mathbb{R}^1$, $y \in \mathbb{R}^n$, $A$ is a matrix whose eigenvalues lie strictly inside the unit circle, $H(0,0) = 0$, $H'_{(y,z)}(0,0) = 0$, $G(0,0) = 0$, $G'_{(y,z)}(0,0) = 0$. Here, the fixed point $O$ is at the origin. It is well known that there exists a $C^r$-smooth invariant center manifold of the form $y = \eta(z)$, where $\eta(0) = 0$, $\eta'(0) = 0$. The map on the center manifold takes the form

$$\bar{z} = z + g(z), \tag{2}$$

where $g(z) \equiv G(\eta(z), z) \in C^r$, $g(0) = 0$, $g'(0) = 0$.

The fixed point $O$ (and the corresponding periodic orbit) is called a *saddle-node* if $g(z)$ has a strict extremum at $z = 0$ (a strict minimum, to be definite); i.e., $g(z) > 0$ at $z \neq 0$. The saddle-node is called *simple* if $g''(0) \neq 0$. In this case equation (2) takes the form

$$\bar{z} = z + l_2 z^2 + \ldots, \tag{3}$$

where $l_2 = g''(0)/2 \neq 0$. Without loss of generality one can assume that $l_2 > 0$.

One can see (Figure 1b) that a small neighborhood of $O$ is split by the strong-stable invariant manifold $\{z = \xi(y)\}$ ($\xi(0) = 0$, $\xi'(0) = 0$) into two regions: the node region $\{x < \xi(y)\}$ and the saddle region $\{z > \xi(y)\}$. All orbits from the node region tend to $O$ along the $z$-axis. The one-dimensional local unstable manifold $\{y = \eta(z),\ z > 0\}$ lies in the saddle region, and all its orbits tend to $O$ with the

iterations of the inverse map. All the other orbits from the saddle region leave the neighborhood of $O$ with the iterations of both map (1) itself and its inverse.

Let $W^u$ denote the global unstable manifold of $O$ (the union of all iterations of the local unstable manifold). As we mentioned, all orbits of $W^u$ are supposed to return to the node region, i.e., $\partial W^u = O$. Thus, here the closure $\overline{W}^u$ is homeomorphic to a circle. It turns out that $\overline{W}^u$ can be a smooth circle (Figure 4) or it can be nonsmooth (Figure 5).

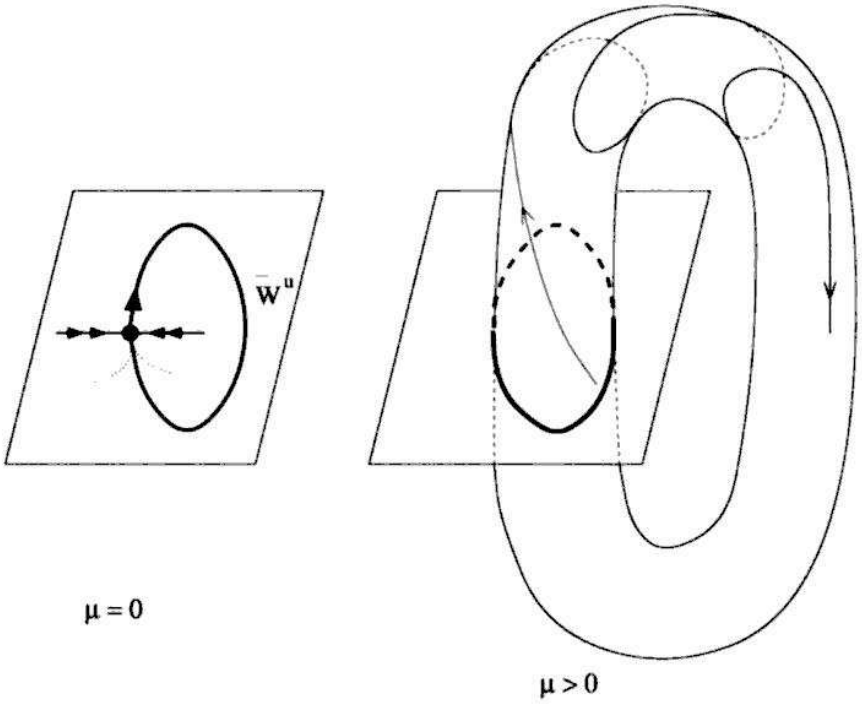


FIGURE 4. When (a) the set $\overline{W}^u$ is smooth, a smooth invariant curve is born (b) at $\mu > 0$.

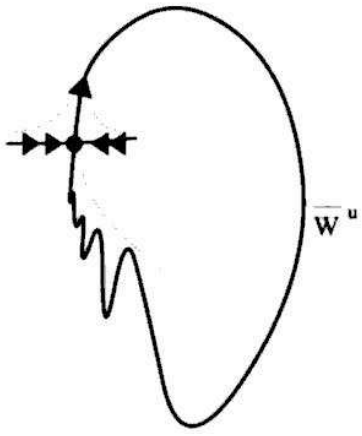


FIGURE 5. The set $\overline{W}^u$ may be nonsmooth; it may have "folds" which accumulate at the saddle-node.

To study bifurcations in a small neighborhood of $\overline{W}^u$ one must introduce a small parameter $\mu$. The functions $H$ and $G$ in (1) should be viewed as depending on $\mu$. We assume that $\mu$ is chosen so that the saddle-node disappears when $\mu > 0$

(Figure 1c). Generically, the saddle-node is simple and for $\mu < 0$, it degenerates into two fixed points, one being a saddle and another being stable (Figure 1a). The map on the center manifold is rewritten in this case as

$$\bar{z} = z + \mu + l_2 z^2 + \dots . \tag{4}$$

For the case where $\overline{W}^u$ is smooth, it was found in [**3**] that when the saddle-node disappears, an attractive smooth invariant curve inherits to $\overline{W}^u$. If the map under consideration is the Poincaré map of a global cross-section for some system of ODE's, then the invariant curve is the line of intersection of an invariant two-dimensional torus with the cross-section (Figure 4b). The Poincaré rotation number on the torus tends to zero as $\mu \to +0$.

This result gave a rigorous explanation of the transition from synchronization to amplitude modulation in periodically forced nonlinear systems: when $\mu < 0$, the only stable regime is the stable periodic orbit which corresponds to synchronization, and the invariant torus that exists at $\mu > 0$ corresponds to the modulation regime (see the discussion in [**4**]).

For the case where $\overline{W}^u$ is a nonsmooth manifold, it was established in [**3**] (under the so-called "big lobe" condition) that there exists a sequence of intervals $(\mu_i, \mu_i')$ accumulating at $\mu = +0$ such that the system has nontrivial hyperbolic sets at $\mu \in (\mu_i, \mu_i')$. Without the big lobe condition (but for one-parameter families of a special kind), this result was proved in [**5**], on the basis of a theorem due to Block on the existence of periodic orbits for endomorphisms of a circle. In [**6**] the results of [**3, 5**] were extended to the general case; there it was also shown that for a "sufficiently small lobe" there exist both intervals of parameter values corresponding to complex dynamics (hyperbolic sets) and those corresponding to simple dynamics (a continuous invariant curve with rational Poincaré rotation number).

An important feature in the nonsmooth case is the existence [**5, 6, 7**] of parameter values arbitrarily close to $\mu = +0$, which correspond to the existence of saddle periodic orbits with *nontransverse homoclinic orbits*. According to our current knowledge (see [**8**]), this leads to extremely complicated dynamics: to the Newhouse phenomenon (persistence of homoclinic tangencies, coexistence of infinitely many sinks) [**9, 10, 11, 12, 13, 14**], to Hénon-like attractors [**15, 16, 7**] and to infinite degeneracies [**17, 14**], which make it impossible to give a complete description of the bifurcations that may occur in this case.

In the present paper we show that if an autonomous system with a saddle-node does *not* have a global cross-section, there may be considerably more different possible cases. Let

$$\dot{x} = X_\mu(x)$$

be a one-parameter family of $n$-dimensional $C^r$-smooth ($r \geqslant 2$) dynamical systems with a saddle-node periodic orbit $L$ at $\mu = 0$. We assume that $\mu$ is the governing parameter for local bifurcations. Thus (Figure 1), at $\mu < 0$ there exist stable and saddle periodic orbits which combine in one orbit $L$ at $\mu = 0$. The local unstable set $W^u_{L,\mathrm{loc}}$ is homeomorphic to the half-cylinder $\mathbb{R}^+ \times S^1$. The orbit $L$ also has a strong-stable manifold $W^{ss}_L$ that divides the neighborhood of $L$ into two regions: saddle and node. When $\mu < 0$, the saddle-node disappears and all orbits leave its small neighborhood. Thus, for the Poincaré map on the center manifold

$$\bar{z} = z + g(z; \mu), \tag{5}$$

the nonlinear part $g(z; \mu)$ is strictly positive at $\mu > 0$.

Note that if the starting point of some orbit lies in the node region, then the time which the orbit spends in a small fixed neighborhood of the saddle-node tends to infinity as $\mu \to +0$, as well as the length of the corresponding piece of the orbit.

Suppose that, at $\mu = 0$, all the orbits of $W_L^u$ return to the node region and tend to $L$ as $t \to +\infty$, not lying in $W_L^{ss}$. The union $\overline{W}^u$ of these orbits may, for instance, be a smooth two-dimensional surface: a torus, or a Klein bottle (the latter can happen if the phase space is nonorientable or if the dimension $n$ of the phase space is not less than four). As in [**3**], the smooth invariant two-dimensional surface is preserved for $\mu > 0$. As above, if the set $\overline{W}^u$ is a nonsmooth torus, then saddle periodic orbits with homoclinic curves may appear at $\mu > 0$; the same can happen if $\overline{W}^u$ is a nonsmooth Klein bottle under some additional conditions [**6**].

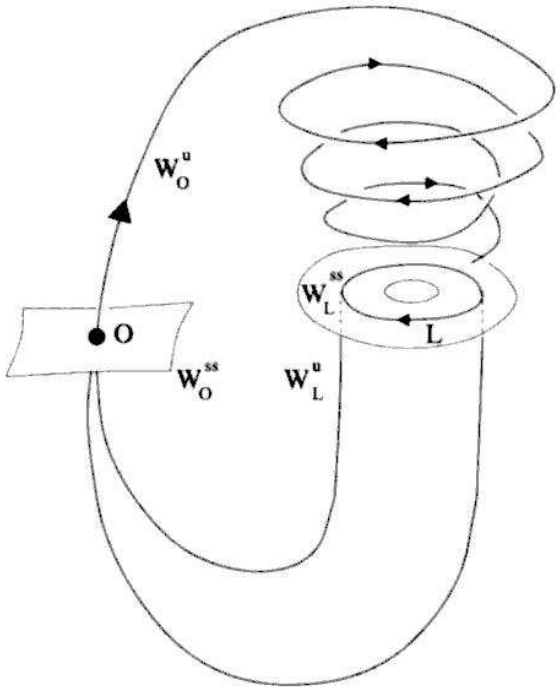


FIGURE 6. A codimension two bifurcation; the unstable manifold of the saddle-node periodic orbit $L$ tends to the saddle-node equilibrium $O$ whose unstable manifold, in turn, tends to $L$. When $O$ disappears, the configuration shown in Figure 3 emerges.

Previously we did not know of any essentially different situation in which the set $\overline{W}^u$ is not a manifold. First, consider the following example. Let a *two*-parameter family of three-dimensional vector fields have, at some value of the parameters, a saddle-node periodic orbit $L$ and a saddle-node equilibrium state $O$ (Figure 6). Suppose that all orbits of $W_L^u$ tend to $O$ as $t \to +\infty$ and that the one-dimensional separatrix of $O$ tends to $L$. If one of the parameters is varied so that $O$ disappears and $L$ does not, then the set $\overline{W}^u$ will have the form shown in Figure 3. The intersection of $\overline{W}^u$ with a local cross-section $S$ to $L$ will be a union of a countable set of circles accumulating at the point $S \cap L$ (Figure 3). Evidently, any neighborhood of this point in the set $\overline{W}^u$ is not homeomorphic to a disk. Therefore, in this case $\overline{W}^u$ is not a manifold.

Systems having a simple saddle-node periodic orbit with the set $\overline{W}^u$ as shown in Figure 3 constitute codimension one surfaces in the space of smooth flows in $\mathbb{R}^n$

($n \geqslant 3$). Below, we shall show (see Theorem 1) how open subsets are distinguished on these surfaces so that *for any one-parameter family $X_\mu$ that intersects such a subset transversely at $\mu = 0$, the system $X_\mu$ has (at all small $\mu > 0$) an attractive periodic orbit whose period and length tend to infinity as $\mu \to +0$.*

Note the relation of this result to the problem of the "blue sky catastrophe" [**18**]. The original formulation was as follows: *Does there exist a continuous one-parameter family of smooth vector fields on a compact manifold that have a closed orbit $L_\mu$ at all $\mu > 0$ and, as $\mu \to +0$, the period of $L_\mu$ tends to infinity so that $L_\mu$ disappears at $\mu = +0$ at a finite distance from the equilibrium states*[1]*?*

Virtual bifurcations of such kind were called *blue sky catastrophes* by Abraham. The first example of such a catastrophe was constructed by Medvedev [**19**] for a one-parameter family of vector fields on a Klein bottle with a saddle-node periodic orbit at $\mu = 0$. Medvedev's family was of a rather special kind: the system that corresponds to $\mu = 0$ is also embedded in a one-parameter family of conservative vector fields all orbits of which are closed. The Poincaré map for this conservative family has the form

$$\bar{\varphi} = -\varphi + \omega(\mu) \mod 1,$$

where $\omega \to \infty$ as $\mu \to +0$. This map has two fixed points, all other points are of period two. Basically, Medvedev used the fact that this family can be perturbed so as to have only two periodic orbits: one stable and one unstable fixed point; the stable fixed point corresponds to a stable periodic orbit whose period and length tend to infinity as $\mu \to +0$.

In a generic perturbation of Medvedev's one-parameter family both fixed points bifurcate infinitely many times as $\mu \to +0$, changing their stability (this was noticed in [**20**] and studied in more detail in [**21**]). Formally, the blue sky catastrophe also takes place here because the structural stability of the periodic orbit under consideration was not required in the original formulation.

The construction proposed in the present paper gives another solution to the blue sky problem. At the same time, it seems to be more adequate because the periodic orbit in Theorem 1 is stable (attractive) for all $\mu > 0$ and this property holds for an open set of one-parameter families.

The precise formulation of the result is given in the next section. The proof is based mostly on the evaluation of the local (Lemma 1) and global (Theorem 2) "through" map at small $\mu > 0$; the proofs are presented in Section 3. The computations are quite straightforward if the smoothness of the system with respect to the phase variables and the parameter is sufficiently high (in this case reduction to an almost autonomous normal form is possible) [**22, 23, 7, 26**]; the general case of low smoothness which we treat in the present paper requires more delicate calculations.

In fact, the results of Section 3 are applied to an arbitrary configuration of the set $\overline{W}^u$, allowing one to establish a correspondence between the dynamics near $\overline{W}^u$ and the dynamics of the one-dimensional *essential* map introduced in Section 2. The essential map is defined on a circle and its degree $m$ defines the topological type of $\overline{W}^u$. Thus, at $m = 1$, the set $\overline{W}^u$ is homeomorphic to a torus, at $m = -1$ it is a Klein bottle. The present paper deals with $m = 0$, which corresponds to $\overline{W}^u$ as shown in Figure 3. In this case, under the conditions of Theorem 1, the dynamics of the essential map is trivial, so the main result follows immediately from

[1]The latter implies that the length of $L_\mu$ also tends to infinity.

the reduction principle of Theorem 2. More examples of the use of this principle can be found in [**6**] for the case in which $\overline{W}^u$ is a torus or a Klein bottle and in [**22, 23**] for other cases.

In particular, these papers treat the case $|m| \geqslant 2$, which is possible in $\mathbb{R}^n$ for $n \geqslant 4$. In this case hyperbolic attractors (the Smale–Williams solenoids) may appear via an analog of the blue sky catastrophe. More examples based on the disappearance of saddle-node invariant tori are given in [**23**].

## 2. Main results

Let $U$ be a small neighborhood of $\overline{W}^u$ and $U_0$ a small neighborhood of $L$, $U_0 \in U$. Let us cut $U_0$ by a local cross-section $S$ and consider the coordinates $(y, z, \varphi)$, where $\varphi \in [0, 1]$ is the angular variable and $(y, z)$ are the normal coordinates, $z \in \mathbb{R}^1$ is a coordinate on the center manifold, $y \in \mathbb{R}^{n-2}$ is a vector of coordinates corresponding to the multipliers less than 1 in absolute value; the values $\varphi = 0$ and $\varphi = 1$ correspond to the points lying on $S$.

The surfaces $\varphi = 0$ and $\varphi = 1$ are assumed to be glued by some involution, namely, by changing the sign of a number of components of the vector $y$. If this number is even (this is the orientable case, where the product of the multipliers of $L$ is positive), then $U_0$ is a solid torus. If this number is odd (the nonorientable case, where the product of the multipliers of $L$ is negative), then $U_0$ is the product of a Möbius strip and a disc $D^{n-2}$.

As it is well-known (see, for example, [**24**]), under an appropriate choice of the involution that glues the surfaces $\{\varphi = 0\}$ and $\{\varphi = 1\}$ one can without loss of smoothness introduce coordinates such that the linear part of the system near $L$ is independent of $\varphi$. Thus, the system near $L$ takes the form

$$\begin{cases} \dot{y} = \mathcal{A}(\mu)y + \widetilde{H}(z, y, \varphi; \mu), \\ \dot{z} = \widetilde{G}(z, y, \varphi; \mu), \\ \dot{\varphi} = 1, \end{cases} \tag{6}$$

where $\widetilde{H}$ and $\widetilde{G}$ vanish at $(y = 0, z = 0, \mu = 0)$ along with the first derivatives with respect to $(z, y)$; the eigenvalues of the matrix $\mathcal{A}$ lie strictly to the left of the imaginary axis. We also assume that the center manifold $W^c$ is locally straightened, so that it takes the form $\{y = 0\}$. Correspondingly,

$$\widetilde{H}(z, 0, \varphi; \mu) \equiv 0. \tag{7}$$

Further, let us straighten the strong stable invariant foliation [**27, 28**] transverse to the center manifold. The leaves of the foliation are given by

$$\{z = \mathcal{Q}(y; \varphi, z', \mu),\ \varphi = \text{const}\},$$

where $z'$ is the coordinate of intersection of a leaf with the center manifold; $\mathcal{Q}$ is a $C^{r-1}$-function (it is $C^r$ with respect to $y$). The straightening is the coordinate transformation $z \mapsto z'$. It brings the invariant foliation to the form $\{z = \text{const}, \varphi = \text{const}\}$. Thus, the last two equations in (6) now become independent of $y$ and the

system can be rewritten as follows:

$$\begin{cases} \dot{y} = \mathcal{A}(\mu)y + \widetilde{H}(z, y, \varphi; \mu), \\ \dot{z} = \widetilde{G}(z, \varphi; \mu), \\ \dot{\varphi} = 1. \end{cases} \tag{8}$$

By construction, the new function $\widetilde{G}$ is the former one taken at $\{y = 0\}$; so it is still a $C^r$-function.

In the new coordinates, the strong-stable invariant manifold $W_L^{ss}$ is the surface $\{z = 0\}$; the node region $U_-$ now corresponds to small negative $z$ and the saddle region $U_+$ corresponds to small positive $z$.

As we mentioned, the invariant foliation is *a priori* $C^{r-1}$. However, but it can be shown (Lemma 4) that it is, in fact, $C^r$ everywhere except for $W_L^{ss}$ at $\mu = 0$. The coordinate transformation which reduces (6) to (8) has the same smoothness.

According to the theorem "on the embedding into an autonomous flow" (see [**5**]; cf. Lemma 5 of the present paper), there exists a transformation of the coordinate $z\colon z \mapsto \mathcal{Z}(z, \varphi)$ which brings the second equation of (8) to an autonomous form for $\mu = 0$ (note that this transformation is identical at $\varphi = 0$ and is *uniquely* defined by the nonlinear part $g$ of the Poincaré map on the center manifold (see (2))). Therefore, we assume

$$\left.\frac{\partial \widetilde{G}}{\partial \varphi}\right|_{\mu=0} \equiv 0. \tag{9}$$

In other words, at $\mu = 0$ the last two equations of (8) take the form

$$\dot{z} = \tilde{g}(z), \qquad \dot{\varphi} = 1, \tag{10}$$

where $\tilde{g}(0) = 0$, $\tilde{g}'(0) = 0$. If $z \neq 0$, then $\tilde{g}(z) > 0$. It is shown in Section 3 (see the comments after Lemma 5) that the "embedding" transformation $z \mapsto \mathcal{Z}$ is $C^r$ at $z \neq 0$. Thus, system (8) after the transformation is $C^{r-1}$ at $z \neq 0$, whereas the flow map between any two cross-sections that do not intersect $\{z = 0\}$ remains $C^r$.

Take small positive $\varepsilon^+$ and $\varepsilon^-$. Consider two cross-sections $S_0\colon \{z = -\varepsilon^-\}$ and $S_1\colon \{z = \varepsilon^+\}$ to the flow $X_\mu$. In the orientable case, $S_0$ and $S_1$ are solid tori $S^1 \times D^{n-2}$; in the nonorientable case, they are homeomorphic to the product of the Möbius strip and the disc $D^{n-3}$.

At $\mu = 0$ (hence, at all small $\mu$), all orbits of $W_L^u$ return to the node region $U^- = \{z < 0\}$ in finite time. Therefore, the flow $X_\mu$ defines a diffeomorphism $T_1$ by which a small neighborhood of the intersection line $l^-\colon \{y = 0\} = W^u \cap S_1$ is mapped into $S_0$. This map has the form

$$y_0 = p(\varphi_1, y_1; \mu), \qquad \varphi_0 = q(\varphi_1, y_1; \mu) \mod 1, \tag{11}$$

where the coordinates on $S_0$ and $S_1$ are denoted by $(\varphi_0, y_0)$ and $(\varphi_1, y_1)$ respectively; $C^r$-smooth functions $p$ and $q \mod 1$ are 1-periodic in $\varphi$.

The curve $l^+ = T_1 l^-\colon \{y_0 = p(\varphi_1, 0; 0),\ \varphi_0 = q(\varphi_1, 0; 0) \mod 1\}$ is the intersection of $\overline{W}^u$ and $S_0$. Note that the function $q$ can be written in the form

$$q(\varphi, y; \mu) = m\varphi + q_0(\varphi, y; \mu), \tag{12}$$

where $q_0$ is periodic in $\varphi$. The integer $m$ defines the homotopy class of $l^+$ in $S_0$ (the sign of $m$ defines the orientation of $l^+$ with respect to $l^-$).

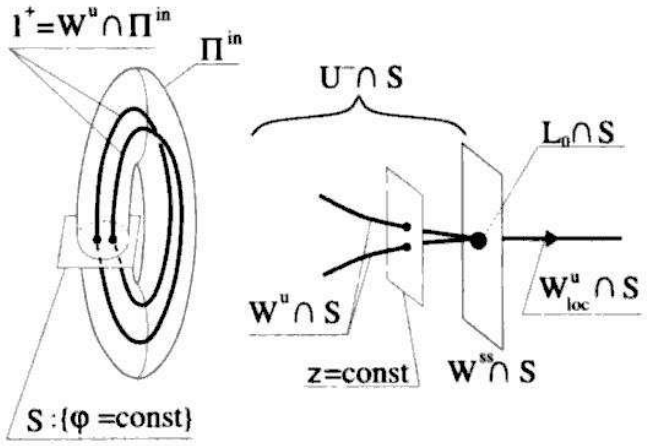


FIGURE 7. If the phase space is four-dimensional (or higher), the cross-sections $S_0$ and $S_1$ are at least three-dimensional. Therefore, the image $l^+$ of the curve $l^-$ may be a multi-round closed curve. The figure gives an example with two rounds ($m = 2$).

If the dimension $n$ of the phase space is greater than three, then $S_0$ is at least three-dimensional and the integer $m$ may be arbitrary (see Figure 7 for the case $m = 2$). At $n = 3$, the cross-section $S_0$ is a two-dimensional annulus in the orientable case. Therefore, in this case $m$ can only take the values 1 (Figures 4, 5), 0 (Figure 3) and $-1$ (the latter only on nonorientable manifolds). In the nonorientable case at $n = 3$, the cross-section $S_0$ is a Möbius strip, therefore, we can only have $m = 0, \pm 1, \pm 2$. At $n = 2$, $S_0$ coincides with $l^+$ and we can only have $m = \pm 1$.

Note that the structure of the set $\overline{W}^u$ is completely determined by the way $\overline{W}^u$ adjoins to $L$ from the side of the node region. It is not hard to see that the intersection of $\overline{W}^u \cap U_-$ with any cross-section of the kind $\{\varphi = \text{const}\}$ consists, at $m \neq 0$, of $|m|$ pieces glued at the point $\{z = 0, y = 0\} = L_0 \cap \{\varphi = \text{const}\}$. It is clear that pieces of $\overline{W}^u$ corresponding to different values of $m$ are mutually nonhomeomorphic. It is also clear that $\overline{W}^u$ is a manifold if and only if $m = \pm 1$ (a torus or a Klein bottle respectively).

We define the *essential map*

$$\varphi \mapsto f(\varphi) \equiv m\varphi + q_0(\varphi, 0; 0). \tag{13}$$

It is defined at $\mu = 0$. By construction, it is obtained as follows: apply the map $T_1$ to the intersection line of the unstable manifold $W^u = W^c \cap U_+$ with the cross-section $S_1$ and then project the image on the center manifold along the leaves of the strong-stable foliation. The projection is done in $S_0$, which lies in the node region where the foliation is defined uniquely [**27, 28**]. Thus, once the cross-sections $S_0$ and $S_1$ are fixed, the essential map is defined uniquely modulo coordinate transformations on the center manifold (the center manifold in the node region is not unique, but systems on different center manifolds are conjugate by the projection along the strong stable invariant foliation, therefore the choice of another center manifold is equivalent to a coordinate transformation on the initial one).

In fact, the set of coordinate transformations which keep the system at $\mu = 0$ in the form (10) is rather small. Indeed, a new coordinate $\varphi$ must satisfy

$$\frac{d}{dt}(\varphi_{\text{new}} - \varphi) = 0,$$

hence the difference $\varphi_{\text{new}} - \varphi$ must be constant along the orbits of the system. In particular, it is constant on $L$. Now, since any orbit on the center manifold tends to $L$ either as $t \to +\infty$ or as $t \to -\infty$, it follows that $\varphi_{\text{new}} - \varphi = \text{const}$ everywhere on $W^c$. Further, since the equation for $z$ in (10) must remain autonomous, one can show that only autonomous (independent on $\varphi$) transformations of the variable $z$ are allowed. Indeed, consider first a transformation which is identical at $\varphi = 0$. By definition, it does not change the Poincaré map of the cross-section $S$: $\{\varphi = 0\}$, therefore, by the uniqueness of the embedding into the flow, if such a transformation keeps the system autonomous, it cannot change the right-hand side. We see that if $z_{\text{new}} \equiv z$ at $\varphi = 0$, then the time evolution of $z_{\text{new}}$ and the time evolution of $z$ is governed by the same equation, which immediately implies that $z_{\text{new}} \equiv z$ for all $\varphi$ in this case. Since an arbitrary transformation is a superposition of an autonomous transformation and a transformation of the kind we have just considered, this proves the claim.

Thus, the only possible coordinate transformation is

$$\varphi \mapsto \varphi + \text{const}, \qquad z \mapsto Z(z). \tag{14}$$

For the essential map, the effect of such a transformation of $z$ is equivalent to the shift of the cross-sections $S_0$ and $S_1$ to $z = Z^{-1}(-\varepsilon^-)$ and $z = Z^{-1}(\varepsilon^+)$ respectively. Since the evolution of $z$ is autonomous, the flight time from a cross-section $\{z = \text{const}\}$ to any other cross-section of this form depends only on the position of the cross-sections, but does not depend on the initial point on the cross-section. Thus, any shift of $S_0$ or $S_1$ is equivalent to a rigid rotation of $\varphi_0$ or $\varphi_1$ respectively. We see finally that

*the essential map is uniquely defined by the system at $\mu = 0$, modulo an arbitrary additive constant and a shift of the origin*:

$$f(\varphi) \to c_0 + f(\varphi + c_1).$$

The essential map carries most of the information on the global saddle-node bifurcations. As we mentioned, its degree $m$ defines the topological type of $\overline{W}^u$. If $|m| = 1$, then $\overline{W}^u$ is smooth if and only if $f(\varphi)$ does not have critical points (cf. with [**5, 6**]). Below (Theorem 2), we give a precise formulation to the following *reduction principle*:

*the bifurcations in $U(\overline{W}^u)$ at $\mu > 0$ follow the bifurcations in the family of one-dimensional maps*

$$\bar{\varphi} = \omega(\mu) + f(\varphi) \mod 1, \tag{15}$$

*where $\omega$ increases to infinity as $\mu \to +0$.*

This was used explicitly in [**6**] for the case $|m| = 1$; the study in [**5**] was essentially based on the same idea. In terms of [**25**], for the orbit of $\overline{W}^u$ intersecting $S_1$ at $\varphi = \varphi_1$, the derivative $f'(\varphi_1)$ equals the conventional multiplier of the orbit, i.e., the quantity which determines the value of the multiplier of a periodic orbit that may be born nearby when $L$ disappears.

The theorem below is another consequence of the reduction principle.

THEOREM 1. *Let $m = 0$ and $|f'(\varphi)| < 1$ for all $\varphi$. Then, for all small $\mu > 0$, the system $X_\mu$ has a stable periodic orbit $L_\mu$ (nonhomotopic to $L$ in $U$) which attracts all orbits of $U$.*

Notice that we do not require here that the saddle-node $L$ be simple. If it is, however, a simple saddle-node, then systems close to $X_0$ and having a saddle-node periodic orbit close to $L$ constitute a codimension one bifurcational surface in the space of dynamical systems. By construction, the function $f$ depends continuously on the system on the bifurcational surface. Thus, if the conditions of Theorem 1 are fulfilled for some system $X_0$, they are also fulfilled for all close systems on the bifurcational surface. This implies that Theorem 1 is valid for any one-parameter family that intersects the surface transversely near $X_0$. In other words, our blue sky catastrophe occurs generically in one-parameter families.

The proof of Theorem 1 is based on the calculation of the Poincaré map $T = T_0 \circ T_1$ of the cross-section $S_1$, which is defined by the orbits of $X_\mu$ for all small $\mu > 0$. Here, $T_1$ is a global map defined by (11) and $T_0 \colon S_0 \to S_1$ is the through map defined locally near $L$ at $\mu > 0$.

Since the last two equations in (8) are independent of $y$, the map $T_0 \colon (y_0, \varphi_0) \mapsto (y_1, \varphi_1)$ is written in the form (for some $C^r$-function $Y$)

$$\begin{cases} y_1 = Y(\varphi_0, y_0, \mu), \\ \varphi_1 = \varphi_0 + \tau(\varphi_0, \mu) \mod 1. \end{cases} \tag{16}$$

The function $\tau$ is the *flight time* from $S_0$ to $S_1$. It is a smooth function periodic in $\varphi_0$. Clearly, we have $\tau(\varphi_0, \mu) \to \infty$ as $\mu \to +0$. In the next section we prove the following

LEMMA 1. *If* (9) *is satisfied at* $\mu = 0$, *then* $\partial\tau/\partial\varphi$ *uniformly tends to zero as* $\mu \to +0$ *in the* $C^{r-1}$*-topology.*

Denoting $\omega(\mu) = \tau(0, \mu)$, from this lemma we obtain

$$\tau(\varphi_0, \mu) = \omega(\mu) + o(1). \tag{17}$$

Besides, since the spectrum of the matrix $\mathcal{A}(\mu)$ from (6) lies strictly to the left of the imaginary axis and since, by (7), $\dot{y} = (\mathcal{A} + \tilde{h})y$ for some $C^{r-1}$-function $\tilde{h}$, it is routinely shown that

$$\|Y\|_{C^{r-1}} \leqslant O(e^{-\alpha\omega}) \tag{18}$$

for some positive $\alpha$. In fact, we shall note in the next section that

$$\|Y\|_{C^r} \to 0 \quad \text{as} \quad \mu \to 0. \tag{19}$$

Collecting formulas (11), (12), (13), (16), (17), and (19) we obtain the following result.

THEOREM 2 (reduction principle). *As* $\mu \to +0$, *the Poincaré map* $T = T_0 \circ T_1$ *approaches* (*along with all derivatives*) *the map*

$$\bar{y} = 0, \qquad \bar{\varphi} = \omega + f(\varphi) \mod 1. \tag{20}$$

In the case of Theorem 1 ($|f'(\varphi)| < 1$), the map (20) is contracting. Hence, for each $\omega$ it has a unique attractor, namely, the unique stable fixed point. The same is clearly valid for all close maps, in particular, for the map $T$ at small $\mu > 0$. Since the map $T$ is defined by the orbits of the flow $X_\mu$, the fixed point corresponds to the attractive periodic orbit $L_\mu$ of $X_\mu$. This proves the theorem: the period of $L_\mu$ grows in proportion to $\omega(\mu)$ and it tends to infinity as $\mu \to +\infty$; since the vector field of $X_\mu$ does not vanish in $U$, it follows that the length of $L_\mu$ also tends to infinity.

The rest of the paper is devoted to the proof of Theorem 2. For greater generality, we consider also the case where the system depends on $\mu$ smoothly: we assume that

*the first derivatives of the right-hand sides of* (6) *with respect to the phase variables* $(y, z, \varphi)$ *are* $C^{r-1}$ *with respect to all variables and* $\mu$.

Assume also that the local Poincaré map near $L$ depends monotonically on $\mu$, i.e.,

$$g'_\mu(0;0) > 0 \tag{21}$$

in (5). In this case we prove that $\partial\omega/\partial\mu \neq 0$, i.e., $\omega$ can be taken as a new parameter and $\mu$ may then be considered as a function of $\omega$ which tends to zero as $\omega \to \infty$. We prove (Lemma 9) that all derivatives of $\mu$ with respect to $\omega$ tend to zero also. Lemma 1 remains valid, concerning now the derivatives of $\tau'_\varphi$ with respect to both $\omega$ and $\varphi_0$. Theorem 2 reads exactly as before, "all the derivatives" now include the derivatives with respect to $\omega$.

## 3. Calculation of the through map

In this section we give the proof of Lemma 1 and other facts related to the proof of Theorem 2. Let $\{z_0, z_1, \dots\}$ be an orbit of the local Poincaré map (5) of the system $X_\mu$ on the center manifold at $\mu \geqslant 0$:

$$z_{j+1} = z_j + g(z_j;\mu), \tag{22}$$

where the variable $z$ is assumed to belong to some small interval $[-\varepsilon^-, \varepsilon^+]$.

Since $g > 0$ if $\mu > 0$ or if $\mu = 0$ and $z \neq 0$, the sequence $\{z_0, z_1, \dots\}$ is monotonically increasing: $z_0 < z_1 < \cdots < z_j < \dots$ in this case. We shall use the following simple estimate:

$$\sum_{i=0}^{j-1} g(z_i;\mu) = \sum_{i=0}^{j-1} (z_{i+1} - z_i) = z_j - z_0 \leqslant \varepsilon^+ + \varepsilon^-. \tag{23}$$

LEMMA 2. *For some smooth function* $\psi$

$$\frac{\partial z_j}{\partial z_0} = \frac{g(z_j;\mu)}{g(z_0;\mu)} \exp\left\{\sum_{i=0}^{j-1} \psi(z_i;\mu)\right\}, \tag{24}$$

*where* $\Psi_j = \sum_{i=0}^{j-1} \psi(z_i;\mu)$ *is uniformly bounded for all* $\mu \geqslant 0$ *and for all* $z_0$ *and* $j \geqslant 0$ *such that* $-\varepsilon^- \leqslant z_0 \leqslant z_j \leqslant \varepsilon^+$.

PROOF. By differentiating (22), we obtain

$$\frac{\partial z_{j+1}}{\partial z_0} = (1 + g'_z(z_j;\mu)) \frac{\partial z_j}{\partial z_0} \quad \text{and} \quad \frac{\partial z_j}{\partial z_0} = \prod_{i=0}^{j-1} (1 + g'_z(z_i;\mu)). \tag{25}$$

Thus relation (24) is fulfilled for

$$\psi = \ln \frac{(1 + g'_z(z;\mu)) g(z;\mu)}{g(\bar{z};\mu)}. \tag{26}$$

Since

$$g(\bar{z};\mu) = g(z + g(z;\mu);\mu) = g(z;\mu)\left(1 + \int_0^1 g'_z(z + sg(z;\mu);\mu)\, ds\right),$$

it follows that

$$|\psi(z_i;\mu)| \leqslant \frac{1}{1+\min_{z\in[-\varepsilon^-,\varepsilon^+]} g'_z(z,\mu)} \max_{s\in[0,1]} |g'_z(z_i;\mu) - g'_z(z_i+sg(z_i;\mu);\mu)| \leqslant Cg(z_i;\mu), \tag{27}$$

where

$$C = \frac{\max_{z\in[-\varepsilon^-,\varepsilon^+]} |g''_{z^2}(z;\mu)|}{1-\max_{z\in[-\varepsilon^-,\varepsilon^+]} |g'_z(z;\mu)|}.$$

Since $g'_z$ is small by assumption, the factor $C$ is finite. Hence, the uniform boundedness of $\sum_{i=0}^{j-1}\psi(z_i,\mu)$ follows from (27) and (23), which yields the lemma. □

LEMMA 3. *At small $\mu \geqslant 0$, for any $k = 1,\dots,r$ the following estimate holds:*

$$\left|\frac{\partial^k z_j}{\partial z_0^k}\right| \leqslant \text{const}\cdot g(z_j;\mu) \tag{28}$$

*uniformly for all $z_0$ and $z_j$ from $[-\varepsilon^-,\varepsilon^+]$, provided $z_0$ is bounded away from zero.*

PROOF. By Lemma 2, the estimate (28) holds at $k=0$. Suppose it is fulfilled for all $k \leqslant k_0$ for some $k_0$. In this case similar estimates holds for the same $k$ for any smooth function of $z_j$. Indeed, if $\beta$ is some $C^k$-function of $z$, then the $k$th derivative of $\beta(z_j)$ with respect to $z_0$ is represented as the sum of terms of the form

$$\text{const}\cdot\frac{\partial^s\beta}{\partial z_j^s}\left(\frac{\partial^{k_1} z_j}{\partial z_0^{k_1}}\right)\cdots\left(\frac{\partial^{k_s} z_j}{\partial z_0^{k_s}}\right),$$

where $1 \leqslant s \leqslant k$, $k_1 \geqslant 1, \dots, k_s \geqslant 1$, and $k_1+\dots+k_s = k$. By (28), the absolute value of such term is estimated from above by $O(g(z_j;\mu)^s)$. Indeed, since $g$ is small, it follows that

$$\left|\frac{\partial^k\beta}{\partial z_0^k}\right| \leqslant \text{const}\cdot g(z_j;\mu).$$

In particular, the validity of (28) at all $k \leqslant k_0 \leqslant r-1$ implies that

$$\left|\frac{\partial^k g(z_j;\mu)}{\partial z_0^k}\right| \leqslant \text{const}\cdot g(z_j;\mu), \tag{29}$$

$$\left|\frac{\partial^k \psi(z_j;\mu|}{\partial z_0^k}\right| \leqslant \text{const}\cdot g(z_j;\mu), \tag{30}$$

where $\psi$ is given by (26). By Lemma 2,

$$\frac{\partial^{k_0+1} z_j}{\partial z_0^{k_0+1}} = \frac{\partial^{k_0}}{\partial z_0^{k_0}}\left(\frac{g(z_j;\mu)}{g(z_0;\mu)}\, e^{\Psi_j(z_0;\mu)}\right).$$

Thus, the $(k_0+1)$th derivative of $z_j$ is represented as the sum of terms

$$\left(\frac{\partial^{k_1}}{\partial z_0^{k_1}} g(z_j;\mu)\right)\left(\frac{\partial^{k_2}}{\partial z_0^{k_2}}\frac{1}{g(z_0;\mu)}\right)\left(\frac{\partial^{k_3}}{\partial z_0^{k_3}} e^{\Psi_j(z_0;\mu)}\right) \tag{31}$$

taken with some constant coefficients; here, $k_1+k_2+k_3 = k_0$.

The first factor in this product is estimated by (29), the second factor is bounded if $z_0$ is bounded away from zero. The derivative $\partial^k\Psi_j/\partial z_0^k$ is uniformly bounded at any $k \leqslant k_0$ because

$$\left|\frac{\partial^s}{\partial z_0^s}\Psi_j\right| = \left|\sum_{i=0}^{j-1}\frac{\partial^k\psi_i}{\partial z_0^k}\right| \leqslant O\left(\sum_{i=0}^{j-1} g(z_i;\mu)\right) \leqslant \text{const}\cdot(\varepsilon^+ + \varepsilon^-) \tag{32}$$

(see (23), (30)). Therefore, the third factor in (31) is uniformly bounded. Thus, estimate (28) remains fulfilled at $k = k_0 + 1$. By induction, this proves the lemma. □

Let us now prove that the strong stable invariant foliation of system (6) is $C^r$-smooth outside the strong stable manifold. This is a necessary step to ensure the transition from (6) to (8) without an essential loss of smoothness.

LEMMA 4. *Let the leaves of the strong stable invariant foliation of system* (6) *be given by* $\{z = \mathcal{Q}(y, z', \varphi, \mu),\ \varphi = \text{const}\}$, *where* $z'$ *is the intersection of the leaf with* $\{y = 0\}$. *Then* $\mathcal{Q}$ *is* $C^r$ *with respect to* $(y, z', \varphi)$ *in the region* $\{\mu > 0\} \cup \{\mu = 0, z \neq 0\}$.

PROOF. It suffices to prove the smoothness of the strong stable foliation $z = Q(y, z', \mu)$ of the local Poincaré map $\Pi\colon S \to S$ given by (1), where the center manifold is straightened to $\{y = 0\}$ so that

$$H(0, z, \mu) \equiv 0, \quad \text{and} \quad G(0, z, \mu) \equiv g(z; \mu). \tag{33}$$

Thus, $\bar{y} = (A(\mu) + h(y, z, \mu))y$ for some $C^{r-1}$-function $h$. Since $z$ grows sub-exponentially, this equation gives

$$\|y_j\|_{C^{r-1}} \leqslant O(e^{-\alpha j}), \tag{34}$$

where $y_j$, as a function of a point $(y, z)$, is the $y$-coordinate of the $j$th iteration of this point by $\Pi$; here $\alpha > 0$ is such that the spectrum of $A$ lies strictly inside the circle $|\cdot| = e^{-\alpha}$ in the complex plane.

By invariance of the strong stable foliation, we must have

$$Q(y, z', \mu) + G(y, Q(y, z', \mu), \mu) = Q(A(\mu)y + H(y, Q(y, z', \mu), \mu), z' + g(z'; \mu)). \tag{35}$$

Thus, at $\mu > 0$, once $Q$ is defined on the fundamental domain $\varepsilon^+ < z' \leqslant \varepsilon^+ + g(\varepsilon^+, \mu)$ in an arbitrary (smooth) way, by applying (35) repeatedly we can define it on a whole neighborhood of the origin (because at $\mu > 0$, for any $z' \in [-\varepsilon^-, \varepsilon^+]$ some iteration of it by (5) enters the fundamental domain). At $\mu = 0$ the function $Q$ is defined by continuity. It is well-known [**27, 28**], that this procedure indeed defines a $C^{r-1}$-function $Q$. Moreover, $Q'_y$ is also $C^{r-1}$. Thus, we have only to prove the existence and continuity of $Q^{(r)}_{(z')^r}$. By (35)

$$\frac{\partial Q}{\partial z'}(y, z', \mu) = \frac{\partial Q}{\partial z'}(y_J, z_J, \mu) \cdot \prod_{j=0}^{J-1} \Phi(y_j, z_j, \mu), \tag{36}$$

where

$$\Phi = \frac{1 + g'_z(z; \mu)}{1 + G'_z(y, Q(y, z, \mu), \mu) - Q'_y(y, z, \mu) H'_z(y, Q(y, z, \mu), \mu)};$$

here, $z_0 = z', z_1, \ldots, z_J$ is the orbit of $z'$ by the map (5) with the last point $z_J$ in the fundamental domain, and $y_j$ is the $y$-coordinate of the $j$th iteration of the point $(y, Q(y, z', \mu))$ by the map (1): $(y_j, Q(y_j, z_j, \mu)) = \Pi^j(y, Q(y, z', \mu))$.

This formula defines $Q'_{z'}$ as a $C^{r-1}$-function at $\mu > 0$ and at $\mu = 0$, $z' > 0$. At $z' < 0$, the limit $\mu \to +0$ gives

$$\frac{\partial Q}{\partial z'}(y, z', 0) = \prod_{j=0}^{+\infty} \Phi(y_j, z_j, 0) \tag{37}$$

(since $y_j$ tends to zero exponentially as $j \to +\infty$, uniformly in $\mu$, it follows from (33) that $\Phi_j$ tends exponentially to 1, i.e., the product converges uniformly; moreover, $Q'_{z'}(y_J, z_J, \mu) \to 1$, because $Q(0, z, \mu) \equiv z$ by definition).

It remains to prove that (37) defines a $C^{r-1}$-function at $z' < 0$. For this, one must show that the series

$$\sum_{j=0}^{J} \frac{\partial^k}{\partial z_0^k} \Phi(y_j, z_j, \mu)$$

converges absolutely and uniformly in $\mu$ for all $k \leqslant r-1$. Since $\Phi$ is smooth, it suffices to prove that the series

$$\sum_{j=0}^{J} \frac{\partial^k (y_j, z_j)}{\partial z_0^k}$$

converges absolutely and uniformly for $k \leqslant r-1$. The latter immediately follows from (34), (23), and (28). The lemma is proved. □

Another result necessary to properly define the essential map (13) is given by the following lemma.

LEMMA 5. *There exists a unique smooth function $\tilde{g}(z)$ such that at $\mu = 0$ the local Poincaré map on the center manifold*

$$\bar{z} = z + g(z; 0) \tag{38}$$

*coincides with the time-1 map of the flow*

$$\dot{z} = \tilde{g}(z). \tag{39}$$

*The function $\tilde{g}$ is given by*

$$\tilde{g}(z) = \begin{cases} g(z;0) \exp\Big\{ -\sum\limits_{i=0}^{\infty} \psi(z_i; 0) \Big\} & \text{at } z < 0, \\ g(z;0) \exp\Big\{ -\sum\limits_{i=-1}^{-\infty} \psi(z_i; 0) \Big\} & \text{at } z > 0, \end{cases} \tag{40}$$

*where $\{z_j\}$ is the orbit of $z = z_0$ by the map (38); $\psi$ is the function defined by (26).*

PROOF. The map (38) coincides with the time-1 map of flow (39) if and only if

$$\int_z^{\bar{z}} \frac{ds}{\tilde{g}(s)} = 1. \tag{41}$$

Since $z = 0$ is the fixed point of (38) with multiplier equal to 1, it follows that $\tilde{g}$ must vanish at zero along with $\tilde{g}'$. Taking the limit $z \to 0$ in (41), we get

$$\lim_{z \to 0} \frac{\tilde{g}(z)}{g(z;0)} = 1 \tag{42}$$

because

$$\int_z^{\bar{z}} \frac{ds}{\tilde{g}(s)} = \frac{\bar{z} - z}{\tilde{g}(z) + \frac{1}{2}\tilde{g}'(z)(\bar{z} - z) + o(\bar{z} - z)} \sim \frac{g(z;0)}{\tilde{g}(z)};$$

here we use the smoothness of $g$ at zero.

Differentiating (41), we obtain

$$\frac{1 + g'_z(z;0)}{\tilde{g}(\bar{z})} - \frac{1}{\tilde{g}(z)} = 0. \tag{43}$$

Therefore, for $z < 0$,

$$\tilde{g}(z_j) = \tilde{g}(z) \prod_{i=0}^{j-1} (1 + g'_z(z_i; 0))$$

which gives (40) by virtue of (42), (24), and (25). The series $\sum_{i=0}^{+\infty} \psi(z_i; 0)$ is convergent due to (27), (23). For $z > 0$ formula (40) is obtained in the same way.

Thus, the smooth function $\tilde{g}$ satisfying (41) is indeed defined uniquely and it must be given by (40). It remains to prove that the function $\tilde{g}$ defined by (40) is smooth (obviously, it satisfies (43) and (42), which implies (41) for *smooth* functions). The first derivative is given, for $z < 0$, by the formula (we use (24)):

$$\tilde{g}'(z) = \left[ g'(z) - \sum_{j=0}^{+\infty} \psi'(z_j; 0) g(z_j; 0) \exp\left\{ \sum_{i=0}^{j-1} \psi(z_i; 0) \right\} \right] \exp\left\{ - \sum_{i=0}^{+\infty} \psi(z_j; 0) \right\}.$$

The series here are uniformly convergent and bounded by virtue of (27) and (23). Moreover, since $g'_z(0; 0) = 0$ and $\psi'_z(0; 0) = 0$, it follows that $\tilde{g}'(-0) = 0$. Analogously, $\tilde{g}'(+0) = 0$; i.e., the function $\tilde{g}$ defined by (40) is smooth at zero. The lemma is proved. □

Note that the series $\sum_{i=0}^{\infty} \psi(z_i; 0)$ for $z_0 < 0$ and $\sum_{i=1}^{+\infty} \psi(z_{-i}; 0)$ for $z_0 > 0$ absolutely converge together with the derivatives with respect to $z_0$ up to the order $r - 1$ because all partial sums are uniformly bounded according to (30), (23). Thus, the function $\tilde{g}$ is $C^{r-1}$ at $z \neq 0$. Hence, the time $t$ shift $z \mapsto \pi_t^*(z)$ of the system (39) is $C^r$ with respect to $z$ at $z \neq 0$.

If $z \mapsto \pi_t(z, \varphi)$ denotes the time $t$ shift of the point $(z, \varphi)$ on the center manifold of system (8), then $\pi_{-\varphi}(z, \varphi)$ is the projection of the point $(z, \varphi)$ onto the local cross-section $S$: $\{\varphi = 0\}$ by the backward orbit. By uniqueness of $\tilde{g}$, the transformation $z \mapsto \mathcal{Z}(z, \varphi) \equiv \pi_\varphi^* \circ \pi_{-\varphi}$ is the unique 1-periodic transformation in $\varphi$ which is identical at $\varphi = 0$ and which brings the last two equations of (8) to the autonomous form (10). As we just showed, the transformation is $C^r$ at $z \neq 0$, hence, the essential map and the map $T$ of Theorem 2 are $C^r$ indeed.

Now let us proceed directly to the proof of Theorem 2 (first, for the case where the right-hand sides depend only continuously on $\mu$). At $\mu > 0$, for any $z_0$ there exists an integer $N(z_0)$ such that the $N$th iteration $z_N$ of $z_0$ belongs to the interval $I_\mu = [0, g(0; \mu)[$ ($N > 0$ for $z_0 < 0$ and $N \leqslant 0$ for $z_0 \geqslant 0$). Let

$$\zeta = \frac{z_N}{g(0; \mu)}.$$

Since $|z_N| \leqslant g(0; \mu)$ and $g(z_N, \mu) = g(0; \mu) + g'_z(0; \mu) z_N + o(z_N)$, it follows that $g(z_N; \mu)/g(0; \mu) \to 1$ as $\mu \to +0$. Thus, by Lemma 3, we get

LEMMA 6. *All derivatives of $\zeta(z; \mu)$ are uniformly bounded.*

Let us define

$$\sigma(z; \mu) = \theta(\zeta(z); \mu) - N(z), \tag{44}$$

where

$$\theta(\zeta; \mu) = \int_0^\zeta \frac{ds}{\mathcal{G}(s; \mu)} + \chi(\zeta) \left[ \int_\zeta^{\zeta + \mathcal{G}(\zeta; \mu)} \frac{ds}{\mathcal{G}(s; \mu)} - 1 \right], \tag{45}$$

$$\mathcal{G}(\zeta; \mu) = \frac{g(\zeta g(0; \mu); \mu)}{g(0; \mu)}, \tag{46}$$

and $\chi(\zeta)$ is a $C^r$-function equal to 1 at $|\zeta| \leqslant \delta$ and equal to zero at $|\zeta - 1| \leqslant \delta$ for some small $\delta > 0$. Note that

$$\lim_{\mu \to +0} \mathcal{G}(\zeta; \mu) = 1, \tag{47}$$

$$\lim_{\mu \to +0} (\theta(\zeta; \mu) - \zeta) = 0, \tag{48}$$

where the limits are taken in $C^r$-topology.

By definition, $\sigma(z; \mu)$ satisfies

$$\sigma(\bar{z}; \mu) = \sigma(z; \mu) + 1. \tag{49}$$

LEMMA 7. *The function $\sigma(z; \mu)$ is $C^r$. Moreover, for small $z \neq 0$ we have*

$$\lim_{\mu \to +0} \frac{d\sigma}{dz} = \frac{1}{\tilde{g}(z)}, \tag{50}$$

*where the limit is taken in the $C^{r-1}$-topology; $\tilde{g}$ is the right-hand side of system* (10), *defined by* (40).

PROOF. By definition, $\sigma$ is $C^r$ on $I_\mu$. Thus, by virtue of (49) it is sufficient to check the continuity of the derivatives of $\sigma$ at $z = 0$. By (45) and (46), for small $z > 0$ we have

$$\sigma(z; \mu) = \int_0^z \frac{ds}{g(s; \mu)} + \chi\left(\frac{z}{g(0; \mu)}\right)\left[\int_z^{\bar{z}} \frac{ds}{g(s; \mu)} - 1\right]$$

and for small $z < 0$

$$\sigma(z; \mu) = \int_0^{\bar{z}} \frac{ds}{g(s; \mu)} - 1 + \chi\left(\frac{\bar{z}}{g(0; \mu)}\right)\left[\int_{\bar{z}}^{\bar{z}+g(\bar{z};\mu)} \frac{ds}{g(s; \mu)} - 1\right].$$

By the definition of $\chi$, in both cases we see that if, say, $|z| < \delta g(0; \mu)/2$, then

$$\sigma(z; \mu) = \int_0^{\bar{z}} \frac{ds}{g(s; \mu)} - 1$$

which is indeed a $C^r$-function for all small $z$.

Now, by (44)–(46) and (24),

$$\sigma'_z(z; \mu) = \theta'_\zeta(\zeta; \mu)\mathcal{G}(\zeta; \mu) \frac{1}{g(z; \mu)} \exp\left\{\sum_{i=0}^{N-1} \psi(z_i; \mu)\right\} \tag{51}$$

for $z < 0$. Thus, (50) follows immediately from (47), (48), Lemma 6 and (40). The case $z > 0$ is analysed in the same way. The lemma is proved. □

Now note that Lemma 1 follows from Lemma 7 almost immediately. Indeed, let the orbit by system (8) of the point $(z = -\varepsilon^-, \varphi_0, y_0)$ of $S_0$ intersect $S_1$ at the point $(z = \varepsilon^+, \varphi_1, y_1)$. Then, by construction

$$\sigma(z^0, \mu) + N(z^0) = \sigma(z^1, \mu) - N(z^1), \tag{52}$$

where

$$z^0 = \pi_{1-\varphi_0}(-\varepsilon^-, \varphi_0), \qquad z^1 = \pi_{-\varphi_1}(\varepsilon^+, \varphi_1); \tag{53}$$

here, as above, $N$ is the number of iterations necessary to get into $I_\mu$, and $\pi_t(z, \varphi)$ is the time $t$ shift of $z$ by the flow (8).

Differentiating (52), we get

$$\sigma_z'(z^0,\mu)\,\frac{\partial z^0}{\partial \varphi_0} = \sigma_z'(z^1,\mu)\,\frac{\partial z^1}{\partial \varphi_1}\,\frac{\partial \varphi_1}{\partial \varphi_0}\,. \tag{54}$$

As $\mu \to +0$,

$$\pi_t(z,\varphi) \underset{C^r}{\longrightarrow} \pi_t^*(z) \tag{55}$$

where $\pi_t^*$ is the time $t$ shift by the autonomous flow (10). Thus, by (50) and (53)–(55)

$$\frac{\partial \varphi_1}{\partial \varphi_0} \underset{C^{r-1}}{\longrightarrow} 1 \tag{56}$$

as $\mu \to +0$, which is exactly the statement of Lemma 1.

To complete the proof of Theorem 2, it remains to prove (19). It is sufficient to show that as $j \to +\infty$,

$$\|y_j\|_{C^r} \to 0 \tag{57}$$

uniformly for all $\mu$; here $y_j$, as a function of $(y_0, z_0)$, is the $y$-coordinate of the $j$th iteration $(y_j, z_j)$ of the point $(y_0, z_0)$ by the local Poincaré map $\Pi\colon S \to S$ given by (1). We assume that the center manifold is straightened, i.e., (33) holds. By differentiating of (1) $r$ times, we obtain

$$\frac{\partial^r y_{j+1}}{\partial (y_0,z_0)^r} = (A + H_y'(y_j,z_j))\,\frac{\partial^r y_j}{\partial (y_0,z_0)^r} + H_{z^r}^{(r)}(y_j,z_j)\left(\frac{\partial z_j}{\partial (y_0,z_0)}\right)^r + O(e^{-\alpha j}) \tag{58}$$

(we have used that by (34), (33) all the derivatives of $y_j$ and of $H$ up to order $r-1$ tend to zero exponentially as $j \to +\infty$). Now note that $\partial z_j/\partial(y_0,z_0)$ is uniformly bounded ($z_j$ is independent of $y_0$ when the strong stable foliation is straightened, and $\partial z_j/\partial z_0$ is bounded according to Lemma 2), whereas $H_{z^r}^{(r)}(y,z)$ tends to zero as $y \to 0$ (by (33)). Thus, since $H_y'$ is small, (58) can be rewritten as

$$\left\|\frac{\partial^r y_{j+1}}{\partial (y_0,z_0)^r}\right\| \leqslant e^{-\alpha}\left\|\frac{\partial^r y_j}{\partial (y_0,z_0)^r}\right\| + o_{j\to+\infty}(1),$$

which gives (57).

This proves Theorem 2 for the case of continuous dependence on $\mu$. Let us now consider the case where the first derivatives of the right-hand sides of the system (6) with respect to $(y,z,\varphi)$ are $C^{r-1}$ with respect to all the variables and $\mu$. By (21), one may assume

$$\mu = g(0;\mu). \tag{59}$$

Let

$$\xi(z^1,z^2) = \int_{z^1}^{z^2} \frac{ds}{g^2(s;\mu)}; \tag{60}$$

this is a well-defined expression for $\mu > 0$. Since $g_z'$ is small, it follows that

$$g(z;\mu) \sim g(z';\mu)$$

for all $z \in [-\varepsilon^-, \varepsilon^+]$ and $z' \in [z, \bar{z} = g(z;\mu)]$ (here "$\sim$" means that the ratio of the left-hand and right-hand sides is bounded away from zero and infinity).

This implies that

$$\sum_{i=0}^{j} \frac{1}{g(z_i;\mu)} \sim \xi(z_0, z_j), \tag{61}$$

where $z_j$ is the $j$th iteration of $z_0$, given by (22). It is immediately seen that as $j \to +\infty$, while $z_0$ and $z_j$ remain in the small interval $[-\varepsilon^-, \varepsilon^+]$, we have

$$\frac{\xi(z_0, z_j)}{j} \to +\infty \tag{62}$$

(because in the left-hand side of (61) $g_i \equiv z_{i+1} - z_i$ must be unboundedly small for an unboundedly large number of $i$'s in order to have an unboundedly large number of iterations on the bounded interval) and

$$\frac{\xi(z_0, z_j)}{\max_{z\in[z_0,z_j]}(1/g(z;\mu))} \to +\infty \tag{63}$$

(because $g'_z$ is small provided $g$ is small, so the number of points $z_i$ for which $g(z_i;\mu) \sim \min_{z\in[z_0,z_j]} g$ grows unboundedly as $\min_{z\in[z_0,z_j]} g \to +0$).

Now note that by (22), we have

$$\frac{\partial z_{j+1}}{\partial \mu} = (1 + g'_z(z_j;\mu)) \frac{\partial z_j}{\partial \mu} + g'_\mu(z_j;\mu).$$

Comparing this formula with (25) we obtain (see (24), (21), and (61))

$$\begin{aligned} \frac{\partial z_j}{\partial \mu} &= \frac{\partial z_j}{\partial z_0} \sum_{i=0}^{j} g'_\mu(z_i;\mu) \Big/ \frac{\partial z_{i+1}}{\partial z_0} \\ &= g(z_j,\mu) \exp\left\{ \sum_{i=0}^{j-1} \psi(z_i;\mu) \right\} \sum_{i=0}^{j} \frac{g'_\mu(z_i;\mu)}{g(z_{i+1};\mu)} \exp\left\{ -\sum_{l=0}^{i} \psi(z_l;\mu) \right\} \\ &\sim g(z_j;\mu)\xi(z_0, z_j). \end{aligned} \tag{64}$$

LEMMA 8. *For any $k_1 = 0, \dots, r-1$, $k_2 = 0, \dots, r$, $k_1 + k_2 \leqslant r$ the estimate*

$$\left| \frac{\partial^{k_1+k_2} z_j}{\partial \mu^{k_1} \partial z_0^{k_2}} \right| \leqslant \text{const} \cdot g(z_j;\mu)\xi^{k_1}(z_0, z_j) \tag{65}$$

*holds uniformly for all $\mu > 0$ and for all $z_0$ and $z_j$ from $[-\varepsilon^-, \varepsilon^+]$, provided $z_0$ is bounded away from zero.*

PROOF. The case $k_1 = 0$ is given by Lemma 3; ($k_1 = 1$, $k_2 = 0$) is given by (64). The lemma is proved by induction, in the same way as Lemma 3. Assume that it is proved for all $k_1 \leqslant k_1^0$. Then for any smooth function $\beta$ of $z$ and $\mu$

$$\left| \frac{\partial^{k_1+k_2} \beta(z_j;\mu)}{\partial \mu^{k_1} \partial z_0^{k_2}} \right| \leqslant \text{const} \cdot g(z_j;\mu)\xi^{k_1}(z_0, z_j) \tag{66}$$

for these values of $k_1$; for a proof note that this derivative is the sum of terms of the kind

$$\text{const} \cdot \frac{\partial^{s_1+s_2}\beta}{\partial \mu^{s_1} \partial z_j^{s_2}} \left( \frac{\partial^{k_{11}+k_{12}} z_j}{\partial \mu^{k_{11}} \partial z_0^{k_{12}}} \right) \cdots \left( \frac{\partial^{k_{s_2 1}+k_{s_2 2}} z_j}{\partial \mu^{k_{s_2 1}} \partial z_0^{k_{s_2 2}}} \right),$$

where $0 \leqslant s_1 \leqslant k_1$, $1 \leqslant s_1 + s_2 \leqslant k_1 + k_2$, $k_{11} + \cdots + k_{s_2 1} = k_1 - s_1$, $k_{12} + \cdots + k_{s_2 2} = k_2$. By (66), the absolute value of such a term is estimated from above by

$O(g(z_j;\mu)^{s_2}\xi^{k_1-s_1}(z_0,z_j))$. Since $g\xi$ is bounded away from zero by (63) and since $g$ is small, (66) follows.

By (66) and (23)

$$\sum_{i=0}^{j}\left|\frac{\partial^{k_1+k_2}\beta(z_i;\mu)}{\partial\mu^{k_1}\partial z_0^{k_2}}\right|\leqslant \text{const}\cdot\xi^{k_1}(z_0,z_j). \tag{67}$$

Applying (66) and (67) with $\beta=\psi$, $\beta=g$ and $\beta=g'_\mu$ to formula (64), we get for $k_1=k_1^0+1$

$$\left|\frac{\partial^{k_1+k_2}z_{j+1}}{\partial\mu^{k_1}\partial z_0^{k_2}}\right|\leqslant \text{const}\cdot g(z_j;\mu)\xi^{k_1-1}(z_0,z_j)\sum_{i=0}^{j-1}\frac{1}{g(z_i;\mu)}$$

which, by (61), coincides with (65). By induction, this gives the lemma. □

Let $\omega(\mu)$ denote the flight time of the point $(z_0=-\varepsilon^-,\varphi_0=0)$ on $S_0$ to the point $(\varepsilon^+,\varphi_1=\omega-M)$ on the cross-section $S^1$, where $M$ denotes the integral part of $\omega$. In our notation

$$z_M(-\varepsilon^-)=\pi_{-(\omega-M)}(\varepsilon^+,\omega-M),$$

where $z_M$ is the $M$th iteration of $z_0$ by the map (5) and $\pi_{-\varphi}(z,\varphi)$ is the projection on the local cross-section $S:\{\varphi=0\}$ by the flow of (8). Thus,

$$\omega'_\mu\sim\frac{\partial z_M}{\partial\mu} \tag{68}$$

or, by (64),

$$\omega'_\mu\sim\xi^*\equiv\xi(-\varepsilon^-,\varepsilon^+) \tag{69}$$

(since $z_M$ is close to $\varepsilon^+$, it follows that $z_M$ and $g(z_M;\mu)$ are bounded away from zero).

By (69), $\omega'_\mu$ never vanishes, so $\omega$ can be regarded as a new parameter and $\mu$ becomes a function of $\omega$. By (68), (65), and (69),

$$\omega^{(k)}_{\mu^k}\leqslant\text{const}\cdot(\omega'_\mu)^k,$$

and it is easy to see (just differentiate the identity $\mu(\omega(\mu))=\mu$ $k$ times) that the following estimate holds.

LEMMA 9. *For all $k\leqslant r-1$, we have*

$$\mu^{(k)}_{\omega^k}\leqslant\text{const}\,\frac{1}{\xi^*}. \tag{70}$$

In particular, this implies that for any function smooth with respect to $\mu$, its derivatives with respect to $\omega$ tend to zero as $\omega\to+\infty$. Moreover, plugging (70) into (66), (67) yields

$$\left|\frac{\partial^{k_1+k_2}\beta(z_j;\mu)}{\partial\omega^{k_1}\partial z_0^{k_2}}\right|\leqslant\text{const}\cdot g(z_j;\mu), \tag{71}$$

$$\sum_{i=0}^{j}\left|\frac{\partial^{k_1+k_2}\beta(z_i;\mu)}{\partial\omega^{k_1}\partial z_0^{k_2}}\right|\leqslant\text{const} \tag{72}$$

for any smooth function $\beta$ of $(z,\mu)$. Revisiting the proof above for the case of continuous dependence on $\mu$, we now immediately reprove Lemmas 4 and 6, in

the sense that $\mathcal{Q}'_{(y,z,\varphi)}$ is $C^{r-1}$ with respect to all variables and $\omega$ (Lemma 4) and all derivatives of $\zeta'_z$ with respect to $z$ and $\omega$ are uniformly bounded (Lemma 6). Formula (19) (where $C^r$ now refers to the space of functions whose first derivatives with respect to $(y_0, \varphi_0)$ have $r-1$ continuous derivatives with respect to $(y_0, \varphi_0, \omega)$) is proved in the same way as before.

To get Lemma 7, where the "$C^{r-1}$-topology" now refers to the derivatives both with respect to $z$ and $\omega$, note that the transition from (51) to (50) is justified by (72) applied with $\beta = \psi$ and by (47) (the relation (48) follows from (47), (45)). To prove the $\omega$-dependent version of the limit (47), note that $\mathcal{G}'_\zeta(\zeta;\mu) = g'_z(\zeta\mu;\mu)$ and $\mathcal{G}(0;\mu) = 1$; hence, the validity of (47) follows from (71) applied with $\beta = g'_z$.

Lemma 7 implies Lemma 1 which (together with (19)) gives the theorem.

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