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On Herman's positive entropy conjecture

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ABSTRACT

We show that any area-preserving C^r-diffeomorphism of a two-dimensional surface displaying an elliptic fixed point can be C^r-perturbed to one exhibiting a chaotic island whose metric entropy is positive, for every $1 \le r \le \infty$. This proves a conjecture of Herman stating that the identity map of the disk can be C^∞-perturbed to a conservative diffeomorphism with positive metric entropy. This implies also that the Chirikov standard map for large and small parameter values can be C^∞-approximated by a conservative diffeomorphisms displaying a positive metric entropy (a weak version of Sinai's positive metric entropy conjecture). Finally, this sheds light onto a Herman's question on the density of C^r-conservative diffeomorphisms displaying a positive metric entropy: we show the existence of a dense set formed by conservative diffeomorphisms which either are weakly stable (so, conjecturally, uniformly hyperbolic) or display a chaotic island of positive metric entropy.

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Introduction

Consider a diffeomorphism f of a two-dimensional surface $\mathbb{M}$. The *maximal Lyapunov exponent* of $x \in \mathbb{M}$ is

$$\lambda(x) = \limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)\| \ . \tag{1}$$

It quantifies the sensitivity to the initial conditions: if $\lambda(x)$ is positive, then the forward orbits of most of the points from a neighborhood of x will diverge exponentially fast from the orbit of x.

Let f preserve a smooth area form ω on $\mathbb{M}$. The *metric entropy*[1] of f is the integral

$$h_\omega(f) := \int_{\mathbb{M}} \lambda(x)\, \omega(dx). \tag{2}$$

Whenever the metric entropy of a dynamical system is positive, points in $\mathbb{M}$ display a positive Lyapunov exponent *with non-zero probability*.

One of the most fundamental questions in conservative dynamics is

Question 0.1. How typical are conservative dynamical systems with positive metric entropy?

Note that a different notion of topological entropy is one of the basic tools in describing chaotic dynamics: positive topological entropy indicates the presence of uncountably many orbits with a positive maximal Lyapunov exponent [36]. However, the positivity of the metric entropy is a much stronger property, as it ensures positive maximal Lyapunov exponent for a non-negligible set of initial conditions. While numerical evidence for a large set of initial conditions corresponding to seemingly chaotic behavior in area-preserving maps is abundant, a rigorous proof for the positivity of metric entropy is available only for a small set of specially prepared examples (see Section 1). Currently no mathematical technique exists for answering Question 0.1 in full generality.

Several prominent conjectures are related to this question. In order to formulate them, let us recall the topologies involved. For $1 \leq r \leq \infty$, let $\mathrm{Diff}^r_\omega(\mathbb{M})$ be the space of

[1] We employ here Pesin formula for the Kolmogorov-Sinai entropy [52].

diffeomorphisms which keep the area form ω invariant. When $r < \infty$, the space $\mathrm{Diff}^r_\omega(\mathbb{M})$ is endowed with the uniform C^r-topology. The space $\mathrm{Diff}^\infty_\omega(\mathbb{M})$ is endowed with the projective limit topology whose base is formed by all C^r-open subsets for all finite r. Let us fix a metric d_r compatible with the C^r-topology (the space $\mathrm{Diff}^r_\omega(\mathbb{M})$ with the metric d_r is complete). The C^∞-topology in $\mathrm{Diff}^\infty_\omega(\mathbb{M})$ is defined by the following metric:

$$d_\infty(f,g) = \sum_{r=0}^{\infty} \frac{1}{r!} \min(1, d_r(f,g))$$

(note that $\mathrm{Diff}^\infty_\omega(\mathbb{M})$ with the metric d_∞ is complete).

Consider the two-dimensional disc $\mathbb{D} := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Let id be the identity map of $\mathbb{D}$. One of the favorite conjectures of Herman can be formulated as follows.

Conjecture 0.2 *(Herman [33]).* For every $\varepsilon > 0$ there exists $f \in \mathrm{Diff}^\infty_\omega(\mathbb{D})$ such that $d_\infty(f, id) < \varepsilon$ and the metric entropy of f is positive: $h_\omega(f) > 0$.

It is linked to his question:

Question 0.3 *(Herman [33]).* Is the set of diffeomorphisms f with positive metric entropy $h_\omega(f)$ dense in $\mathrm{Diff}^\infty_\omega(\mathbb{D})$?

In this work we prove Herman's Conjecture 0.2. This also could be a step towards a positive answer to Question 0.3. Recall that a periodic point P of f is *hyperbolic* if the eigenvalues of $Df^p(P)$ (where p is the period of P) are not equal to 1 in the absolute value. The main result of this work is the following

Theorem A. *For any surface* $(\mathbb{M}, \omega)$*, if a diffeomorphism* $f \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ *has a periodic point which is not hyperbolic, then there is a* C^∞*-small (as small as we want) perturbation of* f *such that the perturbed map* $\hat{f} \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ *has positive metric entropy:* $h_\omega(\hat{f}) > 0$*.*

If $f = id$, then every point in $\mathbb{M}$ is a non-hyperbolic fixed point of f, so Theorem A implies Herman's conjecture immediately. Another immediate consequence employs the notion of the weak stability [41]. The map $f \in \mathrm{Diff}^r_\omega(\mathbb{M})$ is C^r_ω*-weakly stable* if all the periodic points of any C^r-close to f map $\hat{f} \in \mathrm{Diff}^r_\omega(\mathbb{M})$ are hyperbolic.

Corollary B. *A diffeomorphism* $f \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ *is either* C^∞_ω*-weakly stable or* C^∞*-approximated by a diffeomorphism from* $\mathrm{Diff}^\infty_\omega(\mathbb{M})$ *which has positive metric entropy.*

This statement suggests that the answer to Question 0.3 has to be positive. The reason is that the common belief among dynamicists is that any C^∞-weakly stable map of a closed manifold is uniformly hyperbolic (see Section 1.7). If this conjecture is true, then every C^∞_ω-weakly stable diffeomorphism has positive metric entropy (and, moreover, no C^∞_ω-weakly stable diffeomorphisms exist when $\mathbb{M} = \mathbb{D}$).

Another metric entropy conjecture regards the very popular Chirikov standard map family. This is a one-parameter family of area-preserving diffeomorphisms of $\mathbb{T}^2$ defined for $a \in \mathbb{R}$ by

$$T_a(x, y) = (2x - y + a \sin 2\pi x, x). \tag{3}$$

For $a \in (0, \frac{2}{\pi})$ this map has an elliptic fixed point at $(x = 1/2, y = 1/2)$ (a period-p point P of an area-preserving map f is called *elliptic* if the eigenvalues of $Df^p(P)$ are equal to $e^{\pm i\alpha}$ where $\alpha \in (0, \pi)$, i.e. they are complex and lie on the unit circle). When a increases, the elliptic fixed point loses stability and, at a large enough, the numerically obtained phase portraits display a large set where the dynamics is apparently chaotic (the so-called "chaotic sea" [18]). A conjecture due to Sinai can be formulated as follows (cf. [56] P.144):

Conjecture 0.4. There exists a set $\Lambda \subset \mathbb{R}$ of positive Lebesgue measure such that, for $a \in \Lambda$, the metric entropy of T_a is positive.

This conjecture is still completely open despite intense efforts, see e.g. [28]. However, it is shown by Duarte [20] that the map T_a has elliptic periodic points for an open and dense set of sufficiently large values of parameter a. Hence, our main theorem implies the following "approximative version" of Sinai's Conjecture 0.4:

Corollary C. *For every sufficiently large or sufficiently small $a \in \mathbb{R}$, there exists a C^∞-small perturbation $\hat{T} \in \mathrm{Diff}^\infty_\omega$ of the map T_a such that $\hat{T}$ has positive metric entropy.*

We note that a large set (of almost full Lebesgue measure) in a neighborhood of a generic elliptic point of an area-preserving C^r-diffeomorphism with $r \geq 4$ consists of points with zero Lyapunov exponent (the points on KAM-curves [37]). Our result, nevertheless, shows that the Lebesgue measure of the points with positive maximal Lyapunov exponent in a neighborhood of any elliptic point can be positive too.

The proof of Theorem A occupies Sections 2-6 of this paper. In Section 1 we remind certain background information pertinent to this work.

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1. Selected results and conjectures around the positive metric entropy conjecture

The study of the instability and chaos in conservative dynamics enjoys a long tradition since the seminal work by Poincaré [53] on the tree-body problem.

1.1. Uniformly hyperbolic maps

An invariant compact set K of the diffeomorphism f of a manifold $\mathbb{M}$ is uniformly hyperbolic if the restriction $T\mathbb{M}|_K$ of the tangent bundle of $\mathbb{M}$ to K splits into two Df-invariant continuous sub-bundles E^s and E^u such that E^s is uniformly contracted and E^u is uniformly expanded:

$$T\mathbb{M}|_K = E^s \oplus E^u \ , \quad \exists N \geq 1, \quad \|Df^N|_{E^s}\| < 1 \ , \quad \|Df^{-N}|_{E^u}\| < 1 \ .$$

Whenever $K = \mathbb{M}$, the diffeomorphism f is called *uniformly hyperbolic* or Anosov. Note that the maximal Lyapunov exponent of a uniformly hyperbolic dynamical system is positive at every point. Hence, whenever the dynamics is conservative (i.e. f keeps invariant a volume form ω), the metric entropy is positive.

Such dynamics are very well understood. However, the existence of the splitting of $T\mathbb{M}$ into two non-trivial continuous sub-bundles imposes strong restrictions on the topology of $\mathbb{M}$. For instance, there exists no uniformly hyperbolic diffeomorphism of a closed two-dimensional surface different from the torus $\mathbb{T}^2$.

1.2. Stochastic island

The first example of a *non-uniformly hyperbolic* area-preserving map of the disk was given by Katok [35]. His construction started with an Anosov diffeomorphism of $\mathbb{T}^2$ with 4 fixed points and a certain symmetry. Then the diffeomorphism is modified so that the fixed points become non-hyperbolic and sufficiently flat (while the hyperbolicity is preserved outside of the fixed points). After that, the torus is projected to a two-dimensional disc. The singularities of this projection correspond to the fixed points, and their flatness allows for making the resulting map of the disc a diffeomorphism. The positivity of the metric entropy is inherited from the original Anosov map.

This example has been pushed forward by Przytycki [54], where instead of making the fixed points non-hyperbolic he made a surgery to replace each of these fixed points by an elliptic island (in this case - a neighborhood of an elliptic fixed point filled by closed invariant curves) bounded by a heteroclinic link (as defined below). Przytycki's example was put in a more general context by Liverani [39].

Recall that given a hyperbolic periodic point P of a diffeomorphism f, the following sets are immersed smooth submanifolds: the *stable* and, respectively, *unstable manifolds*

$$W^s(P;f) := \{x \in \mathbb{M} : d(f^n(x), P) \underset{n\to+\infty}{\to} 0\} \quad \text{and}$$
$$W^u(P;f) := \{x \in \mathbb{M} : d(f^n(x), P) \underset{n\to-\infty}{\to} 0\}.$$

We will call a C^0-embedded circle L a *heteroclinic N-link* if there exists N hyperbolic periodic points $P_1, \ldots, P_N \in L$ satisfying $L \subset \cup_{i=1,\ldots,N} W^s(P_i) \cup W^u(P_i)$. Note that a heteroclinic N-link is a piecewise C^∞-curve with possible break points at the periodic points P_i.

Przytycki construction gives an example of the stochastic island in the followings sense.

Definition 1.1. A *stochastic island* is a two-dimensional domain $\mathcal{I}$ bounded by finitely many heteroclinic links such that every point in $\mathcal{I}$ has positive maximal Lyapunov exponent.

In this paper (see Section 3) we build one more example of a map $f \in \mathrm{Diff}^\infty_\omega(\mathbb{D})$ with a stochastic island (where ω is the standard area form $dx \wedge dy$ in the unit two-dimensional disc $\mathbb{D}$). To this aim, we adapt to the conservative setting the Aubin-Pujals blow-up construction [5].

One of the main difficulties in the Herman's entropy conjecture is that no other examples of conservative maps of a disc with positive metric entropy are known. All of these constructions are very fragile (sensitive to perturbations): for example, no C^r-generic finite-parameter family of area-preserving diffeomorphisms can have a parameter value for which a heteroclinic or homoclinic link exists; no entire diffeomorphism (including e.g. the standard map and any polynomial diffeomorphism) can have a heteroclinic or homoclinic link [60]. Still, we prove our main theorem by showing that stochastic islands appear near any elliptic point of an area-preserving diffeomorphism after a C^∞-small perturbation.

1.3. Strong regularity

With the aim to extend the available examples of the non-uniformly hyperbolic behavior, Yoccoz launched a program called *Strong Regularity* in his first lecture at Collège de France [12]. The objective was to give a geometric-combinatorial definition of the non-uniformly hyperbolic dynamics which would serve both the one-dimensional (strongly dissipative) case, like e.g. in Jakobson theorem [34] and the 2-dimensional case (e.g. for the positive entropy conjecture). So far there are three examples of such dynamics: one-dimensional quadratic maps (e.g. implying Jakobson theorem) [63], a non-uniformly hyperbolic horseshoe of dimension close to 6/10 [51], and Hénon-like endomorphisms [9] (implying Benedicks-Carleson Theorem [8]).

1.4. Isotopy to identity and renormalization

It is well-known that any symplectic diffeomorphism $F : \mathbb{D} \to \mathbb{R}^2$ is isotopic to identity [17], which implies that it can always be represented as a composition $F = f_n \circ \cdots \circ f_1$ of n symplectic diffeomorphisms f_i, each of which is uniformly $O(1/n)$-close to the identity map. Thus, one could try to prove the Herman's conjecture by using the Ruelle-Takens construction [55]: take as F in this formula an appropriate area-preserving map with a stochastic island, then consider n-disjoint ε-disks $\sqcup_i D_i = \psi_i(\mathbb{D})$ inside $\mathbb{D}$ (where ψ_i are uniform affine contractions) and take a perturbation f of the identity map such that $f(D_i) = D_{i+1}$ and $f|_{D_i}$ is smoothly conjugate to f_i, i.e., $f|_{D_i} = \psi_{i+1} \circ f_i \circ \psi_i^{-1}$ for $i = 1, \dots, n-1$, and $f|_{D_n} = \psi_1 \circ f_n \circ \psi_n^{-1}$. Then, $f^n|_{D_1}$ will be smoothly conjugate to F and, hence, would have positive metric entropy. However, since the conjugates ψ_i^{-1} expand with a rate at least ε^{-1}, we observe that the C^r-norm of f is then $\asymp 1/(n\varepsilon^r)$. As the n discs D_i are disjoint, it follows that $n\varepsilon^2 \le 1$, so f can, a priori, be $\asymp \varepsilon^{2-r}$ far, in the C^r-norm, from the identity.

Therefore, this construction does not produce the result for $r \ge 2$, although one can create C^1-close to identity maps with positive metric entropy in this way (in fact, every area-preserving dynamics can be realized by iterations of C^1-close to identity maps exactly by this procedure). In [47], Newhouse-Ruelle-Takens pushed forward the argument to obtain the C^2-case for torus maps. Fayad [24] also proposed a trick to cover the C^2-case for disk maps, but his method does not work in the C^r-case if $r \ge 3$.

We bypass the problem by using symplectic polynomial approximations of [57] instead of the isotopy. In this way, one C^r-approximates any symplectic diffeomorphism F by the product $f_n \circ \cdots \circ f_1$ of symplectic diffeomorphisms of a very particular form (Hénon-like maps). For these maps, the conjugating contractions ψ_i can be made very non-uniform, allowing for an arbitrarily good approximation of every dynamics by iterations of C^r-close to identity maps for all r, see [59]. The product $f_n \circ \cdots \circ f_1$ is only an approximation of F, and it is still not known if every area-preserving dynamics can be exactly realized by iterations of C^r-close to identity maps. The main technical novelty of this paper is to show that some maps with stochastic islands can.

1.5. Stochastic sea and elliptic islands

The main motivation for the positive metric entropy conjecture is the amazing complexity of dynamics of a typical area-preserving map. Let us stress that no conservative dynamics are understood with certainty, except for those which are semi-conjugate to a rotation [2] or to an Anosov map. The reason is that hyperbolic and non-hyperbolic elements are often inseparable.

Thus, it was discovered by Newhouse [48,50] that a uniformly-hyperbolic Cantor set can be *wild*, i.e., its stable and unstable manifolds can have tangencies, and these non-transverse intersections cannot all be removed by any C^2-small perturbation of the map.

Moreover, Newhouse showed [49] that a C^r-small perturbation of an area-preserving map with a wild set creates elliptic periodic orbits which accumulate to the wild hyperbolic set.

Newhouse theory was applied and further developed by Duarte. He showed in [20] that for all a large enough the "chaotic sea" observed in the standard map (3) contains a wild hyperbolic set K, and for a Baire generic subset of this interval of a values the map has infinitely many generic elliptic periodic points, which accumulate on K. Recall that a generic elliptic point of period k for a map f is surrounded, in its arbitrarily small neighborhood, by uncountably many smooth circles (KAM-curves), invariant with respect to f^k. The map f^k restricted to such curve is smoothly conjugate to an irrational rotation (so the Lyapunov exponent is zero). The set occupied by the KAM curves has positive Lebesgue measure, and their density tends to 1 as the elliptic point is approached. An invariant curve bounds an invariant region that contains the elliptic point, such regions are called *elliptic islands*.

In [21,22], Duarte showed that small perturbations, within the class Diff^r_ω, of any area-preserving surface diffeomorphism with a *homoclinic tangency* (the tangency of the stable and unstable manifolds of a saddle periodic orbit) lead to creation of a wild hyperbolic set and to infinitely many coexisting elliptic points (and elliptic islands). In turn, it was shown in [27,45] that near any elliptic point a homoclinic tangency to some saddle periodic orbit can be created by a C^r-small perturbation.

Altogether, this gives a quite complicated picture of generic conservative dynamics: within the stochastic sea there are elliptic islands, inside elliptic islands there are small stochastic seas, etc. By [27,29], it is impossible to describe such dynamics in full detail. In fact, even most general features are presently not clear: for instance, we have no idea if the observed stochastic sea represents a transitive invariant set, or if it has a positive Lebesgue measure (though, by Gorodetski [31], it may contain uniformly-hyperbolic subsets of Hausdorff dimension arbitrarily close to 2).

The inherent inseparability of the hyperbolic and elliptic behavior even suggests the following provocative question, communicated to us by Fayad.

Question 1.2. Does an open set of area-preserving C^∞-diffeomorphisms exist with the following property: for each diffeomorphism belonging to this subset the complement to the union of the KAM curves has zero Lebesgue measure?

Maps with this property have zero metric entropy, so our Theorem A implies the negative answer to this question (a KAM curve is always a limit of non-hyperbolic periodic points). However, it is still possible that a generic (i.e., belonging to a countable intersection of open and dense subsets) non-hyperbolic map from $\mathrm{Diff}^\infty_\omega$ has zero metric entropy.

In [4], the complement U to the set of all essential invariant curves of a symplectic twist map was considered. Any connected component of U (the Birkhoff "instability zone") is bounded by two invariant topological circles C_1 and C_2. It is shown in [4], that either C_i is a heteroclinic link (a scenario as much improbable as exhibiting a stochastic island, e.g., it is impossible for entire maps), or the Lyapunov exponent of any invariant measure supported by C_i is zero ($i = 1, 2$). By the results of Furman [25], the latter alternative implies that the convergence[2] of any orbit in the instability zone to one of these curves C_i is at most sub-exponential. Thus, it is hard to see how a transitive invariant set with strictly positive maximal Lyapunov exponents can have C_i in its closure, if C_i is not a heteroclinic link.

The twist property is fulfilled near a generic elliptic point, hence the results of [4] hold true there. Therefore, it seems probable that if the Sinai conjecture is correct, then the positive metric entropy is achieved by sets distant from KAM curves. This seems to be consistent with the numerical observations [44].

1.6. Stochastic perturbation of the standard map

In higher dimension, more possibilities exist for creating examples with positive metric entropy. Thus, it was shown in [10] that, for large values of the parameter a, a skew product of the standard map over an Anosov map is non-uniformly hyperbolic and displays non-zero Lyapunov exponents for Lebesgue almost every point. Recently, Blumenthal-Xue-Young [14] used a similar argument for random perturbations of the standard map with large a and also showed the positivity of metric entropy.

1.7. Genericity results

A recent breakthrough by Irie and Asaoka [6] showed, from a cohomological argument, that for any closed surface $(\mathbb{M}, \omega)$ a generic map from $\mathrm{Diff}^\infty_\omega(\mathbb{M})$ has a dense set of periodic points. Hence, if such map is weakly-stable, then it has a dense set of hyperbolic periodic points. A natural conjecture is, then, the structural stability of weakly-stable maps from $\mathrm{Diff}^\infty_\omega(\mathbb{M})$; this would be a counterpart of the "Lambda lemma" from holomorphic dynamics [11,23,40,42].

Another natural conjecture would be that the weakly-stable maps from $\mathrm{Diff}^r_\omega(\mathbb{M})$ are uniformly hyperbolic, $1 \le r \le \infty$. For $r = 1$ this result have been proven by Newhouse [49]. For any $r \ge 2$ this question is open, as well as its dissipative counterpart – a conjecture by Mañé [41], which is also proven only for $r = 1$ [3]. Since uniformly hyperbolic maps from $\mathrm{Diff}^r_\omega(\mathbb{M})$ have positive metric entropy, this conjecture and our Theorem A would imply that maps with positive metric entropy are dense in $\mathrm{Diff}^\infty_\omega(\mathbb{M})$.

Because of the meagerness of the heteroclinic links, the genericity of positive metric entropy does not follow from our result. In fact, one can conjecture that a C^r_ω-generic surface diffeomorphism is either uniformly hyperbolic, or of zero entropy. We do not have an opinion in this regard. In the C^1-topology, this statement was a conjecture by Mañé, now proven by Bochi [15]; in higher regularity it is completely open.

[2] By the works of Birkhoff, Mather, and Le Calvez [13,38,43], there exists an orbit whose α-limit set is in C_1 and the ω-limit set is in C_2.

A milder version of this problem can be formulated as the following question due to Herman [33]:

Question 1.3. Given a surface $(\mathbb{M}, \omega)$, is there an open subset of $\mathrm{Diff}^r_\omega(\mathbb{M})$ where maps with zero metric entropy are dense?

A candidate for such dense set could be a hypothetical set of maps from Question 1.2 (the maps for which the union of all KAM curves would have full Lebesgue measure). Note that by the upper semi-continuity of the maximal Lyapunov exponent, a positive answer to Question 1.3 would also imply the local genericity of maps with zero metric entropy.

1.8. Universal dynamics

In [57], the richness of chaotic dynamics in area-preserving maps was characterized by the concept of a *universal map*. Given a C^r_ω-diffeomorphism f $(r = 1, \dots, \infty)$ of a two-dimensional surface $(\mathbb{M}, \omega)$, its behavior on ever smaller spatial scales can be described by its *renormalized iterations* defined as follows. Let Q be a C^r-diffeomorphism into $\mathbb{M}$ from some disc in $\mathbb{R}^2$. Assume that the domain of definition of Q contains the unit disc $\mathbb{D}$ and the domain of Q^{-1} in $\mathbb{M}$ contains $f^n(Q(\mathbb{D}))$ for some $n \geq 0$. We also assume that the Jacobian $\det(DQ)$ is constant in the chart (x, y) on $\mathbb{M}$ where the area-form ω is standard: $\omega = dx \wedge dy$.

Definition 1.4. The map $\mathbb{D} \to \mathbb{R}^2$ defined as

$$\hat{F}_{Q,n} = Q^{-1} \circ f^n|_{Q(\mathbb{D})} \circ Q$$

is a renormalized iteration of f.

Note that since the Jacobian of Q is constant, all renormalized iterations of f preserve the standard area-form in $\mathbb{R}^2$.

Definition 1.5 *(Universal map).* A diffeomorphism $f \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ is universal if the set of its renormalized iterations is C^∞-dense among all orientation-preserving, area-preserving diffeomorphisms $\mathbb{D} \to \mathbb{R}^2$.

By this definition, the dynamics of a single universal map approximate, with arbitrarily good precision, all symplectic maps of the unit disc.

In the general non-conservative context this notion was used in [58,59]. In C^1 category, the concept of universal dynamics was independently proposed by Bonatti and Diaz [16].

The universal dynamics might sound difficult to materialize, but it is not. It is shown in [29] that an arbitrarily small, in C^∞_ω, perturbation of any area-preserving map with a homoclinic tangency can create universal dynamics. Moreover, universal maps form a

Baire generic subset of the Newhouse domain - the open set in $\mathrm{Diff}^\infty_\omega(\mathbb{M})$ comprised of maps with wild hyperbolic sets. In [27], it was shown that a C^∞_ω-generic diffeomorphism of $\mathbb{M}$ with an elliptic point is universal.[3]

Consequently, any diffeomorphism $f \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ with an elliptic point can be perturbed in such a way that its iterations would approximate any given area-preserving dynamics and, in particular, the dynamics with positive metric entropy. We stress that this observation *is not sufficient for a proof of our Theorem A*. Indeed, if f is a C^∞_ω-diffeomorphism with an elliptic point and g is a C^∞_ω-diffeomorphism of $\mathbb{D}$ with positive metric entropy, the only thing we can conclude from [27,29] is that arbitrarily close to f in $\mathrm{Diff}^\infty_\omega(\mathbb{M})$ there exists a diffeomorphism whose iteration restricted to a certain disc is smoothly conjugate to a map G which is as close as we want to g in $\mathrm{Diff}^\infty_\omega(\mathbb{D})$. However, this map G does not need to inherit the positive metric entropy from g. Overcoming this problem is the main technical point of this paper.

2. Proof of the main theorem

We start with constructing a map with a stochastic island with certain additional properties. In section 3, we give a precise description of the construction similar to those in [5,35,54], which produces a C^∞_ω-diffeomorphism $\check F : \mathbb{D} \to \mathbb{D}$ with a stochastic island $\mathcal{I}$ bounded by four heteroclinic bi-links $\{\check L^a_i \cup \check L^b_i : 0 \le i \le 3\}$. Each $\check L^a_i \cup \check L^b_i$ is a C^∞-embedded circle included in the stable and unstable manifolds of hyperbolic fixed points $\check P_i, \check Q_i$:

$$\check L^a_i \cup \check L^b_i \subset W^u(\check P_i; \check F) \cup W^s(\check Q_i; \check f) \, .$$

The island of $\check F$ is depicted in Fig. 1. For every F which is C^1-close to $\check F$, for every $0 \le i \le 3$, the *hyperbolic continuations* of $\check P_i$ and $\check Q_i$ are the uniquely defined hyperbolic F-periodic orbits close to $\check P_i$ and $\check Q_i$.

We also show the following

Proposition 2.1. *For every conservative map F which is C^2-close to $\check F$, let P_i and Q_i be hyperbolic continuations of $\check P_i, \check Q_i$. If $\{W^u(P_i; F) \cup W^s(Q_i; F) : 0 \le i \le 3\}$ define four heteroclinic bi-links $\{L^a_i \cup L^b_i : 0 \le i \le 3\}$ which are C^2-close to $\{\check L^a_i \cup \check L^b_i : 0 \le i \le 3\}$, then they bound a stochastic island. In particular, the metric entropy of F is positive.*

The proof is given by Corollary E of Theorem D in Section 3. It follows from a new and short argument which implies also some results of [5].

We will call a stochastic island *robust relative link preservation* if it satisfies this property: for every C^r-small perturbation of the map, if the stable and unstable manifolds that form the heteroclinic link do not split, then they bound a stochastic island.

[3] The results in [27,29] were also proven in the space of real-analytic area-preserving maps.

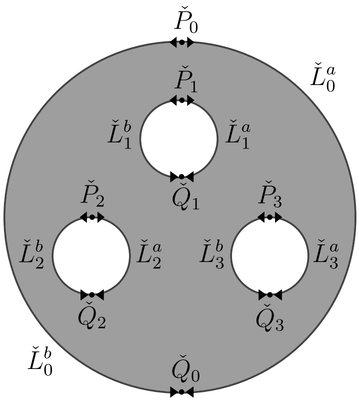

Fig. 1. Stochastic island.

Proposition 2.1 shows that the stochastic island $\check{I}$ of the map $\check{F}$ satisfies this robustness property (with any $r \geq 2$), and the same holds true for the islands bounded by the four heteroclinic bi-links $\{L_i^a \cup L_i^b : 0 \leq i \leq 3\}$ (if these links exist) of any map C^2-close to $\check{F}$.

In Section 4 (see Proposition 4.7), we construct a coordinate transformation $\mathring{\phi} \in \mathrm{Diff}_\omega^\infty(\mathbb{R}^2)$ such that $\mathring{\mathcal{I}} := \mathring{\phi}(\mathcal{I})$ is a "*suitable*" island for $\mathring{F} := \mathring{\phi} \circ \check{F} \circ \mathring{\phi}^{-1}$. The suitability conditions are described in Definition 4.6. They include the requirement that certain segments of the stable and unstable manifolds of the hyperbolic fixed points $\mathring{P}_i := \mathring{\phi}(P_i)$ and $\mathring{Q}_i = \mathring{\phi}(Q_i)$, $0 \leq i \leq 3$, are strictly horizontal, i.e., they lie in the lines $y = const$ where (x, y) are coordinates in $\mathbb{R}^2$. Moreover, the map $\mathring{F}$ near these segments has a particular form, which allows us to establish the following result in Section 5 (this is the central point of our construction):

Proposition 2.2. *Given any finite $r \geq 2$, for every $F \in \mathrm{Diff}_\omega^{r+8}(\mathbb{D})$ which is C^{r+8}-close to $\mathring{F}$, there exists a C^r-small function $\psi : \mathbb{R} \to \mathbb{R}$ such that the map $\bar{F}$ defined as*

$$\bar{F} = S_\psi \circ F \ , \quad \textit{where } S_\psi := (x, y) \mapsto (x, y + \psi(x)) \ ,$$

has the following property: For the hyperbolic continuations P_i and Q_i of the fixed points $\mathring{P}_i$ and, respectively, $\mathring{Q}_i$, the union $W^s(P_i; \bar{F}) \cup W^u(Q_i; \bar{F})$ defines a heteroclinic bi-link $L_i^a \cup L_i^b$ which is C^r-close to $\mathring{L}_i^a \cup \mathring{L}_i^b$, for each $i = 0, \ldots, 3$.

Remark 2.3. We notice that by Proposition 2.1, the map $\bar{F} = S_\psi \circ F$ has a stochastic island, robust relative link preservation, and its metric entropy is positive.

With this information, we can now complete the proof of the main theorem.

Proof of Theorem A. Let $f \in \mathrm{Diff}_\omega^\infty(\mathbb{M})$ have a non-hyperbolic periodic point. By an arbitrarily small perturbation of f one can make this point elliptic. Then, by [27], by a C_ω^∞-small perturbation of f, one can create, in a neighborhood of the elliptic point, a

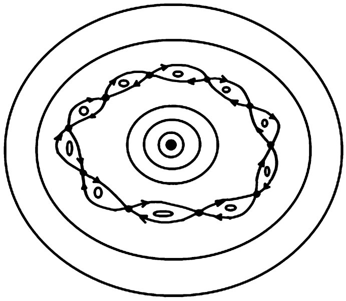

Fig. 2. Construction of a flat homoclinic tangency from [27]: Given an area-preserving map with an elliptic periodic point O, one can add a C^∞-small perturbation such that the first-return map in a small neighborhood of O will be integrable. A change of rotation number at O leads to the birth of a resonant garland where the stable and unstable manifolds of a hyperbolic q-periodic point coincide.

hyperbolic periodic cycle whose stable and unstable manifolds coincide (those define two heteroclinic links), see Fig. 2.

The important thing here is that by a small perturbation of the original map f, we create a periodic point with a flat homoclinic tangency. After that, we apply the following result, proven in Section 6 (see the proof after Corollary 6.3).

Proposition 2.4. *Let $f \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ have a hyperbolic periodic point with a flat homoclinic tangency. Then, there exists a C^∞-dense (residual) subset $\mathcal{F}$ of $\mathrm{Diff}^\infty_\omega(\mathbb{D}, \mathbb{R}^2)$ such that for every $F \in \mathcal{F}$, every $r \geq 2$, every C^r-smooth function $\psi : \mathbb{R} \to \mathbb{R}$, and every $\varepsilon > 0$, there exists a diffeomorphism $\hat{f} \in \mathrm{Diff}^r_\omega(\mathbb{M})$ such that*
- *$d_r(f, \hat{f}) < \varepsilon$, where d_r is the C^r-distance,*
- *the composition $S_\psi \circ F$ is equal to a renormalized iteration of $\hat{f}$, where $S_\psi(x, y) = (x, y + \psi(x))$.*

This statement is an enhanced version of the "rescaling lemma" (Lemma 6) of [29].[4]

This proposition gives us much freedom in varying the renormalized iteration of $\hat{f}$ without perturbing F (by composing with S_ψ for an arbitrarily functional parameter ψ).

The renormalized iterations are described by Definition 1.4. Since a renormalized iteration is C^r-conjugate to an actual iteration of the map $\hat{f}$ restricted to some small disc, it follows that by taking F and ψ exactly as in Proposition 2.2, (so that $S_\psi \circ F$ will have a stochastic island, see Remark 2.3), we will obtain that the map $\hat{f}$ has a stochastic island too, robust relative link preservation.

The stochastic island for the map $\hat{F} := S_\psi \circ F$ is bounded by 4-bi-links, each of which is equal to a C^r-embedded circle. As $S_\psi \circ F$ is smoothly conjugate to $\hat{f}^n$ for some $n > 0$, it follows that the stochastic island for the map $\hat{f}$ is bounded $m = 4n$ heteroclinic bi-links and is robust relative link preservation. We denote them by $(L^a_i \cup L^b_i)^m_{i=1}$. We prove the following result in Section 5.5:

[4] While in [29] the rescaling was done for a single round near the homoclinic tangency, here we do many rounds, similar to [59].

Proposition 2.5. *Given any map $\hat{f} \in \mathrm{Diff}^r_\omega(\mathbb{M})$ with a stochastic island $\mathcal{I}$ bounded by bi-links $(L_i^a \cup L_i^b)_{i=1}^m$ such that each bi-link $L_i^a \cup L_i^b$ is a C^r-embedded circle, arbitrarily close in C^r to $\hat{f}$ there exists a map $\hat{f}_\infty \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ for which the bi-links persist (i.e., the hyperbolic continuations of the stable and unstable manifolds forming each bi-link (L_i^a, L_i^b) comprise a heteroclinic bi-link for the map $\hat{f}_\infty$, C^r-close to (L_i^a, L_i^b)).*

The map $\hat{f}$ with the stochastic island lies in the ε-neighborhood of the original map f in $\mathrm{Diff}^r_\omega(\mathbb{M})$. Since the stochastic island of the map $\hat{f}$ is robust relative link preservation, the map $\hat{f}_\infty$ also has a stochastic island and, hence, positive metric entropy.

This shows, that arbitrarily close, in C^r for any given r, to the original map f there exists a map $\hat{f}_\infty \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ with positive metric entropy. $\square$

3. Stochastic island

In this Section we describe a particular example of a stochastic island (see Fig. 1). It is somewhat similar to the Przytycki's development of the Katok's construction, in the sense that the holes in the island are bounded by heteroclinic links. A difference with the Przytycki's example is that the heteroclinic links form smooth circles in our construction. Similar examples were considered by Aubin-Pujals [5] in the non-conservative case.

3.1. An Anosov map of the torus

Let $\mathbb{T}^2$ be the torus $\mathbb{R}^2/\mathbb{Z}^2$. Let $\mathbb{S}^1$ be the circle $\mathbb{R}/(2\pi\mathbb{Z})$. We endow $\mathbb{R}^2$ and $\mathbb{T}^2$ with the symplectic form $\omega = dx \wedge dy$.

Consider the following linear Anosov diffeomorphism of $\mathbb{T}^2$:

$$F_A : (x, y) \mapsto A(x, y) := (13x + 8y, 8x + 5y) \tag{4}$$

(F_A is the third iteration of the standard Anosov example $(x, y) \mapsto (2x + y, x + y)$). The map F_A preserves the area form ω and is uniformly hyperbolic, e.g. its Lyapunov exponents are non-zero.

The map F_A has four different fixed points $\Omega_0 = (0, 0)$, $\Omega_1 = (\frac{1}{2}, \frac{1}{2})$, $\Omega_2 = (\frac{1}{2}, -\frac{1}{2})$, $\Omega_3 = (-\frac{1}{2}, \frac{1}{2})$. Let $\sigma > 0$ be the logarithm of the unstable eigenvalue of the matrix A: $\sigma = \ln(9 + 4\sqrt{5})$. We may put the origin of coordinates to one of the points Ω_i and make a symplectic linear transformation of the coordinates (x, y) such that the map F_A near Ω_i will become

$$\Omega_i + (x, y) \mapsto \Omega_i + (e^\sigma x,\ e^{-\sigma} y)\ .$$

Observe that F_A is the time-σ map by the flow of the system

$$\dot{x} = \partial_y H_i(x, y), \qquad \dot{y} = -\partial_x H_i(x, y)$$

associated with the Hamiltonian

$$H_i = xy\ .$$

Note that the transition to polar coordinates (ρ, θ) near Ω_i by the rule

$$(x, y) = (\sqrt{2\rho}\cos(\theta), \sqrt{2\rho}\sin(\theta)) \tag{5}$$

preserves the symplectic form, i.e., we have $\omega = d\rho \wedge d\theta$. The map F_A in these coordinates is the time-σ map by the flow defined by the Hamiltonian function

$$H_i(\theta, \rho) = \rho \sin(2\theta)\ . \tag{6}$$

The corresponding Hamiltonian vector field is

$$\frac{d}{dt}\rho = 2\rho\cos(2\theta), \qquad \frac{d}{dt}\theta = -\sin(2\theta).$$

We do not need an explicit expression for the map F_A in the symplectic polar coordinates; just note that near the point Ω_i this map has the form $(\rho, \theta) \mapsto (\bar\rho, \bar\theta)$ where

$$\bar\rho = \rho\, p_0(\theta), \qquad \bar\theta = q_0(\theta), \tag{7}$$

and p_0 and q_0 are C^∞-functions $\mathbb{S}^1 \to \mathbb{R}$. Since this map preserves the symplectic form $d\rho \wedge d\theta$, its Jacobian $p_0(\theta)\partial q_0(\theta)$ equals to 1, so $p_0(\theta) \neq 0$ and $\partial q_0(\theta) \neq 0$.

3.2. Stochastic island in $\mathbb{T}^2$

In order to construct a chaotic island, we shall "blow up" the four points Ω_i. This means that we will take some small $\delta > 0$, consider the closed δ-discs V_i with the center at Ω_i for each $i = 0, 1, 2, 3$, and build a C^∞-diffeomorphism Ψ of $\mathbb{T}^2 \setminus \{\sqcup_{0\le i\le 3} V_i\}$ onto $\mathbb{T}^2 \setminus \{\cup_i \Omega_i\}$. We will do it in such a way that the map $\Psi^{-1} \circ F_A \circ \Psi$ will be smoothly extendable to a C^∞-diffeomorphism $\hat F$ of $\mathbb{T}^2$. Then the invariant set $\hat I = \mathbb{T}^2 \setminus \{\sqcup_i V_i\}$ will be a stochastic island for $\hat F$. Moreover, the points Ω_i will be flat fixed points of $\hat F$ and, importantly, $\hat F$ will inherit the symmetry with respect to $(-id)$ from the map F_A – this will be used at the next step in Section 3.4.

Let $\varepsilon > 0$ be such that the map F_A is the time-σ map of the Hamiltonian flow defined by (6) in the closed ε-disc V_i' about Ω_i, $i = 0, 1, 2, 3$; we assume that the discs V_i' are mutually disjoint. Let $0 < \delta < \varepsilon$ and let $V_i \Subset V_i'$ be the closed δ-discs about Ω_i, $i = 0, 1, 2, 3$.

Let $\psi : \left[\frac{\delta^2}{2}, \frac{\varepsilon^2}{2}\right] \to \left[0, \frac{\varepsilon^2}{2}\right]$ be a C^∞-diffeomorphism such that

$$\psi(\rho) = \rho - \frac{\delta^2}{2} \quad \text{if } \rho \text{ is close to } \frac{\delta^2}{2} \quad \text{and} \quad \psi(\rho) = \rho \quad \text{if } \rho \text{ is close to } \frac{\varepsilon^2}{2}\ . \tag{8}$$

Let $\Psi_i \in C^\infty(\mathbb{T}^2 \setminus int(V_i), \mathbb{T}^2)$ be equal to the identity outside V_i' and let the restriction of Ψ_i to the smaller disc V_i be given by

$$\Psi_i|_{V_i} : (\theta, \rho) \mapsto (\theta, \psi(\rho)) \tag{9}$$

in the polar coordinates (5). The radius-δ circle ∂V_i about Ω_i is sent by Ψ_i to Ω_i. Note that Ψ_i is a diffeomorphism from $\mathbb{T}^2 \setminus V_i$ onto $\mathbb{T}^2 \setminus \Omega_i$ and Ψ_i commutes with $(-id)$. Note also that in a neighborhood of ∂V_i the map Ψ_i preserves the form $\omega = d\rho \wedge d\theta$.

Define

$$\Psi = \Psi_i \text{ in } V_i' \setminus V_i \text{ for } i = 0, 1, 2, 3 \text{ and } \Psi = id \text{ in } \mathbb{T}^2 \setminus \{\sqcup_i V_i'\}. \tag{10}$$

This map is a C^∞-diffeomorphism from $\hat{I} = \mathbb{T}^2 \setminus \{\sqcup_i V_i\}$ onto $\mathbb{T}^2 \setminus \{\cup_i \Omega_i\}$ and it commutes with $(-id)$.

Denote $\hat{F} = \Psi^{-1} \circ F_A \circ \Psi$. By construction, this is a C^∞-diffeomorphism of $\hat{I}$, which commutes with $(-id)$. In a small neighborhood of the circles ∂V_i the map $\hat{F}$ preserves symplectic form $d\rho \wedge d\theta$ and, by (6),(8),(9), it coincides in this neighborhood with the time-σ map of the flow defined by the symplectic form $d\rho \wedge d\theta$ and the Hamiltonian

$$\hat{H}_i = (\rho - \delta^2/2)\sin(2\theta)\ .$$

We can, therefore, smoothly extend $\hat{F}$ inside V_i (i.e., to $\rho \le \delta^2/2$) as the time-σ map of the flow defined by the smoothly extended Hamiltonian $\hat{H}_i$:

$$\hat{H}_i = (\rho - \delta^2/2)\sin(2\theta)\xi(\rho)\ , \tag{11}$$

where ξ is a C^∞-function, equal to zero at all ρ close to zero and equal to 1 at all $\rho \ge \delta^2/2$.

We summarize some relevant properties of the map $\hat{F}$ in the following

Proposition 3.1. *The map $\hat{F}$ is a C^∞-diffeomorphism of $\mathbb{T}^2$ such that:*

1. *$\hat{F}|_{\mathbb{T}^2 \setminus \{\sqcup_i V_i\}}$ is conjugate to $F_A|_{\mathbb{T}^2 \setminus \{\cup_i \Omega_i\}}$ via a C^∞-diffeomorphism Ψ;*
2. *the set $\hat{I} := \mathbb{T}^2 \setminus \{\sqcup_i V_i\}$ is invariant with respect to $\hat{F}$;*
3. *$\hat{F}$ preserves a smooth symplectic form $\hat{\omega}$;*
4. *$\hat{F}$ commutes with $(-id)$, and $\hat{\omega}$ is invariant with respect to $(-id)$;*
5. *$\hat{F}$ equals to the identity in a small neighborhood of the points Ω_i;*
6. *each circle ∂V_i is a heteroclinic 4-link.*

Proof. Claim 1 is given just by construction of $\hat{F}$. Claim 2 follows from it by continuity of $\hat{F}$: since Ω_i are fixed points of F_A, each disc V_i is invariant with respect to $\hat{F}$. Claim 5 follows since $\hat{H}_i$ is constant near Ω_i, so the corresponding vector field is identically zero there.

Claim 3: By claim 1, the map $\hat{F}$ preserves the symplectic form $\hat{\omega} = \Psi^*\omega$ in $\mathbb{T}^2 \setminus \{\sqcup_i V_i\}$. Since $\Psi^*\omega = d\rho \wedge d\theta = \omega$ near ∂V_i (as it follows from (8),(9)), we can smoothly extend $\hat{\omega}$ onto the whole torus by putting $\hat{\omega} = \omega$ in $\sqcup_i V_i$. Since $\hat{F}|_{V_i}$ is the time-σ map by a Hamiltonian flow, the form $\hat{\omega} = \omega$ inside the discs V_i is preserved by $\hat{F}$.

Claim 4 follows since both the original map F_A and the conjugacy Ψ commute with $(-id)$, the Hamiltonians $\hat{H}_i$ that defines $\hat{F}$ inside the discs V_i (see (11)) are invariant with respect to $(-id) : \theta \mapsto \theta + \pi$, and the symplectic form ω is invariant with respect to $(-id)$.

In order to prove claim 6, notice that by (11) the map $\hat{F}$ near $\partial V_i : \{\rho = \delta^2/2\}$ is the time-σ map of the system

$$\frac{d}{dt}\rho = 2(\rho - \delta^2/2)\cos(2\theta), \qquad \frac{d}{dt}\theta = -\sin(2\theta). \tag{12}$$

This system has 4 saddle equilibria on the circle $\rho = \delta^2/2$: $\theta = 0, \pi/2, \pi, 3\pi/2$. These equilibria are hyperbolic fixed points of $\hat{F}$, and the invariant arcs of the circle $\rho = \delta^2/2$ between these points are formed by their stable or unstable manifolds. $\square$

3.3. The island is robust relative link preservation

The following statement establishes that the set $\hat{I}$ is a stochastic island for $\hat{F}$. It also concerns the dynamics of perturbations of $\hat{F}$. Similar results were obtained by Aubin-Pujals in [5] for the non-conservative case. The proof is given by a new and shorter argument.

Theorem D. *For the map $\hat{F}$, as well as for every, not necessarily conservative, diffeomorphism $\tilde{F}$ which is C^2-close to $\hat{F}$ and keeps the circles ∂V_i invariant ($i = 0, 1, 2, 3$), all points in $\hat{I}$ have positive maximal Lyapunov exponent. The maps $\tilde{F}$ and $\hat{F}$ are topologically conjugate on $\hat{I}$; all such maps are transitive.*

Proof. By (7),(8),(9), the map $\hat{F}$ near ∂V_i can be written as $(\rho, \theta) \mapsto (\bar{\rho}, \bar{\theta})$ where

$$\bar{\rho} = \frac{\delta^2}{2} + (\rho - \frac{\delta^2}{2})p_0(\theta), \qquad \bar{\theta} = q_0(\theta)\ .$$

A C^2-small perturbation $\tilde{F}$ of $\hat{F}$ which keeps the circle ∂V_i invariant must send $\rho = \frac{\delta^2}{2}$ to $\bar{\rho} = \frac{\delta^2}{2}$, so it has the form

$$\bar{\rho} = \frac{\delta^2}{2} + (\rho - \frac{\delta^2}{2})(p_0(\theta) + p(\theta, \rho)), \qquad \bar{\theta} = q_0(\theta) + q(\theta, \rho)\ ,$$

where the function p is C^1-small and q is C^2-small.

By reversing our surgery (10), we obtain a diffeomorphism $\tilde{F}_A = \Psi \circ \tilde{F} \circ \Psi^{-1}$ of $I_A := \mathbb{T}^2 \setminus \cup_i\{\Omega_i\}$, which takes the following form near Ω_i (see (8),(9)):

$$\bar\rho = \rho \cdot (p_0(\theta) + p(\theta, \rho + \frac{\delta^2}{2})), \qquad \bar\theta = q_0(\theta) + q(\theta, \rho + \frac{\delta^2}{2}) \,.$$

The following Lemma enables us to compare $\tilde F_A$ with F_A defined in (4).

Lemma 3.2. *In the Cartesian coordinates, the restrictions $\tilde F_A$ and F_A to $I_A = \mathbb{T}^2 \backslash \cup_i \{\Omega_i\}$ are uniformly C^1-close.*

Proof. The transformation $(\rho, \theta) \mapsto (x, y) = (\sqrt{2\rho}\cos\theta, \sqrt{2\rho}\sin\theta)$ to Cartesian coordinates near Ω_i has the following property

$$\|\partial_{(x,y)}\rho\| \le \sqrt{2\rho}, \qquad \|\partial_{(x,y)}\theta\| \le \frac{1}{\sqrt{2\rho}}. \tag{13}$$

Thus, the map $\tilde F_A$ near Ω_i takes the form $(x, y) \mapsto (\bar x, \bar y)$ where

$$\bar x = \sqrt{2\rho}\sqrt{p_0 + p}\cos(q_0 + q), \qquad \bar y = \sqrt{2\rho}\sqrt{p_0 + p}\sin(q_0 + q).$$

The uniformly-hyperbolic map F_A is given by

$$\bar x = \sqrt{2\rho}\sqrt{p_0}\cos(q_0), \qquad \bar y = \sqrt{2\rho}\sqrt{p_0}\sin(q_0)$$

(see (7)). Recall that $p_0(\theta) \ne 0$ for all θ.

It follows that, $\tilde F_A(x, y) = F_A(x, y) + \nu\sqrt{2\rho}\ \phi(\rho, \theta)$ near Ω_i where $\nu = \|(p, q)\|_{C^1} \sim \|\tilde F - \hat F\|_{C^2}$ is small and ϕ is uniformly bounded along with its first derivatives with respect to ρ and θ. By (13), this gives us that near the points Ω_i

$$\|\partial_{(x,y)}(\tilde F_A - F_A)\| = \nu \left\| \frac{\partial_{(x,y)}\rho}{\sqrt{2\rho}}\ \phi + \sqrt{2\rho}\ \partial_{(\rho,\theta)}\phi\ \partial_{(x,y)}(\rho, \theta) \right\| = O(\nu). \quad \square$$

This lemma implies the following:

Lemma 3.3. *For the map $\tilde F$, every $z \in \hat I$ displays a positive Lyapunov exponent.*

Proof. We have found that the map $\tilde F_A : I_A \to I_A$ is uniformly close in C^1 to the uniformly-hyperbolic map F_A in a neighborhood of the fixed points Ω_i (even though the derivative of $\tilde F_A$ may be not defined at the points Ω_i). Since the 4 fixed points Ω_i are the only singularities of the surgery transformation Ψ, the derivative of $\tilde F_A$ is uniformly close to the derivative of F_A everywhere on I_A, i.e., $D\tilde F_A$ is uniformly close to $(x, y) \mapsto A(x, y) = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}(x, y)$ (see (4)). In particular, $D\tilde F_A$ takes every vector with positive coordinates to a vector with positive coordinates and at least 4 times larger norm. Hence,

$$\|D\tilde F_A^n(P)\| \ge 4^n \tag{14}$$

for every point $P \in I_A$.

The map $\tilde{F} : \hat{I} \to \hat{I}$ is smoothly conjugate to $\tilde{F}_A = \Psi \circ \tilde{F} \circ \Psi^{-1}$, so

$$\|D\tilde{F}^n(\Psi^{-1}(P))\| \geq \|D\tilde{F}_A^n(P)\|/(\|D\Psi(\tilde{F}_A^n(P)\| \cdot \|D\Psi^{-1}(P)\|)$$

for every $P \in I_A$. If the point P is chosen such that the iterations $\tilde{F}^n(\Psi^{-1}(P))$ do not converge to $\cup_i \partial V_i$, then there is a sequence $n_j \to +\infty$ such that the iterations $\tilde{F}_A^{n_j}(P)$ stay away from the points Ω_i - the only singularities of the conjugacy map Ψ^{-1}. Thus, both $\|D\Psi(\tilde{F}_A^{n_j}(P)\|$ and $\|D\Psi^{-1}(P)\|$ are bounded away from zero in this case. It follows then from (14) that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D\tilde{F}^n(\Psi^{-1}(P))\| \geq \limsup_{n_j\to\infty} \frac{1}{n_j} \log \|D\tilde{F}_A^{n_j}(P)\| \geq \ln 4 \ .$$

By definition, this means that the maximal Lyapunov exponent of $\Psi^{-1}(P)$ for the map $\tilde{F}$ is strictly positive.

In the remaining case, if all iterations of the point $\Psi^{-1}(P)$ by $\tilde{F}$ converge to $\cup_i \partial V_i$, they must converge to one of the saddle fixed points that lie in $\cup_i \partial V_i$ (since every $\hat{F}$-pseudo-orbit which remains close to ∂V_i, is necessarily eventually close to one of the saddle point of V_i). In this case, the maximal Lyapunov exponent of $\Psi^{-1}(P)$ equals to the maximal Lyapunov exponent of the saddle point, i.e., it is positive. Thus, in any case, every point of $\hat{I}$ has positive maximal Lyapunov exponent for the map $\tilde{F}$. □

By the uniform hyperbolicity of F_A (see Lemma 3.2), using a variation of the Moser's technique [46] of the proof of Anosov structural stability theorem [1], we prove:

Lemma 3.4. *There exists a homeomorphism h of $\mathbb{T}^2$ which conjugates F_A and $\tilde{F}_A$, and leaves each Ω_i invariant.*

This lemma implies the topological conjugacy between $\tilde{F}|_{\hat{I}}$ and $\hat{F}|_{\hat{I}}$. The transitivity of $\tilde{F}|_{\hat{I}}$ follows from the transitivity of $F_A|_{\mathbb{T}^2\setminus\{\cup_i\Omega_i\}}$ by the conjugacy. This completes the proof of Theorem D. □

Proof of Lemma 3.4. The map F_A induces an automorphism on the Banach space Γ of bounded continuous vector fields γ vanishing at Ω_i, $i = 0, 1, 2, 3$:

$$F_A^\sharp : \gamma \mapsto DF_A \circ \gamma \circ F_A^{-1} \ .$$

The hyperbolicity of F_A (it uniformly expands in the unstable direction and uniformly contracts in the stable direction) implies that the linear operator $id - F_A^\sharp$ has a bounded inverse.[5] By the implicit function theorem, this implies that the fixed point $\gamma = 0$ of $F_A^\sharp$ is unique, and every (nonlinear) operator on Γ which is C^1-close to $F_A^\sharp$ has a unique fixed point close to $\gamma = 0$. In particular, the operator

$$\gamma \mapsto \tilde{F}_A \circ (id + \gamma) \circ F_A^{-1} - id\ ,$$

has a unique fixed point γ. By the construction, the map $h = id + \gamma$ satisfies:

$$h \circ F_A = \tilde{F}_A \circ h\ .$$

Thus, $\tilde{F}_A$ and F_A are semi-conjugate, and we have that

$$h \circ F_A^n = \tilde{F}_A^n \circ h \tag{15}$$

for every integer n, positive and negative.

In order to prove the topological conjugacy between F_A and $\tilde{F}_A$, it remains to show that the continuous map h is injective. This is done as follows: if $h(P) = h(Q)$, then $h(F_A^n P) = h(F_A^n Q)$ for all $n \in \mathbb{Z}$, by (15). Since h is uniformly close to identity, it follows that $F_A^n P$ is uniformly close to $F_A^n Q$, i.e., $A^n(P - Q)$ is uniformly small for all $n \in \mathbb{Z}$. By the hyperbolicity of the matrix A, this gives $P = Q$, as required. □

3.4. Stochastic island in the disc

It is easy to see that the result of the factorization π of the 4-punctured torus $\mathbb{T}^2 \setminus \cup_{i=0}^3 \{\Omega_i\}$ over $(-id) : (x, y) \mapsto (-x, -y)$ is a 4-punctured sphere. One can realize the smooth map $\pi : \mathbb{T}^2 \setminus \cup_i \{\Omega_i\} \to \mathbb{S}^2$ e.g. by a Weierstrass elliptic function[6]; see also [35].

Each fiber of π is a pair of points (x, y) and $-(x, y)$. Since $\hat{F}$ commutes with $(-id)$ in our construction, and $\hat{F}$ is the identity map in a neighborhood of each of the points Ω_i ($i = 0, 1, 2, 3$) where π is singular, the push-forward $\check{F} = \pi \circ \hat{F} \circ \pi^{-1}$ of $\hat{F}$ is a well-defined C^∞-diffeomorphism of the sphere $\mathbb{S}^2$. As the symplectic form $\hat{\omega}$, which is preserved by $\hat{F}$, is invariant with respect to $(-id)$, it follows that the push-forward of $\hat{\omega}$ by π is a smooth symplectic form $\check{\omega}$ on $\mathbb{S}^2 \setminus \cup_i \{\Omega_i\}$, and $\check{\omega}$ is invariant by $\check{F}$. Note that the form $\check{\omega}$ can get singular at the points $\pi\Omega_i$, but we can smoothen $\check{\omega}$ in an arbitrary way near these points; since $\check{F}$ is the identity map there, it preserves any smooth area form near $\pi\Omega_i$. Hence $\check{F}$ leaves invariant a smooth symplectic form $\check{\omega}$.

By construction, the set $\check{I} = \pi\hat{I}$ is the stochastic island for the map $\check{F}$ (the island $\hat{I}$ is at a bounded distance from the singularities Ω_i, so π^{-1} realizes a smooth conjugacy

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[5] As F_A is a linear map, it is easy to provide an explicit formula for $id - F_A^\sharp$: if $(id - F_A^\sharp)\gamma = \beta$, then $\gamma = \sum_{n=0}^\infty e^{-n\sigma} \beta_s \circ F_A^{-n} - \sum_{n=1}^\infty e^{-n\sigma} \beta_u \circ F_A^n$, where β_s and β_u are the projections of β to the stable and, respectively, unstable directions of F_A, and $e^{\pm\sigma}$ are the eigenvalues of F_A, $\sigma > 0$.

[6] or, if we realize the 4-punctured sphere as the surface $\{Z^2 + \eta(X, Y) = 1,\ |X| \le 1,\ |Y| \le 1\}$ in $\mathbb{R}^3$, where $\eta = (X^2 + Y^2 + \sqrt{1 - X^2 - Y^2 + X^4 + Y^4 - X^2Y^2})/2$, then π can be explicitly defined as $\pi(x, y) = (X = 2|x| - 1, Y = 2|y| - 1, Z = \mathrm{sign}(xy)\sqrt{1 - \eta(X, Y)}),\quad |x| \le 1/2,\ |y| \le 1/2$.

between $\check{F}|_{\check{I}}$ and $\hat{F}|_{\hat{I}}$, which takes heteroclinic links to heteroclinic links and keeps the maximal Lyapunov exponent positive). Note that the 4-links ∂V_i are invariant with respect to $(-id)$. The map π glues the opposite points of ∂V_i together, hence the circles $\pi(\partial V_i)$ that bound the island $\check{I}$ are heteroclinic bi-links.

Now we will transform the stochastic island $\check{I}$ on $\mathbb{S}^2$ to a stochastic island for a map of the plane. Let us identify $\mathbb{S}^2$ with the one-point compactification of $\mathbb{R}^2$, where $\pi\Omega_0$ is identified with ∞; this can be done e.g. by the stereographic projection $\pi_0 : (\mathbb{S}^2 \setminus \pi\Omega_0) \to \mathbb{R}^2$. As $\check{F}$ is equal to the identity at a neighborhood of Ω_0, after the projection to $\mathbb{R}^2$, the map $\check{F}$ will be a C^∞-diffeomorphism and will be equal to the identity at a neighborhood of infinity. The form $\check{\omega}$ will become a symplectic form on $\mathbb{R}^2$ and it will be preserved by $\check{F}$. Let $\check{\omega} = \beta(x,y)dx \wedge dy$ for some smooth function $\beta \neq 0$. The diffeomorphism $\pi_1 : (x,y) \mapsto (X = x, Y = \int_0^y \beta(x,s)ds)$ of $\mathbb{R}^2$ onto a domain $D \subset \mathbb{R}^2$ transforms $\check{\omega}$ to the standard symplectic form $dX \wedge dY$. The map $\check{F}$ takes D to D in the coordinates (X,Y) and is equal to identity near the boundary ∂D, so it can be extended onto the whole of $\mathbb{R}^2$ as the identity map outside of D; it will still preserve the standard form $dX \wedge dY$. By performing an additional scaling $\pi_2 : (X,Y) \mapsto (\kappa X, \kappa Y)$ we can achieve that $\check{F} = id$ everywhere outside the unit disc $\mathbb{D}$. Thus the image by $\pi_2 \circ \pi_1 \circ \pi_0$ of the stochastic island $\check{I}$ on $\mathbb{S}^2$ will lie inside $\mathbb{D}$; it is a stochastic island $\mathcal{I}$ for the map $\check{F}$ (because $\check{I}$ is separated from $\pi\Omega_0$, so the map $\pi_2 \circ \pi_1 \circ \pi_0$ is a smooth conjugacy).

We have shown the existence of a diffeomorphism $\check{F} \in \mathrm{Diff}^\infty_{dX\wedge dY}(\mathbb{D})$ with a stochastic island $\mathcal{I}$. Note that Proposition 2.1 from Section 2 is satisfied for this island, as easily follows from Theorem D:

Corollary E. *The stochastic island $\mathcal{I}$ for the map $\check{F} \in \mathrm{Diff}^\infty(\mathbb{D})$ is robust relative link preservation.*

Proof. The island $\mathcal{I}$ is bounded by four heteroclinic bi-links $\mathcal{L}_i = \pi_2 \circ \pi_1 \circ \pi_0 \circ \pi(\partial V_i)$, $i = 0,1,2,3$. For every F which is C^2-close to $\check{F}$, if F does not split the links, then it is C^2-conjugate to a C^2-diffeomorphism of $\mathbb{D}$ which is C^2-close to $\check{F}$ and keeps the links $\mathcal{L}_i$ invariant. Lifting this diffeomorphism to the torus $\mathbb{T}_2$ by $\pi^{-1} \circ \pi_0^{-1} \circ \pi_1^{-1} \circ \pi_2^{-1}$, we obtain a diffeomorphism $\tilde{F}$ of $\mathbb{T}_2 \setminus \{\cup_{i=0,1,2,3}\Omega_i\}$ which preserves the links ∂V_i and is C^2-close to $\hat{F}$ on $\hat{I}$. By Proposition D, the map $\tilde{F}$ has positive maximal Lyapunov exponent at every point of $\hat{I}$. Since the smooth conjugacy does not change the Lyapunov exponent, the map F has positive maximal Lyapunov exponent a every point of the island bounded by the continuations of the links $\mathcal{L}_i$. $\square$

4. Geometric model for a suitable stochastic island

Now, we choose a particular coordinate system in $\mathbb{R}^2$ such that the map $\check{F}$ and its stochastic island $\mathcal{I}$ we just constructed will acquire certain suitability properties (as given by Definition 4.6; see Fig. 4).

Let $F \in \mathrm{Diff}^\infty_\omega(\mathbb{R}^2)$ have a heteroclinic link L.

Definition 4.1. A *fundamental interval* of L for the map F is a closed segment $D_1 \subset L$ such that $F(D_1) \cap D_1$ is exactly one point – an endpoint both to D_1 and $F(D_1)$. Given $m \geq 1$, an *m-fundamental interval* D_m of L is the union $D_m = \cup_{i=0}^{m-1} F^i(D_1)$ of the m first iterates of a certain fundamental interval D_1.

Let (x, y) be symplectic coordinates in $\mathbb{R}^2$, so $\omega = dx \wedge dy$. Below we always fix the orientation in $\mathbb{R}^2$ such that the x-axis looks to the right and the y-axis looks up.

Definition 4.2. An m-fundamental interval D_m of a heteroclinic link L will be called *straight* if D_m is included in a straight line $y = const$.

It is a well-known fact (see [26]) that any m-fundamental interval D_m can be straightened and the so-called time-energy coordinates can be introduced in its neighborhood, i.e., the map near D_m becomes a translation to a constant vector. We formulate this result as

Lemma 4.3. *If L is a link between two hyperbolic fixed points P and Q for $F \in \mathrm{Diff}^\infty_\omega(\mathbb{R}^2)$ and $D_m \in L$ is an m-fundamental interval, then there exists a symplectic C^∞-diffeomorphism ϕ from a neighborhood of D_m into $\mathbb{R}^2$ such that $\phi(D_m)$ is a straight m-fundamental interval for $\phi \circ F \circ \phi^{-1}$ and, in a neighborhood of $\phi(D_m)$,*

$$\phi \circ F \circ \phi^{-1} =: (x, y) \mapsto (x + \tau, y)$$

for some constant $\tau \neq 0$.

Proof. Put P to the origin of coordinates and bring the map to the Birkhoff normal form by a symplectic C^∞ coordinate transformation [17, Thm. 1]. This means that we introduce symplectic coordinates (x, y) near P such that the map F near P will be given by

$$(x, y) \mapsto (\exp(q(xy)) \cdot x, \exp(-q(xy)) \cdot y) \tag{16}$$

for some function $q \in C^\infty(\mathbb{R}, \mathbb{R})$ with $q(0) > 0$. Note that the unstable manifold of P is given by $y = 0$ in these coordinates. By iterating F forward, we can extend the domain of the Birkhoff coordinates to a small neighborhood of any compact subset of $W^u(P)$. In particular, we may assume that the map F is given by (16) near the m-fundamental interval D_m. Observe that $D_m := \{(x, y) : \tilde{x} \in [x_0, e^{mq(0)} x_0],\ y = 0\}$ in these coordinates, for some $x_0 \neq 0$ (by making, if necessary, the coordinate change $(x, y) \to -(x, y)$, we can always make $x_0 > 0$).

Let $h = \int q$. Obviously, F is the time-1 map of the flow defined by the Hamiltonian $H(x, y) = h(xy)$. Put $X(x, y) := \dfrac{\ln x}{q(xy)}$ and $Y(x, y) = h(xy)$. The map $(x, y) \mapsto (X, Y)$ is a C^∞_ω-coordinate change near D_m which conjugates F with $(X, Y) \mapsto (X + 1, Y)$. $\square$

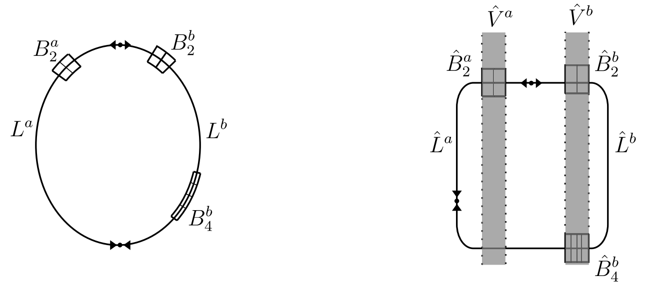

Fig. 3. Suitable intersection of a heteroclinic bi-link with two strips.

Note that the map $(x, y) \mapsto (x + \tau, y)$ is the time-τ map by the vector filed $\dot{x} = 1$, $\dot{y} = 0$ defined by the Hamiltonian $H(x, y) = y$. Therefore, x plays the role of time and the conserved quantity y can be viewed as energy, which justifies the "time-energy" terminology.

4.1. Making a bi-link suitable

A *vertical strip* V is the region $\{(x, y) : x \in [c_1, c_2], y \in \mathbb{R}\}$ in $\mathbb{R}^2$ for some $c_1 < c_2$.

Definition 4.4 *(Suitable intersection of a heteroclinic bi-link with two strips).* Two vertical strips V^a and V^b intersect a heteroclinic bi-link $(\mathring{L}^a, \mathring{L}^b)$ of a dynamics $\mathring{F}$ in a *suitable way* if:

- the intersection of V^a with $\mathring{L}^a$ is a straight 2-fundamental interval D_2^a;
- the intersection of V^b with $\mathring{L}^b$ is the disjoint union of a straight 2-fundamental interval $\mathring{D}_2^b$ and a straight 4-fundamental interval $\mathring{D}_4^b$;
- the strip V^b does not intersect $\mathring{L}^a$;
- there exists $\tau > 0$ such that the map $\mathring{F}$ in restriction to a neighborhood of $\mathring{D}_2^a$ is given by $(x, y) \mapsto (x - \tau, y)$ and, in restriction to a neighborhood of $\mathring{D}_2^b$, it is given by $(x, y) \mapsto (x + \tau, y)$;
- there exists $n \geq 1$ such that $\mathring{F}^n$ sends $\mathring{D}_2^b \cup \mathring{F}^2(\mathring{D}_2^b)$ to $\mathring{D}_4^b$, and the restriction of $\mathring{F}^n$ to a neighborhood of $\mathring{D}_2^b$ is

  $$\mathring{F}^n : (x, y) \mapsto \theta - (\frac{1}{2}x, 2y)$$

  for some $\theta \in \mathbb{R}^2$.

See Fig. 3 for an illustration.

In Lemma 4.5 below, we are going to show that any conservative diffeomorphism F with a bi-link (L^a, L^b) is smoothly conjugate to a conservative diffeomorphism $\mathring{F}$ with a suitable bi-link $(\mathring{L}^a, \mathring{L}^b)$. This means that we have a lot of geometric freedom in the choice of the bi-link $(\mathring{L}^a, \mathring{L}^b)$, that we shall explain.

Take any two parallel straight lines in $\mathbb{R}^2$: $\{y = y_1\}$ and $\{y = y_2\}$ with $y_2 < y_1$. Let $\tau > 0$ and $x_a < x_b$ so that $x_a + \tau < x_b - \tau$. Consider the two vertical strips:

$$V^a := [x_a - \tau, x_a + \tau] \times \mathbb{R} \quad \text{and} \quad V^b := [x_b - \tau, x_b + \tau] \times \mathbb{R}.$$

Consider any C^∞-smooth circle $\mathring{L}$ in $\mathbb{R}^2$ equal to the union of two curves $\mathring{L}^a$, $\mathring{L}^b$ satisfying:

- $\mathring{L} = \mathring{L}^a \cup \mathring{L}^b$ and $\mathring{L}^a \cap \mathring{L}^b = \partial \mathring{L}^a = \partial \mathring{L}^b$.
- $\mathring{L}^a \cap V^a = [x_a - \tau, x_a + \tau] \times \{y_1\}$ and $\mathring{L}^a \cap V^b = \varnothing$.
- $\mathring{L}^b \cap V^b = [x_b - \tau, x_b + \tau] \times \{y_1, y_2\}$.

Now we can state:

Lemma 4.5. *Let $F \in \mathrm{Diff}^\infty_\omega(\mathbb{R}^2)$ have a heteroclinic bi-link (L^a, L^b). Then there exists a symplectic C^∞-diffeomorphism $\mathring{\phi}$ of a small neighborhood of $L^a \cup L^b$ into $\mathbb{R}^2$ such that:*

- $\mathring{\phi}(L^a \cup L^b) = \mathring{L}$, $\mathring{\phi}(L^a) = \mathring{L}^a$, $\mathring{\phi}(L^b) = \mathring{L}^b$;
- *the vertical strips V^a and V^b intersect the bi-link $(\mathring{L}^a, \mathring{L}^b)$ of $\mathring{\phi} \circ F \circ \mathring{\phi}^{-1}$ in a suitable way (with $n = 4$ in Definition 4.4).*

Proof. Let us take fundamental intervals D^a of L^a and D^b of L^b. Observe that:

- $D^a_2 := D^a \cup F(D^a)$ and $D^b_2 := D^b \cup F(D^b)$ are 2-fundamental intervals of L^a and L^b, respectively;
- $D^b_4 := \cup_{i=4}^7 F^i(D^b)$ is a 4-fundamental interval of L^b.

We remark that D^b_2 and D^b_4 are included in the 8-fundamental interval $D^b_8 := \cup_{i=0}^7 F^i(D^b)$ of L^b.

By Lemma 4.3, there exist symplectic diffeomorphisms ϕ^a and ϕ^b acting from a neighborhood of $\hat{D}^a_2 := D^a_2 \cup F(D^a_2)$ and, respectively, a neighborhood of $\hat{D}^b_8 = D^b_8 \cup F(D^b_8)$ into $\mathbb{R}^2$ such that:

- $\phi^a(D^a_2) = [-2, 0] \times \{0\}$ and $\phi^a \circ F \circ (\phi^a)^{-1}$ is the translation to $(-1, 0)$ in a neighborhood of $\phi^a(D^a_2)$;
- $\phi^b(D^b_8) = [0, 8] \times \{0\}$ and $\phi^b \circ f \circ (\phi^b)^{-1}$ is the translation to $(1, 0)$ in a neighborhood of $\phi^b(D^b_8)$.

Observe that $\phi^b(D^b_2) = [0, 2] \times \{0\}$ and $\phi^b(D^b_4) = [4, 8] \times \{0\}$.

Let $\delta > 0$ be small and denote $J := [-\delta\tau, \delta\tau]$ and $J' := [-\frac{\delta\tau}{2}, \frac{\delta\tau}{2}]$. Let

$$B^a_2 := (\phi^a)^{-1}([-2, 0] \times J), \qquad B^b_2 := (\phi^b)^{-1}([0, 2] \times J), \quad B^b_4 := (\phi^b)^{-1}([4, 8] \times J') .$$

For $\delta > 0$ small enough, the sets B_2^b, B_4^b and B_2^a are disjoint. Take two linear area-preserving maps:

$$A_2 := (x, y) \mapsto (\tau x, \frac{y}{\tau}) \quad \text{and} \quad A_4 := (x, y) \mapsto -\,(\frac{\tau}{2}x, \frac{2}{\tau}y)\,.$$

Notice that the maps

$$\mathring{\phi}_2^a := A_2 \circ \phi^a, \qquad \mathring{\phi}_2^b := A_2 \circ \phi^b, \qquad \mathring{\phi}_4^a := A_4 \circ \phi^b$$

send, respectively, B_2^a, B_2^b and B_4^b onto translations of $R := [-\tau, \tau] \times [-\delta, \delta]$.

Let $\mathring{\phi}_0$ be a symplectic embedding of a neighborhood of the disjoint union $B_0 := B_2^a \cup B_2^b \cup B_4^b$ into $\mathbb{R}^2$ such that:

- B_2^a, B_2^b and B_4^b are sent by $\mathring{\phi}_0$ onto, respectively, (see Fig. 3):
$$\mathring{B}_2^a := R + (x_a, y_1), \quad \mathring{B}_2^b := R + (x_b, y_1), \quad \mathring{B}_4^b := R + (x_b, y_2);$$
- the restriction of $\mathring{\phi}_0$ to neighborhoods of B_2^a, B_2^b and B_4^b is the composition of respectively $\mathring{\phi}_2^a$, $\mathring{\phi}_2^b$ and $\mathring{\phi}_4^b$ with some translations.

Note that $\mathring{\phi}_0$ sends the fundamental intervals D_2^a, D_2^b and D_4^b onto, respectively,:

$$\mathring{D}_2^a := [-\tau, \tau] \times \{0\} + (x_a, y_1), \quad \mathring{D}_2^b := [-\tau, \tau] \times \{0\} + (x_b, y_1),$$
$$\mathring{D}_4^b := [-\tau, \tau] \times \{0\} + (x_b, y_2),$$

so these images lie in the curve $\mathring{L}$, in the intersection with the vertical strips V^a and V_b.

The map $\mathring{\phi}_0 \circ F^4 \circ \mathring{\phi}_0^{-1}$ sends $\mathring{D}_2^b$ into $\mathring{D}_4^b$ and its restriction to a neighborhood of $\mathring{D}_2^b$ is the composition of a translation with the linear map $(x, y) \mapsto (-x/2, -2y)$, as required by Definition 4.4 with $n = 4$. Therefore, to prove the lemma, it suffices to construct a symplectic C^∞-diffeomorphism $\mathring{\phi}$ of a small neighborhood of $B_0 \cup L^a \cup L^b$ into $\mathbb{R}^2$ which would send L^a, L^b to $\mathring{L}^a, \mathring{L}^b$, such that its restriction to a neighborhood of B_0 would be equal to $\mathring{\phi}_0$.

Without the symplecticity requirement, the map $\mathring{\phi}$ would be given by Whitney extension theorem [62]. Making the diffeomorphism $\mathring{\phi}$ symplectic requires an extra effort, as it is done below.

Consider an annulus $\mathbb{A} := (\mathbb{R}/\mathbb{Z}) \times [-\eta, \eta]$ for a sufficiently small $\eta > 0$. Let $t \in \mathbb{R}/\mathbb{Z}$ and $h \in [-\eta, \eta]$ be coordinates in $\mathbb{A}$. By the Weinstein's Lagrangian neighborhood theorem [61], if η is sufficiently small, then there exists an area-preserving diffeomorphism N from the annulus $\mathbb{A}$ to a neighborhood of the bi-link $L^a \cup L^b$, which sends the central circle $S := \{h = 0\}$ to $L^a \cup L^b$. Similarly, there exists an area-preserving diffeomorphism $\mathring{N}$ from $\mathbb{A}$ to a small neighborhood of the curve $\mathring{L} = \mathring{L}^a \cup \mathring{L}^b$, which sends S to $\mathring{L}$.

Let $\delta > 0$ be small enough, so that the sets B_0 and $\mathring{\phi}_0(B_0)$ will be contained in $N(\mathbb{A})$ and, respectively, $\mathring{N}(\mathbb{A})$. By Whitney extension theorem, there exists $G \in \mathrm{Diff}^\infty(\mathbb{A}, \mathbb{A})$

such that $G(S) = S$ and $\mathring{N} \circ G \circ N^{-1}$ restricted to a neighborhood U of B_0 is $\mathring{\phi}_0$. In particular, $\det DG|_{N^{-1}(U)} = 1$. The map G is orientation-preserving but it is not, a priori, area-preserving outside of U.

Our goal is to correct G in order to make it area-preserving. More precisely, we are going to construct a C^∞-diffeomorphism Ψ of $\mathbb{A}$ such that $\det D\Psi = \det DG$, $\Psi(S) = S$, and the restriction of Ψ to $N^{-1}(B_0)$ is the identity. Then the Lemma will be proved by taking $\mathring{\phi} := \mathring{N} \circ G \circ \Psi^{-1} \circ N^{-1}$.

Let us keep fixed the neighborhood U of B_0 where G is area-preserving, and let us take $\delta > 0$ small. Then B_0 can be made as close as we want to $D_2^b \cup D_4^b \cup D_2^a$. Hence, for $\delta > 0$ small enough, if the image by N of a vertical segment $\{t = const,\ |h| \leq \eta\}$ intersects B_0, then it lies entirely in U, i.e., $\det DG = 1$ everywhere on this segment. Therefore, if we define the map

$$\Psi : (t, h) \mapsto (t, \int_0^h \det DG(t, s) ds),$$

then $\Psi = id$ in the restriction to $N^{-1}(B_0)$. It is also obvious, that $\Psi = id$ in restriction to the central circle $S = \{h = 0\}$, and $\det D\Psi = \det DG$. $\square$

4.2. Making the stochastic island suitable

Consider the map $\check{F} \in \mathrm{Diff}^\infty_\omega$ with the stochastic island $\mathcal{I}$. Recall that $\mathcal{I}$ is bounded by 4 heteroclinic bi-links (L_i^a, L_i^b), $i = 0, 1, 2, 3$, each of which is a C^∞-smooth circle. We take a convention that $L_0^a \cup L_0^b$ is the outer circle, i.e., the bi-links (L_i^a, L_i^b) with $i = 1, 2, 3$ lie inside the region bounded by $L_0^a \cup L_0^b$.

Below, we will construct symplectic coordinates $\mathring{\phi}$ in $\mathbb{R}^2$ such that the island $\mathring{\phi}(\mathcal{I})$ will satisfy the following suitability conditions.

Definition 4.6 *(Suitable intersection of 4 heteroclinic bi-link-s with 4 pairs of vertical strips).* We say that 4 pairs of vertical strips (V_i^a, V_i^b), $i = 0, 1, 2, 3$, intersect bi-links $(\mathring{L}_j^a, \mathring{L}_j^b)$, $j = 0, 1, 2, 3$ in a suitable way if the following conditions hold true.

- (H_1) For every $0 \leq i \leq 3$, the intersection of $V_i^a \sqcup V_i^b$ with $(\mathring{L}_i^a, \mathring{L}_i^b)$ is suitable in the sense of Definition 4.4, with $n = 4$.
- (H_2) For every $j \geq 1$ and every $i \neq j$, the strips V_i^a and V_i^b do not intersect the circle $\mathring{L}_j^a \cup \mathring{L}_j^b$

See Fig. 4 for an illustration.

Proposition 4.7. *There exist $\mathring{\phi} \in \mathrm{Diff}^\infty_\omega(\mathbb{R}^2)$ and 4 pairs of vertical strips V_i^a, V_i^b, $i = 0, 1, 2, 3$, which intersect the heteroclinic bi-links $(\mathring{L}_j^a = \mathring{\phi}(L_j^a), \mathring{L}_j^b = \mathring{\phi}(L_j^b))$, $j = 0, 1, 2, 3$, of the map $\mathring{F} = \mathring{\phi} \circ \check{F} \circ \mathring{\phi}^{-1}$ in a suitable way.*

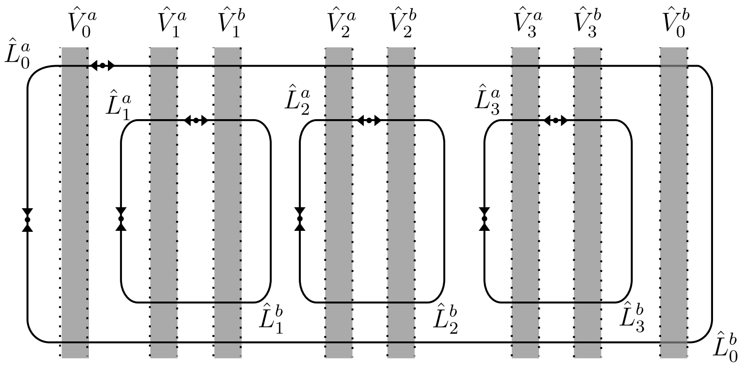

Fig. 4. Suitable intersection of the stochastic island and 8 vertical strips.

Proof. By Lemma 4.5, for every $i = 0, 1, 2, 3$ there exists a pair of vertical strips V_i^a, V_i^b and a symplectic C^∞ diffeomorphism $\mathring{\phi}_i$ of a small neighborhood of the bi-link (L_i^a, L_i^b) such that the strips V_i^a, V_i^b intersect $(\mathring{L}_i^a, \mathring{L}_i^b) := (\mathring{\phi}_i(L_i^a), \mathring{\phi}_i(L_i^b))$ of the map $\mathring{F}_i := \mathring{\phi}_i \circ \check{F} \circ \mathring{\phi}_i^{-1}$ in a suitable way, ensuring the fulfillment of Condition (H_1) of Definition 4.6.

Note that in Lemma 4.5 there is a freedom in the choice of the curve $\mathring{L}_i = \mathring{\phi}_i(L_i^a \cup L_i^b)$. So, we take $\mathring{L}_i$ such that it will bound a disc of the same area as $L_i = L_i^a \cup L_i^b$ does. Also, by choosing the constants y_1, y_2, x_0, x_1 in Lemma 4.5 in an appropriate way for each i, we can assure that $\mathring{\phi}_i(L_i)$ do not intersect for different i and $\mathring{\phi}_1(L_1) \cup \mathring{\phi}(L_2) \cup \mathring{\phi}(L_3)$ lies inside the disc bounded by $\mathring{\phi}_0(L_0)$, and the strips V_i^a and V_i^b are positioned where we wish, thus ensuring Condition (H_2) of Definition 4.6.

Let $\mathbb{A}_i$ be a sufficiently small closed annulus around L_i, $i = 0, 1, 2, 3$. Let us prove the proposition by showing the existence of a symplectic extension $\mathring{\phi}$ of the symplectic maps $\mathring{\phi}_i$ from a neighborhood of the annuli $\mathbb{A}_i$ to the whole of $\mathbb{R}^2$, i.e., a diffeomorphism $\mathring{\phi} \in \mathrm{Diff}^\infty_\omega$ such that $\mathring{\phi}|_{\mathbb{A}_i} = \mathring{\phi}_i|_{\mathbb{A}_i}$ for all $i = 0, 1, 2, 3$. Since the curve $\mathring{L}_i$ bounds the disc of the same area as $L_i = L_i^a \cup L_i^b$ does, for each i, the annuli $\mathbb{A}_i$ is necessarily such that the area of each of the connected components of $\mathbb{R}^2 \setminus \sqcup_i \mathbb{A}_i$ equals to the area of a corresponding component of $\mathbb{R}^2 \setminus \sqcup_i \mathring{\phi}_i(\mathbb{A}_i)$. Then the existence of the sought symplectic extension $\mathring{\phi}$ is a standard consequence of Dacorogna-Moser theorem [19], as given by

Corollary 4.8 *(Cor. 4 [7]). Let $K \subset \mathbb{R}^2$ be a compact set, U be a neighborhood of K and let $\psi \in C^\infty_\omega(U, \mathbb{R}^2)$ be close to the identity. Assume that every bounded connected component W of $\mathbb{R}^2 \setminus U$ and its corresponding one in $\mathbb{R}^2 \setminus \psi(U)$ have the same area. Then there exists $\phi \in Diff^\infty_\omega(\mathbb{R}^2)$ which is C^∞-close to the identity and such that $\phi|K = \psi|K$.* □

5. Restoration of broken heteroclinic links

Let $\mathring{F} \in Diff^\infty_\omega(\mathbb{R}^2)$ be the map constructed in the previous Section. It has a stochastic island $\mathring{\mathcal{I}}$ bounded by 4 smooth circles - heteroclinic bi-links $(\mathring{L}_i^a, \mathring{L}_i^b)$, $i = 0, 1, 2, 3$. The bi-link $\mathring{L}_0^a \cup \mathring{L}_0^b$ forms the outer boundary of $\mathring{\mathcal{I}}$. By construction, there exist 4 pairs of vertical strips (V_i^a, V_i^b) which intersect $(\mathring{L}_i^a, \mathring{L}_i^b)$ in a suitable way in the sense of Definition 4.6. We denote $V_i^a = I_i^a \times \mathbb{R}$ and $V_i^b = I_i^b \times \mathbb{R}$, where I_i^a, I_i^b are closed disjoint intervals in the x-axis.

In this Section we consider perturbations of the map $\mathring{F}$ and prove Proposition 2.2. Namely, we show that *for every $r \geq 1$, for every $\eta > 0$, for every $F \in \mathrm{Diff}_\omega^{r+8}$ which is sufficiently close to $\mathring{F}$ in C^{r+8}, there exists $\psi \in C^r(\mathbb{R}, \mathbb{R})$, supported in $\cup_i(I_i^a \cup I_i^b)$ and with C^r-norm smaller than η, such that the map $\bar{F} = S_\psi \circ F$ has 4 heteroclinic bi-links (L_i^a, L_i^b) close to $(\mathring{L}_i^a, \mathring{L}_i^b)$*. We recall that given a function ψ, we denote

$$S_\psi : (x, y) \mapsto (x, y + \psi(x)) \ .$$

It is pretty much obvious that to prove this statement, it suffices to show

Proposition 5.1. *Let $r \geq 1$. Let $\mathring{F} \in \mathrm{Diff}_\omega^{r+4}$ have a heteroclinic bi-link $(\mathring{L}^a, \mathring{L}^b)$ which intersects two vertical strips $V^a = I^a \times \mathbb{R}$ and $V^b = I^b \times \mathbb{R}$ in a suitable way. For every $F \in \mathrm{Diff}_\omega^{r+4}$ which is C^{r+4}-close to $\mathring{F}$, there exists $\psi \in C^r(\mathbb{R}, \mathbb{R})$ which is C^r-small and supported in $I^a \sqcup I^b$, such that the map $S_\psi \circ F$ has a bi-link (L^a, L^b) close to $(\mathring{L}^a, \mathring{L}^b)$.*

Proposition 2.2 is inferred from this statement as follows.

Proof of Proposition 2.2. Let F be a C^{r+8}-small perturbation of $\mathring{F}$. By Proposition 5.1, for each $i = 1, 2, 3$ there exists a C^{r+4}-small function ψ_i supported in $I_i^a \sqcup I_i^b$ such that the map $S_{\psi_i} \circ \mathring{F}$ has a bi-link $(\bar{L}_i^a, \bar{L}_i^b)$ close to $(\mathring{L}_i^a, \mathring{L}_i^b)$. By property (H_2) of the suitable intersection (see Definition 4.6), the vertical strips V_i^a and V_i^b do not intersect the bi-links $(\mathring{L}_j^a, \mathring{L}_j^b)$ for $j \neq i$, $j > 0$. Thus the map S_{ψ_i} is identity near the bi-links $(\mathring{L}_j^a, \mathring{L}_j^b)$ with $j \neq i$, $j > 0$. Therefore, the map $S_{\psi_1+\psi_2+\psi_3} \circ F$ has 3 bi-links (L_i^a, L_i^b) close to $(\mathring{L}_i^a, \mathring{L}_i^b)$, respectively.

The map $S_{\psi_1+\psi_2+\psi_3} \circ F$ is an ω-preserving diffeomorphism and is C^{r+4}-close to $\mathring{F}$. Therefore, by applying Proposition 5.1 to this map and the link $(\mathring{L}_0^a, \mathring{L}_0^b)$, we obtain that there exists a C^r-small function ψ_0 localized in $I_0^a \cup I_0^b$ such that the map $S_{\psi_0} \circ S_{\psi_1+\psi_2+\psi_3} \circ F$ has a bi-link (L_0^a, L_0^b) close to $(\mathring{L}_0^a, \mathring{L}_0^b)$. Since V_0^a and V_0^b do not intersect the bi-links (L_i^a, L_i^b) for $i > 0$ (by property (H_2) of the suitable intersection), the map S_{ψ_0} is identity near these bi-links, hence it does not destroy them. Thus, the map $\bar{F} = S_\psi \circ F$ with $\psi = \psi_0 + \psi_1 + \psi_2 + \psi_3$ has all 4 bi-links (L_i^a, L_i^b) as required. $\square$

Proof of Proposition 5.1. This Proposition follows from the two lemmas below which we prove in Sections 5.3 and 5.4 respectively.

Lemma 5.2. *Under the hypotheses of Proposition 5.1, for every $F \in \mathrm{Diff}_\omega^k$ which is C^k-close to $\mathring{F}$, $k \geq 3$, there exists a C^{k-2}-small function ψ_a supported in I^a and such that the map $S_{\psi_a} \circ F$ has a link L^a close to $\mathring{L}^a$.*

Lemma 5.3. *Under the hypotheses of Proposition 5.1, for every $F \in \mathrm{Diff}^k_\omega$ which is C^k-close ($k \geq 3$) to $\mathring{F}$ and has a link L^a close to $\mathring{L}^a$, there exists a C^{k-2}-small function ψ^b supported in I^b and such that the map $S_{\psi_b} \circ F$ has a link L^b close to $\mathring{L}^b$.*

Indeed, if an ω-preserving diffeomorphism F is a C^{r+4}-small perturbation of F, then, by Lemma 5.2, there exists a C^{r+2}-small ψ_a such that the map $S_{\psi_a} \circ F$ has the link L_a. This map is ω-preserving and is C^{r+2}-close to $\mathring{F}$. Therefore, applying Lemma 5.3 to this map, we find that there exists a C^r-small ψ_b supported in I^b and such that the map $S_{\psi_b} \circ S_{\psi_a} \circ F = S_{\psi_a+\psi_b} \circ F$ has the link L_b. As the strip $I^b \times \mathbb{R}$ does not intersect L^a, the link L^a also persists for the map $S_{\psi_a+\psi_b} \circ \hat{F}$, which gives Proposition 5.1 with $\psi = \psi_a + \psi_b$. $\square$

5.1. Evaluation of the link splitting

The map $\mathring{F}$ has two saddle fixed points P and Q on the circle $\mathring{L}^a \cup \mathring{L}^b$. The point P is repelling on the circle, while Q is attracting on the circle.

Let $W^a(P;\mathring{F})$ and $W^b(P;\mathring{F})$ denote the halves of the unstable manifolds of P equal to, respectively, $\mathring{L}^a \setminus \{Q\}$ and $\mathring{L}^b \setminus \{Q\}$. Let $W^a(Q;\mathring{F})$ and $W^b(Q;\mathring{F})$ be the halves of the stable manifolds of Q equal to respectively $\mathring{L}^a \setminus \{P\}$ and $\mathring{L}^b \setminus \{P\}$.

The points P and Q persist for every C^1-close map F, and depend continuously on F. The corresponding manifolds $W^a(P;F)$, $W^b(P;F)$, $W^a(Q;F)$ and $W^b(Q;F)$ also persist, and depend continuously on F as embedded curves of the same smoothness as F. To avoid ambiguities, we will fix a sufficiently small neighborhood of the point Q and then $W^a(P;F)$ and $W^b(P;F)$ will denote the two arcs of $W^u(P,F)$ which connect P with the boundary of this neighborhood and are close, respectively, to $W^a(P;\mathring{F})$ and $W^b(P;\mathring{F})$. Similarly, $W^a(Q;F)$ and $W^b(Q;F)$ are the arcs in $W^s(Q,F)$ which connect Q with the boundary of a small neighborhood of P and are close, respectively, to $W^a(Q;\mathring{F})$ and $W^b(Q;\mathring{F})$.

In general, the links are broken when the map $\mathring{F}$ is perturbed, so $W^a(P;F)$ and $W^b(P;F)$ do not need to coincide with, respectively, $W^a(Q;F)$ and $W^b(Q;F)$.

In the next two Sections we will show, for a given $F \in \mathrm{Diff}^\omega_k(\mathbb{D})$ which is C^k-close to $\mathring{F}$, how to find a C^{k-2}-function ψ such that each of the unions $W^a(P;S_\psi\circ F) \cup W^a(Q;S_\psi\circ F)$ and $W^b(P;S_\psi \circ F) \cup W^b(Q;S_\psi \circ F)$ forms a heteroclinic link between P and Q.

In this Section, we obtain formulas for the defect of coincidence between $W^a(P;S_\psi\circ F)$ and $W^a(Q;S_\psi \circ F)$ or $W^b(P;S_\psi \circ F)$ and $W^b(Q;S_\psi \circ F)$. In order to do that, we shall use the so-called time-energy coordinates near the fundamental interval $\mathring{D}^a_2 = V_a \cap L_a$ of the link L_a and the fundamental domain $\mathring{D}^b_2 \subset V_b \cap L_b$ of the link L_b. Recall that by the suitability conditions (see Definition 4.4) there exist $\tau > 0$ and $(x_a, y_a) \in \mathbb{D}$, $(x_b, y_b) \in \mathbb{D}$ such that $\mathring{D}^a_2 = \{x \in [x_a - 2\tau, x_a]\} \times \{y = y_a\}$, $\mathring{D}^b_2 = \{x \in [x_b, x_b + 2\tau]\} \times \{y = y_b\}$, and the map $\mathring{F}$ restricted to a small neighborhood N^a of $\mathring{D}^a_2$ or a small neighborhood N^b of $\mathring{D}^b_2$ is given by

$$\mathring{F}|_{N^a} := (x,y) \mapsto (x-\tau, y), \qquad \mathring{F}|_{N^b} := (x,y) \mapsto (x+\tau, y)\ . \tag{17}$$

Definition 5.4. For an ω-preserving map F, which is C^k-close to $\mathring{F}$, an N^a time-energy chart ϕ^a is an ω-preserving diffeomorphism from $N^a \cup F(N^a)$ to $\mathbb{D}$ which is C^{k-1}-close to identity and satisfies

$$\phi^a \circ F|_{N^a} = \mathring{F} \circ \phi^a|_{N^a}\ . \tag{18}$$

An N^b time-energy chart ϕ^b is an ω-preserving diffeomorphism from $N^b \cup F(N^b)$ to $\mathbb{D}$ which is C^{k-1}-close to identity and satisfies

$$\phi^b \circ F|_{N^b} = \mathring{F} \circ \phi^b|_{N^b}\ . \tag{19}$$

We notice that the identity map is a time-energy chart for $\mathring{F}$. The time-energy charts are not uniquely defined, so we will fix their choice below. In our construction the time-energy charts will be identity near $\{x = x_a\}$ and $\{x = x_b\}$.

Once certain time-energy coordinates are introduced in $N^a \cup F(N^a)$, the curves $W^a(P; F) \cap \{N^a \cup F(N^a)\}$ and $W^a(Q; F) \cap \{N^a \cup F(N^a)\}$ become graphs of τ-periodic functions: the manifolds $W^a(P; F)$ and $W^a(Q; F)$ are invariant with respect to F which means that in the time-energy coordinates they are invariant with respect to the translation to $(-\tau, 0)$, see (18),(17). We denote as $w^u_a(F, \phi^a)$ and $w^s_a(F, \phi^a)$ the τ-periodic functions whose graphs are the curves $\phi^a(W^a(P; F))$ and $\phi^a(W^a(Q; F))$, respectively.

Definition 5.5. The *link-splitting function* $M^a(F, \phi^a)$ associated to (N^a, F, ϕ^a) is the τ-periodic function equal to $w^u_a(F, \phi^a) - w^s_a(F, \phi^a)$ at $x \in [x_a - \tau, x_a]$.

Similarly, let $w^s_b(F, \phi^b)$ and $w^u_b(F, \phi^b)$ be the τ-periodic functions whose graphs are the curves $\phi^b(W^b(Q; F)$ and $\phi^b(W^b(P; F))$.

Definition 5.6. The *link-splitting function* $M^b(F, \phi^b)$ associated to (N^b, F, ϕ^b) is the τ-periodic function equal to $w^u_b(F, \phi^b) - w^s_b(F, \phi^b)$ at $x \in [x_b, x_b + \tau]$.

By the definition, the link L^a or L^b is restored when the function M^a or, respectively, M^b is identically zero.

We start with constructing a C^{k-1}-smooth time-energy chart for the map F.

Lemma 5.7. *There exists a small neighborhood N^a of $\mathring{D}^a_2$ and a small neighborhood N^b of $\mathring{D}^b_2$ such that for every ω-preserving diffeomorphism F which is C^k-close to $\mathring{F}$, $k \geq 3$, there exists C^{k-1}-smooth time-energy chart ϕ^a and ϕ^b, which depend continuously on F and equal to identity if $F = \mathring{F}$.*

Proof. We will show the proof only for the existence of ϕ^a. The proof for ϕ^b is identical up to the exchange of index a to b and $(-\tau)$ to τ.

Let $\rho \in C^\infty(\mathbb{R}, [0,1])$ be zero everywhere near $x = x_a$ and 1 everywhere near $x = x_a - \tau$. Let $\phi_0(x,y) := (x,y)(1-\rho(x)) + \rho(x)\mathring{F} \circ F^{-1}(x,y)$. The map ϕ_0 is a C^k-diffeomorphism from a small neighborhood of $\{x \in [x_a - \tau, x_a],\ y = y_a\}$ into $\mathbb{D}$, it is C^k-close to identity and equals to the identity near (x_a, y_a) and to $\mathring{F} \circ F^{-1}$ near $(x_a - \tau, y_a)$.

Thus, ϕ_0 satisfies

$$\phi_0 \circ F \circ \phi_0^{-1}(x,y) = (x - \tau, y) \ .$$

in a neighborhood of (x_a, y_a) (see (17)). Take a small neighborhood of D^a and define there $\phi^a(x,y) = \phi_0(x, \sigma(x,y))$ where the C^{k-1}-function σ satisfies $\sigma(x, y_a) = y_a$ and $\partial_y \sigma = \det D\phi_0^{-1}(x,\sigma)$. By construction, $\det D\phi^a \equiv 1$, i.e., it is an ω-preserving C^{k-1}-diffeomorphism and, since ϕ_0 is C^k-close to the identity, ϕ^a is C^{k-1}-close to the identity. Since $\det D\phi_0 = 1$ everywhere near (x_a, y_a) and $(x_a - \tau, y_a)$, we have that $\sigma \equiv y$ near these points, so $\phi^a \equiv \phi_0$ there. In particular,

$$\phi^a \circ F = \mathring{F} \circ \phi^a$$

near (x_a, y_a).

It follows that we obtain the required time-energy chart if we extend ϕ^a to a small neighborhood of $\mathring{D}_2^a \cup \mathring{F}\mathring{D}_2^a$ by the rule

$$\phi^a =: \begin{cases} \mathring{F} \circ \phi^a \circ F^{-1} & \text{if } x \in [x_a - 2\tau, x_a - \tau], \\ \mathring{F}^2 \circ \phi^a \circ F^{-2} & \text{if } x \leq x_a - 2\tau \ . \end{cases} \qquad \square$$

Now, take some sufficiently small $\delta > 0$. Consider any map F close enough to $\mathring{F}$ and let $\phi^{a,b}$ be the C^{k-1} time-energy charts for F, defined in Lemma 5.7. Given any close to zero smooth function $\psi(x)$ supported inside $[x_a - 2\tau + \delta, x_a - \delta]$, we consider the map

$$\bar{F} := S_\psi \circ F \tag{20}$$

and define for it the time-energy chart ϕ^a_ψ in the open set $N^a \cup F(N^a)$ such that

$$\phi^a_\psi = \begin{cases} \phi^a \circ F \circ \bar{F}^{-1} = \phi^a \circ S_{-\psi} & \text{if } x \geq x_a - \tau, \\ \phi^a \circ F^2 \circ \bar{F}^{-2} & \text{if } x \leq x_a - \tau. \end{cases} \tag{21}$$

Recall that ψ vanishes for x close to x_a and for x close to $x_a - 2\tau$. Furthermore, if x is close to $x_a - \tau$, then the x-coordinate of $\bar{F}^{-1}(x,y)$ is close to x_a. Thus, on a neighborhood of $\bar{F}^{-1}(x,y)$, it holds $F \circ \bar{F}^{-1} = id$ and so

$$\phi^a \circ F^2 \circ \bar{F}^{-2}(x,y) = \phi^a \circ F \circ \bar{F}^{-1}(x,y) = \phi^a \circ S_{-\psi}(x,y) \ .$$

Hence ϕ^a_ψ has no discontinuities at $x = x_a - \tau$ and the following required conjugacy holds true:

$$\phi^a_\psi \circ \bar{F}|_{N^a} = \mathring{F} \circ \phi^a_\psi|_{N^a} \ . \tag{22}$$

Similarly, for any close to zero smooth function $\psi(x)$ which is supported inside $[x_b + \delta, x_b + 2\tau - \delta]$, for the map $\bar{F}$ given by (20), we define the time-energy chart ϕ^b_ψ in $N^b \cup F(N^b)$ by the rule

$$\phi^b_\psi = \begin{cases} \phi^b \circ F \circ \bar{F}^{-1} = \phi^b \circ S_{-\psi} & \text{if } x \le x_b + \tau, \\ \phi^b \circ F^2 \circ \bar{F}^{-2} & \text{if } x \ge x_b + \tau, \end{cases} \tag{23}$$

such that the identity

$$\phi^b_\psi \circ \bar{F}|_{N^b} = \mathring{F} \circ \phi^b_\psi|_{N^b} \tag{24}$$

holds.

With this choice of the time-energy charts, for small ψ we have the link-splitting function $M^a(S_\psi \circ F, \phi^a_\psi)$ (if ψ is supported inside $[x_a - 2\tau, x_a]$) or $M^b(S_\psi \circ F, \phi^b_\psi)$ (if ψ is supported inside $[x_b, x_b + 2\tau]$). This defines the operators $\mathcal{M}^a : \psi \to M^a(S_\psi \circ F, \phi^a_\psi)$ and $\mathcal{M}^b : \psi \to M^b(S_\psi \circ F, \phi^b_\psi)$ acting from the space of smooth functions supported inside $[x_a - 2\tau, x_a]$ or, respectively, inside $[x_b, x_b + 2\tau]$, to the space of τ-periodic functions of the same smoothness.

5.2. Regularity of the graph transform operator

The regularity of the operators $\mathcal{M}^a$ and $\mathcal{M}^b$ depends on the smoothness class of ψ. We choose it to be C^{k-2}. For $\delta > 0$ small, for every $x \in \mathbb{R}$ we denote by $C^{k-2}_0([x - 2\tau + \delta, x - \delta], \mathbb{R})$ the Banach space of real C^{k-2}-functions supported inside $[x - 2\tau + \delta, x - \delta]$, endowed with the C^{k-2}-norm. We have the crucial result:

Proposition 5.8. *The operators*

$$\mathcal{M}^a : \psi \in C^{k-2}_0([x_a - 2\tau + \delta, x_a - \delta], \mathbb{R}) \mapsto M^a(S_\psi \circ F, \phi^a_\psi) \in C^{k-2}(\mathbb{R})$$
$$\mathcal{M}^b : \psi \in C^{k-2}_0([x_b + \delta, x_a + 2\tau - \delta], \mathbb{R}) \mapsto M^b(S_\psi \circ F, \phi^b_\psi) \in C^{k-2}(\mathbb{R})$$

are of class C^1 in a small neighborhood of zero, and depend continuously on the map F.

Proposition 5.8 follows immediately from the two following lemmas:

Lemma 5.9. *The functions $w^u_a(S_\psi \circ F, \phi^a_\psi)$ and $w^u_b(S_\psi \circ F, \phi^b_\psi)$ are independent of ψ, and depend continuously on F:*

$$w^u_a(S_\psi \circ F, \phi^a_\psi) = w^u_a(F, \phi^b) \quad \text{and} \quad w^u_b(S_\psi \circ F, \phi^b_\psi) = w^u_b(F, \phi^b) \ .$$

Lemma 5.10. *The following operators are of class C^1 and depends continuously on F:*

$$\psi \in C_0^{k-2}([x_a - 2\tau + \delta, x_a - \delta], \mathbb{R}) \mapsto w_a^s(S_\psi \circ F, \phi_\psi^a) \in C^{k-2}(\mathbb{R}) ,$$
$$\psi \in C_0^{k-2}([x_b + \delta, x_b + 2\tau - \delta], \mathbb{R}) \mapsto w_b^s(S_\psi \circ F, \phi_\psi^b) \in C^{k-2}(\mathbb{R}) .$$

Lemma 5.10 was rather unexpected: the proof works because S_ψ induces a graph transform which is a translation (which is a smooth operator).

Proof of Lemma 5.9. Let ψ be supported inside $[x_a - 2\tau + \delta, x_a - \delta]$. Denote $\bar{F} := S_\psi \circ F$. The graph of the function w_a^u is the curve $\phi_\psi^a(W^a(P; \bar{F})) \cap [x_a - \tau, x_a] \times \mathbb{R}$. It follows from our choice of the chart ϕ_ψ^a (see the first line of (21)) that this curve is the image by $\phi^a \circ F$ of an arc of the curve $\ell_a^u = \bar{F}^{-1}(W^a(P; \bar{F}) \cap [x_a - \tau, x_a] \times \mathbb{R})$, which is an arc of $W^a(P; \bar{F})$ lying at $x \geq x_a$. The set $W^a(P; \bar{F}) \cap \{x \geq x_a\}$ is a part of the unstable manifold of P which depends only on the dynamics at $\{x \geq x_a\}$. Since ψ is zero at $x \geq x_a$, the map S_ψ is identity there, hence $\bar{F}|_{\{x \geq x_a\}}$ equals $F|_{\{x \geq x_a\}}$, and $W^a(P; \bar{F}) \cap \{x \geq x_a\}$ equals $W^a(P; F) \cap \{x \geq x_a\}$, in particular the curve ℓ_a^u does not depend on ψ, i.e. it is the same for F and $\bar{F}$.

Thus,

$$\begin{aligned}\phi_\psi^a(W^a(P; \bar{F}))|_{x \in [x_a - \tau, x_a]} &= (\phi^a \circ S_{-\psi}) \circ (S_\psi \circ F)(\ell_a^u) = \phi^a \circ F(\ell_a^u) \\ &= \phi^a(W^a(P; F))|_{x \in [x_a - \tau, x_a]} ,\end{aligned}$$

so $w_a^u(\bar{F}, \phi_\psi^a) = w_a^u(F, \phi^a)$ is the same for all small ψ.

Exactly in the same way, just by changing the index a to b and the interval $[x_a - \tau, x_a]$ to $[x_b, x_b + \tau]$, we obtain that when ψ is supported inside $[x_b + \delta, x_b + 2\tau - \delta]$ the function $w_b^u(\bar{F}, \phi_\psi^b)$ is independent of ψ. $\square$

Proof of Lemma 5.10. For $c \in \{a, b\}$, let us show that $w_c^s(\bar{F}, \phi_\psi^c)$ is a C^1 function of $\psi \in C^{k-2}(\mathbb{R})$, with $\bar{F} := S_\psi \circ F$. Again, we start with the case $c = a$ and ψ supported inside $[x_a - 2\tau + \delta, x_a - \delta]$ and derive the expression for $w_a^s(\bar{F}, \phi_\psi^a)$ in this case.

The graph of $w_a^s(\bar{F}, \phi_\psi^a)$ is the curve $\phi_\psi^a \bar{\ell}_s^a$ where $\bar{\ell}_s^a$ is the piece of $W^a(Q; \bar{F})$ in the intersection with $(\phi_\psi^a)^{-1}([x_a - \tau, x_a] \times \mathbb{R})$. The curve $W^s(Q; \bar{F})$ (a halve of the stable manifold of Q) is obtained by iterations of its small, adjoining to Q part by the map $\bar{F}^{-1}$. The maps $\bar{F}^{-1}$ and F^{-1} coincide at $x \leq x_a - 2\tau + \delta$, and both are close to the map $\mathring{F}^{-1}$ which takes the line $\{x = x_a - 2\tau + \delta\} \cap N^a$ into $x = x_a - \tau + \delta$, so the piece of $W^s(Q; \bar{F})$ between Q and $x = x_a - \tau + \delta/2$ does not move as ψ varies, i.e., it is the same as for the map F.

This means that for all small ψ, the image by $\bar{F}$ of the curve $\bar{\ell}_a^s$ lies inside the piece of $W^s(Q; F)$ to the left of $x = x_a - \tau + \delta/2$, i.e., $\bar{F}\bar{\ell}_a^s \subset F\ell_a^s$, where the ψ-independent curve ℓ_a^s is a piece of $W^s(Q; F)$ in the intersection with $\{x \in [x_a - \tau - \delta/2, x_a + \delta/2]\}$. So

$$\phi^a_\psi(W^a(Q;\bar F))\cap\{x\in[x_a-\tau,x_a]\}\subset \phi^a_\psi\circ\bar F^{-1}\circ F\ \ell^s_a=\phi^a\circ S_{-\psi}\circ F^{-1}\circ S_{-\psi}\circ F\ \ell^s_a\ , \quad (25)$$

as given by (21),(20).

Formula (25) states that the graph of $w^s_a(\bar F,\phi^a_\psi)$ is the image of the ψ-independent curve ℓ^s_a by the map $\phi^a\circ S_{-\psi}\circ F^{-1}\circ S_{-\psi}\circ F$. Thus, the regularity of the map $\psi\mapsto w^s_a(\bar F,\phi^a_\psi)$ is determined by the regularity of the corresponding *graph transform* operators.

The following defines the *graph-transform operator* $\mathcal{F}^\#$ associated with a smooth map $\mathcal{F}$.

Fact 5.11 *(See Thm 2.2.5. P. 145 [32]).* Consider a curve $\mathcal{L}=\{(x,y): y=w(x)\}$ in an (x,y)-plane, where w is a C^s-smooth function defined on some closed interval, and a C^n-smooth map $\mathcal{F}:(x,y)\mapsto(p(x,y),q(x,y))$ $(n\geq s)$ defined in a neighborhood U of $\mathcal{L}$. Then, under the condition $\partial_x p(x,y)+\partial_y p(x,y)Dw(x)\neq 0$ everywhere in U, the image $\mathcal{F}\mathcal{L}$ is a curve of the form $y=\tilde w(x)$ where $\tilde w\in C^s$. Moreover the operator $\mathcal{F}^\#$ which takes the C^s-function w to the C^s-function $\tilde w$ is of regularity class C^{n-s}.

Since $\phi^a(\ell^s_a)$ is the graph of $w^s_a(F,\phi^a)$, we have from (25) that

$$w^s_a(\bar F,\phi^a_\psi)=(\phi^a)^\#\circ(S_{-\psi})^\#\circ(F^{-1})^\#\circ(S_{-\psi})^\#\circ(F\circ(\phi^a)^{-1})^\#\ w^s_a(F,\phi^a)\ . \quad (26)$$

Since ϕ^a and F are at least of class C^{k-1}, the graph-transform operators $(\phi^a)^\#$, $(F^{-1})^\#$, and $(F\circ(\phi^a)^{-1})^\#$ have regularity at least C^1 when act from C^{k-2}-smooth functions to C^{k-2}-smooth functions. We cannot use the same fact for the graph transform operator induced by $S_{-\psi}$ since the latter map is only of class C^{k-2}. However, the map $S_{-\psi}$ is given by $(x,y)\mapsto(x,y-\psi(x))$, so the associated graph transform operator $(S_{-\psi})^\#$ sends a function w to $w-\psi$. Thus, it is linear in both w in ψ, i.e., it is of class C^∞ with respect to both w and ψ (irrespective of their class of smoothness).

Altogether, this implies that the map $\psi\mapsto w^s_a(\bar F,\phi^a_\psi)$ given by (26) is of class C^1.

Now, let us handle the case $c=b$.

We need to derive the expression for $w^s_b(\bar F,\phi^b_\psi)$. The graph of this function is the curve $\phi^b_\psi\bar\ell^s_b$ where $\bar\ell^s_b$ is the piece of $W^b(Q;\bar F)$ in the intersection with $(\phi^b_\psi)^{-1}(\{x\in[x_b,x_b+\tau]\})$. The curve $W^b(Q;\bar F)$ is close to $W^b(Q;\mathring F)$. Recall that $W^b(Q,\mathring F)$ coincides with $W^b(P,\mathring F)$ and forms a heteroclinic link $\mathring L_b$. It intersects the vertical strip $V_b=\{x\in[x_b,x_b+2\tau]\}$ twice, along two straight line segments $\mathring D^b_2=[x_b,x_b+2\tau]\times\{y=y_b\}$ and $\mathring D^b_4=\mathring F^4(\mathring D^b_2)\cup\mathring F^6(\mathring D^b_2)=[x_b,x_b+2\tau]\times\{y=y'_b\}$ for some $y'_b<y_b$. The piece of $W^b(Q,\mathring F)$ between Q and the left end of $\mathring D^b_4$ lies entirely in the region $x\leq x_b$, i.e., to the left of the vertical strip V_b. Also, the map $\mathring F$ in a small neighborhood of $\mathring D^b_4$ is given by

$$\mathring F:(x,y)\mapsto(x-\tau/2,y) \quad (27)$$

(as implied by the link suitability Definition 4.4). It follows that the curve $W^b(Q,\bar F)$ intersects the straight line $x=x_b$ at a point Z_b with the y-coordinate close to y_b such that the piece of $W^b(Q,\bar F)$ between Q and $\bar F^7 Z_b$ lies entirely in the region $x < x_b + \tau/2 + \delta/2$. Since ψ is supported inside $[x_b+\delta, x_b+2\tau-\delta]$, it follows that $F = \bar F$ near this piece of $W^b(Q,\bar F)$. Therefore, the piece of $W^b(Q,\bar F)$ between Q and $x = x_b + \tau/2 + \delta/2$ is unmoved as ψ varies, so it coincides with the corresponding piece of $W^b(Q,F)$.

This piece contains the point $\bar F^7 Z_b$, hence it contains the curve $\bar F^7 \bar\ell^b_s$ for all small ψ. It follows that there exists a ψ-independent curve ℓ^s_b so that $\bar F^7 \bar\ell^s_b \subset F^7 \ell^s_b$ and ℓ^s_b is a piece of $W^s(Q;F)$ in the intersection with $\{x \in [x_b - \delta/2, x_b + \tau + \delta/2]\}$. So

$$\phi^b_\psi(W^b(Q;\bar F))|_{x\in[x_b,x_b+\tau]} \subset \phi^b_\psi \circ \bar F^{-7} \circ F^7\ \ell^s_b = \phi^b \circ S_{-\psi} \circ (\bar F^{-1} \circ S_{-\psi})^7 \circ F^7\ \ell^s_b\ , \quad (28)$$

see (23),(20).

Since $\phi^b(\ell^s_b)$ is the graph of $w^s_b(F,\phi^b)$, we have from (28) that

$$w^s_b(\bar F,\phi^b_\psi) = (\phi^b)^\# \circ (S_{-\psi})^\# \circ ((F^{-1})^\# \circ (S_{-\psi})^\#)^7 \circ (F^7 \circ (\phi^b)^{-1})^\#\ w^s_b(F,\phi^b)\ . \quad (29)$$

Like we did it for the function w^s_a given by (26), we obtain that the map $\psi \mapsto w^s_b(\bar F,\phi^b_\psi)$ defined by (29) is of class C^1. □

The continuity of the operators $\mathcal{M}^a$ and $\mathcal{M}^b$ with respect to F allows us to obtain enough information about them by computing them for $F = \mathring{F}$, which we do in the two following lemmas.

Lemma 5.12. *When $F = \mathring{F}$:*

$$\mathcal{M}^a(\psi)(x) = \psi(x) + \psi(x-\tau) \qquad \text{for } x \in [x_a - \tau, x_a]. \quad (30)$$

Proof. For $F = \mathring{F}$, the link L^a exists and the fundamental interval D^a is straight, so the curves $W^a(P;\mathring{F})$ and $W^a(Q;\mathring{F})$ coincide for $x \in [x_a - \tau, x_a]$ and lie in the straight line $y = y_a$. The map ϕ^a for $\mathring{F}$ is identity, so we have $w^u_a(\mathring{F},\phi^a) = w^s_a(\mathring{F},\phi^a) = y_a$. The map $\mathring{F}$ is the translation to $(-\tau, 0)$ (see (17)). Plugging this information into (26) gives (30) immediately. □

Lemma 5.13. *When $F = \mathring{F}$,*

$$\begin{aligned}\mathcal{M}^b(\psi)(x) = \psi(x) + \psi(x+\tau) - \ &\tfrac{1}{2}\left(\psi(\tfrac{3x_b+\tau-x}{2}) + \psi(\tfrac{3x_b+2\tau-x}{2}) + \psi(\tfrac{3x_b+3\tau-x}{2})\right.\\ &\left.+ \psi(\tfrac{3x_b+4\tau-x}{2})\right)\end{aligned} \quad (31)$$

for $x \in [x_b, x_b+\tau]$.

Proof. When $F = \mathring{F}$, we have $\phi^b = id$, so

$$\mathcal{M}^b : \psi \mapsto w^u_b(\mathring{F}, id) - (S_{-\psi})^\# \circ ((\mathring{F}^{-1})^\# \circ (S_{-\psi})^\#)^7 \circ (\mathring{F}^7)^\#\ w^s_b(\mathring{F}, id) \quad (32)$$

(see (29)). Recall that $W^b(Q,\mathring{F})$ coincides with $W^b(P,\mathring{F})$ and intersects the vertical strip V_b on two straight line segments, $\mathring{D}_2^b=[x_b,x_b+2\tau]\times\{y=y_b\}$ and $\mathring{D}_4^b=[x_b,x_b+2\tau]\times\{y=y_b'\}$, so

$$w_b^u(\mathring{F},id)=w_b^s(\mathring{F},id)\equiv y_b \quad \text{at } x\in[x_b,x_b+\tau]. \tag{33}$$

Also, the map $\mathring{F}^7$ from a small neighborhood of $[x_b,x_b+\tau]\times\{y=y_b\}$ to a small neighborhood of $[x_b,x_b+\tau]\times\{y=y_b'\}$ acts as

$$\mathring{F}^7:(x,y)\mapsto(-\frac{x}{2}+\frac{3x_b}{2}+\frac{\tau}{2},-2y+2y_b+y_b')$$

(see Definition 4.4). Therefore,

$$(\mathring{F}^7)^{\#}\ w_b^s(\mathring{F},id)\equiv y_b' \quad \text{at } x\in[x_b,x_b+\tau/2]\ ,$$

and

$$w_7(x):=(S_{-\psi})^{\#}(\mathring{F}^7)^{\#}\ w_b^s(\mathring{F},id)(x)=y_b'-\psi(x) \quad \text{at } x\in[x_b,x_b+\tau/2]\ .$$

Next, by (27), we obtain

$$w_6(x):=(S_{-\psi})^{\#}\circ(\mathring{F}^{-1})^{\#}w_7(x)=y_b'-\psi(x-\tau/2)-\psi(x) \quad \text{at } x\in[x_b+\tau/2,x_b+\tau]\ .$$

Repeating the same procedure two more times, we obtain

$$w_4(x):=((S_{-\psi})^{\#}\circ(\mathring{F}^{-1})^{\#})^3w_7(x)=y_b'-\psi(x-\frac{3}{2}\tau)-\psi(x-\tau)-\psi(x-\frac{\tau}{2})-\psi(x) \text{ at } x\in[x_b+\frac{3}{2}\tau,x_b+2\tau]\ .$$

The map $\mathring{F}^{-1}$ takes the rectangle $K:=[x_b+3\tau/2,x_b+2\tau]\times[y_b'-\varepsilon,y_b'+\varepsilon]$ (for some small ε) to the right of the strip V_b, and its next image $\mathring{F}^{-2}K$ also lies to the right of V_b. We recall that the function ψ vanishes there and $S_{-\psi}=id$. Therefore, $F^{-1}\circ(S_{-\psi}\circ F^{-1})^2=F^{-3}$ on K. As the curve $\{y=w_4(x)\}$ lies in K for small ψ, this implies that

$$(F^{-1}\circ(S_{-\psi}\circ F^{-1})^2)^{\#}w_4=(F^{-3})^{\#}w_4\ .$$

By the link suitability conditions (Definition 4.4), the map F^{-3} takes K into a small neighborhood of $\mathring{D}_2^b$ and is given by

$$\mathring{F}^{-3}:(x,y)\mapsto(-2x+3x_b+5\tau,-\frac{y}{2}+\frac{y_b'}{2}+y_b).$$

This gives

$$((S_{-\psi}\circ F^{-1})^6)^{\#}w_7(x) = y_b + \tfrac{1}{2}(\psi(\tfrac{3x_b+2\tau-x}{2}) + \psi(\tfrac{3x_b+3\tau-x}{2}) + \psi(\tfrac{3x_b+4\tau-x}{2}) \\ +\psi(\tfrac{3x_b+5\tau-x}{2})) - \psi(x) \ \text{ at } x\in[x_b+\tau, x_b+2\tau]\ .$$

Since $\mathring{F}^{-1}$ is the translation to $(-\tau,0)$ near $\mathring{F}(D_2^b)$, we finally obtain that

$$((S_{-\psi}\circ F^{-1})^7)^{\#}w_7(x) = y_b + \tfrac{1}{2}(\psi(\tfrac{3x_b+\tau-x}{2}) + \psi(\tfrac{3x_b+2\tau-x}{2}) + \psi(\tfrac{3x_b+3\tau-x}{2}) \\ +\psi(\tfrac{3x_b+4\tau-x}{2})) - \psi(x+\tau) - \psi(x) \quad \text{at } x\in[x_b+\tau, x_b+2\tau]\ ,$$

which implies (31) by (32),(33). $\square$

5.3. Proof of Lemma 5.2: restoring the link L^a

In order to prove Lemma 5.2, we show that for every ω-preserving F which is C^k-close to $\mathring{F}$ there exists a C^{k-2}-smooth function ψ, supported in $x\in(x_a-2\tau, x_a)$, such that

$$M^a(S_\psi\circ F, \phi^a_\psi) \equiv 0\ , \tag{34}$$

where the C^{k-2}-smooth time-energy chart ϕ^a_ψ for the map $S_\psi\circ F$ is defined by (21). We will solve this equation for ψ by reducing it to a fixed point problem for a certain contracting operator.

First, we restrict the class of perturbation functions ψ; namely, we will take them in the form

$$\psi(x) = \rho(x)\tilde\psi(x)\ ,$$

where $\tilde\psi$ is a τ-periodic function, and $\rho\in C^\infty(\mathbb{R},[0,1])$ has support in $[x_a-2\tau, x_a]$ and satisfies $\rho(x)+\rho(x-\tau)=1$ for every $x\in[x_a-\tau, x_a]$. Then the operator

$$\mathcal{M}^a_\rho : \tilde\psi \mapsto \mathcal{M}^a(\rho\tilde\psi)\ ,$$

which provides a correspondence between $\tilde\psi$ and the link-splitting function for the map $S_{\rho\tilde\psi}\circ F$, takes a small ball around zero in the space $C^{k-2}(\mathbb{R}/(\tau\mathbb{Z}),\mathbb{R})$ of τ-periodic functions into the same space.

By Lemma 5.8, the operator $\mathcal{M}^a_\rho$ is of class C^1 on $C^{k-2}(\mathbb{R}/(\tau\mathbb{Z}),\mathbb{R})$. For $F=\mathring{F}$, we have $\mathcal{M}^a_\rho = id$. Indeed, by (30),

$$\mathcal{M}^a_\rho(\tilde\psi)(x) = \rho(x)\tilde\psi(x) + \rho(x-\tau)\tilde\psi(x-\tau) = [\rho(x)+\rho(x-\tau)]\tilde\psi(x) = \tilde\psi(x)\ .$$

Due to a continuous dependence on F, the operator

$$\tilde\psi \mapsto \tilde\psi - \mathcal{M}^a_\rho(\tilde\psi)$$

is a contraction in a neighborhood of zero in $C^{k-2}(\mathbb{R}/(\tau\mathbb{Z}),\mathbb{R})$ for all $F\in \mathrm{Diff}^k_\omega(\mathbb{D})$ which are C^k-close to $\mathring{F}$. Therefore, it has a unique fixed point $\tilde\psi$ near zero in $C^{k-2}(\mathbb{R}/(\tau\mathbb{Z}),\mathbb{R})$, for each such F.

The operator $\mathcal{M}_\rho^a$ vanishes at this fixed point. By construction, the corresponding function $\psi = \rho\tilde{\psi}$ solves equation (34), i.e., the link-splitting function is identically zero for the map $\bar{F} = S_\psi \circ F$, meaning that the link L^a persists for this map. Lemma 5.2 is proven. $\square$

5.4. Proof of Lemma 5.3: restoring the link L^b

In order to prove Lemma 5.3, we show that for every ω-preserving F which is C^k-close to $\mathring{F}$, if the link L_a persists for the map F, then there exists a C^{k-2}-smooth function ψ, supported in $x \in (x_b, x_b + 2\tau)$, such that

$$M^b(S_\psi \circ F, \phi_\psi^b) \equiv 0 \,, \tag{35}$$

where the C^{k-2} time-energy chart ϕ_ψ^b for the map $\bar{F} = S_\psi \circ F$ is defined by (23).

In order to resolve equation (35), we will use the following property of the link-splitting function M^b.

Lemma 5.14. *If the link L^a is persistent for the map F, then the link-splitting function $M^b(S_\psi \circ F, \phi_\psi^b)$ has zero mean for every ψ supported by $(x_b, x_b + 2\tau)$:*

$$\int\limits_{x_b}^{x_b+\tau} M(\bar{F}, \phi_\psi^b)dx = 0 \,.$$

Proof. We may always assume that the map ϕ_ψ^b is the restriction to N_a of an area-preserving diffeomorphism or $\mathbb{R}^2$ (the possibility of the symplectic extension of an area-preserving diffeomorphism from a disc to the whole $\mathbb{R}^2$ is a standard fact; see e.g. Corollary 4 in [7]).

Let the points $Z^u \in W^b(P, \bar{F}) \cap N^b$ and $Z^s \in W^b(Q, \bar{F}) \cap N^b$ have the same x-coordinate x_b, i.e., $Z^u = (x_b, w^u(\bar{F}, \phi_\psi^b)(x_b))$, $Z^s = (x_b, w^s(\bar{F}, \phi_\psi^b)(x_b))$. As the map $\bar{F}$ is the translation to $(\tau, 0)$ in the time-energy coordinates in N^b, the points $\bar{F}Z^s$ and $\bar{F}Z^u$ have the same x-coordinate $x_b + \tau$, and the image of the vertical segment connecting Z^s and Z^u is the vertical segment connecting $\bar{F}Z^s$ and $\bar{F}Z^u$.

Since the support of ψ lies in $(x_b, x_b + 2\tau)$, the maps F and $\bar{F}$ coincide in a neighborhood of L_a, so the link L_a is not split for all ψ under consideration. Thus, we may consider a region $\mathcal{D}$ bounded by the link L^a, the piece of $W^b(P, \bar{F})$ between Z^u and P, the piece of $W^b(Q, \bar{F})$ between Q and Z^s, and the vertical segment that connects Z^s and Z^u. The region $\bar{F}\mathcal{D}$ has the same area as $\mathcal{D}$. It is bounded by the link L^a, the piece of $W^b(P, \bar{F})$ between $\bar{F}Z^u$ and P, the piece of $W^b(Q, \bar{F})$ between Q and $\bar{F}Z^s$, and the vertical segment that connects $\bar{F}Z^s$ and $\bar{F}Z^u$. The equality of the areas means that the area of the region between the curves $W^b(P, \bar{F}) : \{y = w^u(\bar{F}, \phi_\psi^b)(x)\}$ and $W^b(Q, \bar{F}) : \{y = w^s(\bar{F}, \phi_\psi^b)(x)\}$ at $x \in [x_b, x_b + \tau]$ is zero (see Fig. 5). This means that

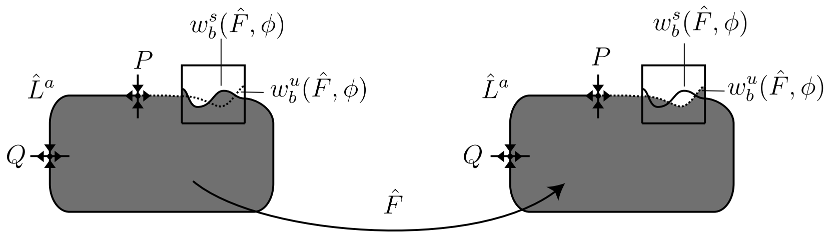

Fig. 5. The integral of the function $M(\bar F,\phi) = w_b^u(\bar F,\phi) - w_b^s(\bar F,\phi)$ is null.

$\int_{x_b}^{x_b+\tau}(w^u(\bar F,\phi_\psi^b) - w^s(\bar F,\phi_\psi^b))dx = 0$, which proves the lemma (see Definition 5.6 of the link-splitting function). $\square$

Now, like in the previous Section, we restrict the class of perturbation functions ψ:

$$\psi(x) = \rho(x)\tilde\psi(x) ,$$

where $\tilde\psi$ is a τ-periodic function with the zero mean, and $\rho \in C^\infty(\mathbb{R},[0,1])$ has support in $[x_b+\delta, x_b+2\tau-\delta]$ for $\delta > 0$ small, and satisfies $\rho(x)+\rho(x+\tau) = 1$ for every $x \in [x_b, x_b+\tau]$. Then, if the link L_a persists for the map F, the operator

$$\mathcal{M}_\rho^b : \tilde\psi \mapsto \mathcal{M}^b(\rho\tilde\psi) ,$$

takes a small ball around zero in the space $C_0^{k-2}(\mathbb{R}/(\tau\mathbb{Z}),\mathbb{R})$ of τ-periodic functions with the zero mean into the same space (by Lemma 5.14).

By Lemma 5.8, the operator $\mathcal{M}_\rho^b$ is of class C^1 on $C_0^{k-2}(\mathbb{R}/(\tau\mathbb{Z}),\mathbb{R})$. It follows easily from (31) that if $F = \mathring{F}$, then

$$\mathcal{M}_\rho^b(\tilde\psi)(x) = \tilde\psi(x) - \frac{1}{2}\,(\tilde\psi(\frac{3x_b+\tau-x}{2}) + \tilde\psi(\frac{3x_b+2\tau-x}{2})). \tag{36}$$

Note that the space $C_0^{k-2}(\mathbb{R}/(\tau\mathbb{Z}),\mathbb{R})$ is a Banach space when endowed with the following norm:

$$\|\psi\|_{C_0^{k-2}} = \max_{1\le i\le k-2} \|D^i\psi_1 - D^i\phi_2\|_{C^0} . \tag{37}$$

This norm includes evaluation of the derivatives only (still it is a well-defined norm - if two functions with the zero mean have the same derivative, they coincide).

By (31) and by the continuous dependence of $\mathcal{M}^b$ on F (see Lemma 5.8), the operator $id - \mathcal{M}_\rho^b$ is C^1-close to the linear operator $\tilde\psi \mapsto \bar\psi$ where

$$\bar\psi(x) = \frac{1}{2}\,(\tilde\psi(\frac{3x_b+\tau-x}{2}) + \tilde\psi(\frac{3x_b+2\tau-x}{2})).$$

We have

$$D^i\bar{\psi}(x) = \frac{(-1)^i}{2^{i+1}} \left(D^i\tilde{\psi}(\frac{3x_b+\tau-x}{2}) + D^i\tilde{\psi}(\frac{3x_b+2\tau-x}{2})\right),$$

so this operator is, obviously, a contraction in the norm (37). Therefore, $id - \mathcal{M}^b_\rho$ is a contraction on $C^{k-2}_0(\mathbb{R}/(\tau\mathbb{Z}), \mathbb{R})$ for all F which are C^k-close to $\mathring{F}$, provided F is area-preserving and the link L_a persists for F.

Thus, $id - \mathcal{M}^b_\rho$ has a fixed point $\tilde{\psi}$. The corresponding function $\psi = \rho\tilde{\psi}$ solves equation (35); Lemma 5.3 is proven. □

5.5. *Proof of Proposition 2.5*

The previous results allow for construction of C^r-maps with stochastic islands for any finite r. Below we prove Proposition 2.5 which deals with the C^∞ case. Let $\hat{f} \in \mathrm{Diff}^r_\omega(\mathbb{M})$ have a stochastic island $\mathcal{I}$ bounded by bi-links $(L^a_i \cup L^b_i)^m_{i=1}$ so that each bi-link $C_i := L^a_i \cup L^b_i$ is a C^r-smooth circle (without break points). Let us show that arbitrarily close in C^r to $\hat{f}$ there exists a map $\hat{f}_\infty \in \mathrm{Diff}^\infty_\omega(\mathbb{M})$ for which the bi-links persist.

Choose a map $f \in \mathrm{Diff}^\infty_\omega(\mathbb{R}^2)$ which is sufficiently close in C^r to $\hat{f}$; such exists by Zehnder smoothing theorem [64, Thm. 1]. The bi-links do not need to persist for f. To restore them, the idea is to smoothen the circles C_i to C^r-close circles $\tilde{C}_i$ which are of class C^∞. Then we will perform a local surgery to construct $\hat{f}_\infty$ of class C^∞_ω which is C^r-close to f and such that $\hat{f}_\infty(\sqcup_i \tilde{C}_i) = \sqcup_i \tilde{C}_i$. By local maximality of the hyperbolic continuation $(\tilde{P}_i, \tilde{Q}_i)_i$ of saddle points $(P_i, Q_i)_i$ defining the bi-links $(L^a_i \cup L^b_i)^m_{i=1}$, it follows that each $\tilde{P}_i, \tilde{Q}_i$ belongs to $\tilde{C}_i$. Moreover, as the unique invariant curves which contain $\tilde{P}_i$ or $\tilde{Q}_i$ are their local stable and unstable manifolds, we obtain that the circles $\tilde{C}_I$ are heteroclinic bi-links, i.e., the bi-links are persistent for $\hat{f}_\infty$.

Consequently, we need only to prove the following

Lemma 5.15. *There exists a collection of C^∞-circles $\tilde{C}_i$ which are C^r-close to C_i, and a map $\hat{f}_\infty$ of class C^∞_ω which is C^r-close to f, such that $\hat{f}_\infty(\sqcup_i \tilde{C}_i) = \sqcup_i \tilde{C}_i$.*

Proof. Choose $\varepsilon > 0$ small enough. Let $(x_i(t), y_i(t))_{t\in\mathbb{S}^1}$, where $(\frac{dx_i}{dt})^2 + (\frac{dy_i}{dt})^2 \neq 0$, be a parameterization of a closed C^∞-curve, which is sufficiently close to C_i in the C^r-topology. Consider the map $(t, h) \mapsto (x, y)$ defined as

$$x = x_i(t) - z(t,h)\frac{dy_i(t)}{dt}, \qquad y = y_i(t) + z(t,h)\frac{dx_i(t)}{dt},$$

where $z(t, 0) = 0$ and

$$\partial_h z = \left((\frac{dx_i}{dt})^2 + (\frac{dy_i}{dt})^2 + z(\frac{d^2x_i}{dt^2}\frac{dy_i}{dt} - \frac{d^2y_i}{dt^2}\frac{dx_i}{dt}) \right)^{-1}.$$

It is easy to check that $\det(\partial_{t,h}(x, y)) \equiv 1$, so this map is a symplectic C^∞-diffeomorphism from $\mathbb{S}^1 \times \{|h| \leq \varepsilon\}$ onto a small neighborhood V_i of C_i in $\mathbb{R}^2$ (we think of (x, y) as the

Cartesian coordinates in $\mathbb{R}^2$). Thus, (t, h) are symplectic coordinates in V_i and the curve C_i is given by an equation $h = \sigma_i(t)$ where σ_i is C^r-small.

Let $\tilde{C}_i$ be a curve $h = \tilde{\sigma}_i(t)$ where σ_i is a C^∞-function, which is sufficiently close to σ_i in C^r. By adding a small constant to $\tilde{\sigma}_i$, if necessary, we can ensure that the disc bounded by the curve $\tilde{C}_i$ in $\mathbb{R}^2$ has the same area as the disc bounded by the curve C_i. We do this for all $i = 1, \ldots, m$. Recall that the collection of curves C_i is invariant with respect to the map $\hat{f}$ and every curve C_i is invariant with respect to some iteration of $\hat{f}$. So, there is j such that $C_i = \hat{f}(C_j)$, and C_j is the image of C_i by some iteration of $\hat{f}$. In particular, C_j and C_i bound discs of equal area in $\mathbb{R}^2$. Therefore, the curves $\tilde{C}_i$ and $\tilde{C}_j$ also bound discs of the same area.

Since $\tilde{C}_j$ is close to C_j and $\tilde{C}_i$ is close to $C_i = \hat{f}(C_j)$, we can always assume that the symplectic C^∞-diffeomorphism f (that C^r-approximates the map $\hat{f}$) is chosen such that the closed curve $f(\tilde{C}_j)$ lies in V_i and has an equation $h = \mathring{\sigma}_i(t)$ with some C^r-small function $\mathring{\sigma}_i$. The discs bounded by the curves $\tilde{C}_i$ and $f(\tilde{C}_j)$ in $\mathbb{R}^2$ have the same area, so the area between the curves $\tilde{C}_i$ and $f(\tilde{C}_j)$ is zero, which means that

$$\int_{\mathbb{S}^1} (\tilde{\sigma}_i(t) - \mathring{\sigma}_i(t))dt = 0.$$

Therefore, on $\mathbb{S}^1$ there exists a (C^{r+1}-small) function $\Phi_i(t)$ such that

$$\frac{d}{dt}\Phi_i(t) = \tilde{\sigma}_i(t) - \mathring{\sigma}_i(t).$$

Now, we define a symplectic diffeomorphism $\phi_i : V_i \to V_i$ by a generating function $S(t, \bar{h}) = t\bar{h} - \Phi_i(t)\xi(\bar{h})$, where ξ is a C^∞-function which vanishes identically for $|\bar{h}| \geq \varepsilon$ and equals identically to 1 on the range of values of the function $\tilde{\sigma}_i$. Namely, we define $\phi_i : (t, h) \mapsto (\bar{t}, \bar{t})$ implicitly by the rule $\bar{t} = \partial_{\bar{h}} S$, $h = \partial_t S$, or

$$\bar{t} = t - \Phi_i(t)\frac{d\xi(\bar{h})}{d\bar{h}}, \qquad h = \bar{h} - (\tilde{\sigma}_i(t) - \mathring{\sigma}_i(t))\xi(\bar{h}). \tag{38}$$

These formulas indeed define a single-valued map, C^r-close to identity, because $\tilde{\sigma}_i(t) - \mathring{\sigma}_i(t)$ is C^r-small. By construction, the map ϕ_i equals to identity at the boundary of V_i (at $|h| = \varepsilon$), so ϕ_i is indeed a diffeomorphism of V_i; it is also immediate that it preserves the symplectic form $dt \wedge dh$.

It remains to note that ϕ_i sends the curve $f(\tilde{C}_j) : \bar{h} = \mathring{\sigma}_i(\bar{t})$ to the curve $\tilde{C}_i : h = \tilde{\sigma}_i(t)$ (since ξ is identically 1 everywhere in the range of $\tilde{\sigma}_i$, it follows from (38) that $\bar{t} = t$, $h - \mathring{\sigma}_i(t) = \bar{h} - \tilde{\sigma}_i(\bar{t})$ for every point $(\bar{t}, \bar{h}) \in \tilde{C}_i$). Thus, if we perform this construction for all i and define a C^r-close to identity symplectic C^∞-diffeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ as ϕ_i inside V_i and identity outside $\sqcup_i V_i$, then the map $\hat{f}_\infty = \psi \circ f$ will send the set of the C^∞-smooth curves $\tilde{C}_i$ into itself, as required. $\square$

6. Rescaling Lemma

We recall that given $U \subset \mathbb{R}^2$, the set $Diff^\infty_\omega(U, \mathbb{R}^2)$ denotes the set of symplectic diffeomorphisms ϕ from U onto their image. We endow this space with the complete metric:

$$d_{C^\infty}(\phi, \tilde{\phi}) = \sum_r 2^{-r} \min(1, \|\phi - \tilde{\phi}\|_{C^r}) \,.$$

Consider a symplectic C^∞-diffeomorphism f of $\mathbb{R}^2$ which has a saddle periodic point O. Assume that there exists a *homoclinic band*, i.e., the intersection of the stable and unstable manifolds of O contains a closed interval J of non-zero length. We assume that J is sufficiently small (if not, take its sufficiently small subinterval), so that J lies entirely inside a fundamental domain, implying that $f^m(J) \cap J = \emptyset$ for all $m \neq 0$. By considering, if necessary, an iterate of J, we also assume that J lies close to O in a local stable manifold of O.

Given a function $\psi \in C^r(\mathbb{R})$, we define a symplectic Hénon-like map H_ψ as

$$H_\psi(x, y) = (y, -x + \psi(y)). \tag{39}$$

The following statement enables to produce a perturbation of the dynamics displaying a renormalization arbitrarily close to any composition of Hénon-like maps. It is an improved and more developed[7] version of the rescaling lemma from [29]. We recall that $\mathbb{D}$ denotes the closed unit disk of $\mathbb{R}^2$.

Rescaling Lemma 6.1. *For every odd $N \in \mathbb{N}$, for every $r \geq 1$, for all $L, \delta > 0$, and for every neighborhood U_J of J, there exist:*

- *an integer $n > 1$,*
- *symplectic diffeomorphisms $\Phi_1, \ldots, \Phi_N \in Diff^\infty_\omega(\mathbb{R}^2)$ which are δ-close to the identity.*
- *a C^∞-diffeomorphism $Q : \mathbb{R}^2 \to \mathbb{R}^2$ with constant Jacobian satisfying $Q(\mathbb{D}) \subset U_J$,*

such that for any functions $\psi_1, \ldots, \psi_N \in C^r(\mathbb{R})$ whose C^r-norms are bounded by L, there exists a symplectic C^r-diffeomorphism $\hat{f}$ which coincides with f on the complement of U_J and satisfies:

$$\|\hat{f} - f\|_{C^r} < \delta, \tag{40}$$

$$Q^{-1} \circ \hat{f}^n \circ Q|_{\mathbb{D}} = H_{\psi_N} \circ \Phi_N \circ \cdots \circ H_{\psi_1} \circ \Phi_1|_{\mathbb{D}} \ , \tag{41}$$

where H_{ψ_i} are the symplectic Hénon-like maps defined by (39). Moreover:

[7] The current formulation includes many rounds near a homoclinic tangency; see also Remark 6.2.

$$\hat{f}^m \circ Q(\mathbb{D}) \cap Q(\mathbb{D}) = \varnothing \quad \textit{for al} \quad m = 1, \ldots, n-1. \tag{42}$$

Remark 6.2. It is important for us that the maps Φ_i in formula (41) are independent of the choice of the functions $\psi_1, \ldots, \psi_N$.

Corollary 6.3. *Let $r \geq 1$. Take any symplectic C^∞-diffeomorphism $F : \mathbb{D} \to \mathbb{R}^2$. For any $\delta > 0$, $L > 0$, and a neighborhood U_J of J, there exist:*

- *a C^∞-diffeomorphism $Q : \mathbb{R}^2 \to \mathbb{R}^2$ with constant Jacobian satisfying $Q(\mathbb{D}) \subset U_J$,*
- *a symplectic diffeomorphism $\hat{F} : \mathbb{D} \to \mathbb{R}^2$ which is δ-close to F in the C^∞-metric,*

such that for every function $\psi \in C^r(\mathbb{R})$ with C^r-norm bounded by L, there exists a symplectic diffeomorphism $\hat{f}$, which is δ-close to f in the C^r-norm, coincides with f on the complement of U_J, and its renormalization at the disc $Q(\mathbb{D})$ for some $n \geq 1$ satisfies:

$$Q^{-1} \circ \hat{f}^n \circ Q|_{\mathbb{D}} = S_\psi \circ \hat{F}, \qquad \textit{with } S_\psi : (x, y) \mapsto (x, y + \psi(x)), \tag{43}$$

$$\hat{f}^m \circ Q(\mathbb{D}) \cap Q(\mathbb{D}) = \varnothing \quad \textit{for all} \quad m = 1, \ldots, n-1. \tag{44}$$

Proof. By Theorem 2 of [57], every symplectic C^∞-diffeomorphism of any two-dimensional disc to $\mathbb{R}^2$ can be arbitrarily well approximated by a composition of an *even* number of Hénon-like maps of the form (39). In particular, for every $\delta > 0$, for the given C^∞-diffeomorphism F there exists an even number N' of functions $\psi_1, \ldots, \psi_{N'}$ such that

$$d_{C^\infty}(H_0^{-1} \circ F, H_{\psi_{N'}} \circ \cdots \circ H_{\psi_1}) < \frac{\delta}{2},$$

where $H_0(x, y) = (y, -x)$ is the Hénon-like map associated to the zero function ψ in (39). Since H_0 is just a linear rotation, it follows that

$$d_{C^\infty}(F, H_0 \circ H_{\psi_{N'}} \circ \cdots \circ H_{\psi_1}) < \frac{\delta}{2}, \tag{45}$$

Note that the map S_ψ defined in (43) satisfies $H_\psi = S_\psi \circ H_0$. Since $N' + 1$ is odd, we can apply the Rescaling Lemma with $N = N' + 1$ and with the sequence of functions $\psi_1, \ldots, \psi_{N'}$ and $\psi_{N'+1} \equiv \psi$. This gives us the perturbation $\hat{f}$ to f such that (41) is fulfilled:

$$Q^{-1} \circ \hat{f}^n \circ Q|_{\mathbb{D}} = H_{\psi_{N'+1}} \circ \Phi_{N'+1} \circ \cdots \circ H_{\psi_1} \circ \Phi_1|_{\mathbb{D}} = S_\psi \circ H_0 \circ \Phi_{N'+1} \circ \cdots \circ H_{\psi_1} \circ \Phi_1|_{\mathbb{D}}, \tag{46}$$

for some $n > 1$. We recall that the Rescaling Lemma asserts that $\hat{f}$ coincides with f on the complement of a small neighborhood U_J of J and that Q is a diffeomorphism with constant Jacobian such that $Q(\mathbb{D}) \subset U_J$. By (46), we notice that (43) is satisfied with:

$$\hat{F} = H_0 \circ \Phi_{N'+1} \circ H_{\psi_{N'}} \cdots \circ H_{\psi_1} \circ \Phi_1|_{\mathbb{D}}.$$

Since the maps Φ_i can be made as close to identity as we want, we can make

$$d_{C^\infty}(\hat{F}, H_0 \circ H_{\psi_{N'}} \circ \cdots \circ H_{\psi_1}) < \frac{\delta}{2},$$

which implies, by (45), that $\hat{F}$ lies in the δ-small C^∞-neighborhood of F. Since the maps Φ_i are independent of the choice of ψ, it follows that $\hat{F}$ is independent of the function ψ either. Note that (44) follows from (42). □

Proof of Proposition 2.4. Take any $L > 0$, $\varepsilon > 0$, and any integer $r > 0$. Consider the set $\mathcal{F}_{r,\varepsilon,L}$ of maps $\hat{F} \in Diff^\infty_\omega(\mathbb{D}, \mathbb{R}^2)$ such that for every function $\psi \in C^r(\mathbb{R})$ whose C^r-norm is bounded by L, in the ε-neighborhood of f in C^r there exists a symplectic diffeomorphism $\hat{f}$ for which some renormalized iteration equals to $S_\psi \circ \hat{F}$. By Corollary 6.3, the set $\mathcal{F}_{r,\varepsilon,L}$ is dense in $Diff^\infty_\omega(\mathbb{D}, \mathbb{R}^2)$.

Claim 6.4. *The set $\mathcal{F}_{r,\varepsilon,L}$ is open in $Diff^\infty_\omega(\mathbb{D}, \mathbb{R}^2)$.*

Proof. This is inferred from (44) by the following observation. Let $\hat{F}$ satisfies (43) for some ψ, Q, and $\hat{f}$:

$$\hat{f}|_{Q(\mathbb{D})} = \hat{f}^{-(n-1)} \circ Q \circ S_\psi \circ \hat{F} \circ Q^{-1}|_{Q(\mathbb{D})}.$$

Then, for any map $\tilde{F}$ close to $\hat{F}$, we take a map $\tilde{f}$ such that it equals to $\hat{f}$ outside a small neighborhood of $Q(\mathbb{D})$ and

$$\tilde{f}|_{Q(\mathbb{D})} = \hat{f}^{-(n-1)} \circ Q \circ S_\psi \circ \tilde{F} \circ Q^{-1}|_{Q(\mathbb{D})}.$$

If $\tilde{F}$ is close enough to $\hat{F}$, then $\tilde{f}$ is close to $\hat{f}$ too. So, by (44), the discs $\tilde{f}^m \circ Q(\mathbb{D})$ stay at a sufficient distance from $Q(\mathbb{D})$ for all $m = 1, \dots, n-1$, which means that $\tilde{f} \equiv \hat{f}$ in the disc $\tilde{f}^m \circ Q(\mathbb{D})$ for all $m = 1, \dots, n-1$. By construction, this gives us

$$Q^{-1} \circ \tilde{f}^n \circ Q|_{\mathbb{D}} = S_\psi \circ \tilde{F},$$

i.e., $\tilde{F}$ is a renormalized iteration of $\tilde{f}$. As the map $\tilde{f}$ can be taken as close as we want to $\hat{f}$ if $\tilde{F}$ is close enough to $\hat{F}$, we also have that $d_{C^\infty}(\tilde{f}, f) < \varepsilon$. Thus, every $\tilde{F}$ which is close enough to $\hat{F}$ lies in $\mathcal{F}_{r,\varepsilon,L}$. □

Now, we define the topologically generic set $\mathcal{F} \in Diff^\infty_\omega(\mathbb{D}, \mathbb{R}^2)$ from Proposition 2.4 as

$$\mathcal{F} = \bigcap_{(r,m,L)\in\mathbb{N}^3} \mathcal{F}_{r,\frac{1}{m},L}. \quad \square$$

In the following, we complete the proof of the main theorem by proving the Rescaling Lemma.

6.1. Local behavior near a nonlinear saddle

Denote as T_0 the restriction of f^s to a sufficiently small neighborhood of O. One can introduce symplectic C^∞-coordinates (x, y) such that the local stable and unstable manifolds $W^s_{loc}(O)$ and $W^u_{loc}(O)$ get straightened (i.e., they acquire equations $y = 0$ and $x = 0$, respectively) and the restrictions of T_0 onto $W^s_{loc}(O)$ and $W^u_{loc}(O)$ become linear (see Section 2.1 in [29]). This means that the map $T_0 : (x, y) \mapsto (\bar{x}, \bar{y})$ takes the following form

$$\bar{x} = \lambda x + p(x, y)x, \qquad \bar{y} = \lambda^{-1} y + q(x, y)y, \tag{47}$$

where $0 < |\lambda| < 1$ and

$$p(x, 0) = 0, \qquad\qquad q(0, y) = 0. \tag{48}$$

Since T_0 is symplectic, $\det(DT_0) \equiv 1$, so

$$(\lambda + \partial_x(p(x, y)x))(\lambda^{-1} + \partial y(q(x, y)y)) - \partial_y p(x, y)\partial_x q(x, y)xy \equiv 1,$$

and it is a trivial exercise to check that this identity and (48) imply

$$p(0, y) = 0, \qquad q(x, 0) = 0. \tag{49}$$

Identities (48), (49) are important, because they imply nice uniform estimates for arbitrarily long iterations of T_0. Namely, the following result holds true:

Lemma 6.5 *(Lem. 7 in [30]). Let (47)-(49) hold. There exist $\alpha > 0$ and a sequence of functions ξ_k, η_k from $U_0 := [-\alpha, \alpha]^2$ into $\mathbb{R}^1$, such that for all k large enough, we have for all (x, y), $(\tilde{x}, \tilde{y}) \in U_0$:*

$$T_0^k(x, y) = (\tilde{x}, \tilde{y}) \qquad \Longleftrightarrow \qquad \left[\tilde{x} = \lambda^k x + \xi_k(x, \tilde{y}),\;\; y = \lambda^k \tilde{y} + \eta_k(x, \tilde{y})\right] . \tag{50}$$

Moreover, the functions $\lambda^{-k}\xi_k$ and $\lambda^{-k}\eta_k$ are C^∞-small when k is large:

$$\forall \rho \geq 1, \quad \|\xi_k, \eta_k\|_{C^\rho} = o(\lambda^k) \text{ as } k \to +\infty . \tag{51}$$

This lemma means that arbitrarily long iterations of T_0 are well approximated by the iterations of its linearization.

We will look for the perturbed map $\hat{f}$ in the form $\hat{f} = g \circ f$ where g is a close to identity symplectic diffeomorphism. In our construction $(g - id)$ will be supported outside of $f(U_0)$, so formulas (50) will stay valid for the perturbed map $\hat{f} = g \circ f$ with the same ξ_k, η_k, and λ.

6.2. Formulas for the iterations near the homoclinic band

Recall that the homoclinic band J is assumed to be small enough, so that $J \cap f^m(J) = \emptyset$ for every $m \neq 0$. This also implies that there exists an iteration J^+ of J that lies in $W^s_{loc}(O) := (-\alpha, \alpha) \times \{0\}$ and an iteration J^- that lies in $W^u_{loc}(O) := \{0\} \times (-\alpha, \alpha)$ such that $f^{\bar{m}}(J^-) = J^+$ for some $\bar{m} > 0$ and $f^j(J^-) \cap U_0 = \emptyset$ for $j = 1, \ldots, \bar{m} - 1$. We take α sufficiently small, so $\bar{m}$ is sufficiently large, in particular $\bar{m} > s$ (where s is the period of O). Let $U_+ \subset U_0$ and $U_- \subset U_0$ be sufficiently small neighborhoods of J_+ and J_-. As $f^{-1}(J_+)$ lies outside U_0, it follows that $U_+ \cap f(U_0) = \emptyset$; similarly, $f^{-1}(U_0) \cap U_- = \emptyset$. Note also that

$$f^j(U_-) \cap U_0 = \emptyset \text{ for } j = 1, \ldots, \bar{m} - 1. \tag{52}$$

As $\bar{m} > s$, we also have that

$$U_+ \cap T_0(U_0) = \emptyset. \tag{53}$$

Denote

$$T_1 := f^{\bar{m}}|_{U_-} .$$

Choose N different points $M_i^- := (0, y_i^-) \in J^-$ and put $M_i^+ := (x_i^+, 0) := T_1 M_i^-$. The perturbation $g = \hat{f} \circ f$ will be supported in a small neighborhood of the points M_i^+. Thus, $\hat{f}$ will coincide with f outside of U_+. We could from the very beginning choose J lying in $W^s_{loc}(O)$ sufficiently close to O, i.e., assume that $J_+ = J$, Then the neighborhood U_J in the formulation of the Rescaling Lemma is the neighborhood U_+, so the claim of the lemma that $\hat{f}$ coincides with f in the complement of U_J will be fulfilled.

As $J^- \subset \{0\} \times (-\alpha, \alpha)$ is sent to $J^+ \subset (-\alpha, \alpha) \times \{0\}$, the map $T_1 : (\tilde{x}, \tilde{y}) \mapsto (x, y)$ near the point $M_i^- = (0, y_i^-)$ can be written in the form

$$x = x_i^+ + b_i(\tilde{y} - y_i^-) + \varphi_{1\,i}(\tilde{x}, \tilde{y} - y_i^-), \qquad y = (c_i + d_i(\tilde{y} - y_i^-) + \varphi_{2\,i}(\tilde{x}, \tilde{y} - y_i^-))\, \tilde{x}, \tag{54}$$

where the C^∞-functions $\varphi_{1\,i}$ and $\varphi_{2\,i}$ satisfy

$$\varphi_{1\,i}(0) = 0, \qquad \partial_y \varphi_{1\,i}(0) = 0, \qquad \varphi_{2\,i}(0) = 0, \qquad \partial_y \varphi_{2\,i}(0) = 0. \tag{55}$$

Obviously, this implies that

$$\begin{aligned} &\varphi_{j\,i}(\tilde{x}, \tilde{y} - y_i^-) = O(|\tilde{x}|) + o(|\tilde{y} - y_i^-|), \\ &\partial_y \varphi_{j\,i}(\tilde{x}, \tilde{y} - y_i^-) = o(1)_{(\tilde{x}, \tilde{y} - y_i^-) \to 0}, \qquad j = 1, 2. \end{aligned} \tag{56}$$

Note also that the area-preservation property of T_1 implies that the coefficients b_i, c_i in (54) satisfy

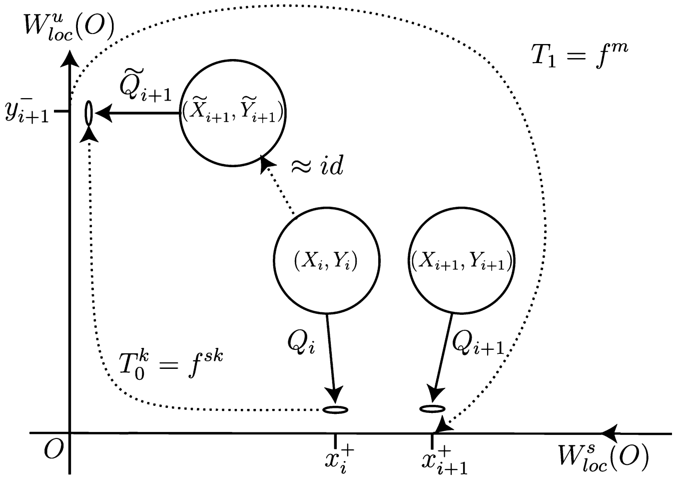

Fig. 6. Rescaling coordinates.

$$b_i c_i = -\det(DT_1(M_i^-)) = -1. \tag{57}$$

6.3. Scaling transformation

We will further assume that the indices i are defined modulo N, i.e. hereafter $i+1 \equiv 1$ if $i = N$ and $i - 1 \equiv N$ if $i = 1$. We will need a sequence of positive real numbers R_i, $i = 1, \dots N$, that satisfy

$$R_1 = 1, R_{i+1} = -c_{i+1} b_i R_{i-1}. \tag{58}$$

Such sequence indeed exists when N is odd: we define R_i by (58) inductively: R_1, R_3, ... , R_N, R_2, ... , R_{N-1}, until we arrive to $R_{N+1} = (-1)^N \prod_{i=1}^N c_i \prod_{i=1}^N b_i$ and notice that the constraint $R_{N+1} = R_1$ is satisfied by virtue of (57).

We now perform affine rescaling of the coordinates (x, y) near the points M_i^+ and $(\tilde{x}, \tilde{y})$ near the points M_i^-. First, we chose a decreasing sequence $(\mu_k)_k$ of scaling constants $\mu_k \in (0, 1)$ which converges to 0 sub-exponentially. In particular, for every $\rho \geq 1$, it holds:

$$\mu_k \to 0 \quad \text{and} \quad \lambda^k \cdot \mu_k^{-\rho} \to 0 \text{ when } k \to \infty\ . \tag{59}$$

Next, we take a sufficiently large integer k and define coordinate transformations

$$Q_{ik} : (X_i, Y_i) \mapsto (x, y) \quad \text{and} \quad \tilde{Q}_{ik} : (\tilde{X}_i, \tilde{Y}_i) \mapsto (\tilde{x}, \tilde{y})\ ,$$

by the following rule:

$$x = x_i^+ + b_i R_{i-1} \mu_k X_i, \qquad y = \lambda^k (y_{i+1}^- + \gamma_{i\,k} + R_i \mu_k Y_i), \tag{60}$$

$$\tilde{x} = \lambda^k (x^+_{i-1} + \beta_{i\,k} + b_{i-1} R_{i-2} \mu_k \tilde{X}_i), \qquad \tilde{y} = y^-_i + R_{i-1} \mu_k \tilde{Y}_i, \tag{61}$$

where the constant terms $\gamma_{i\,k}$ and $\beta_{i\,k}$ are given by

$$\beta_{i\,k} = \xi_k(x^+_{i-1}, y^-_i)\lambda^{-k}, \qquad \gamma_{i\,k} = \eta_k(x^+_i, y^-_{i+1})\lambda^{-k}. \tag{62}$$

It is obvious that given any real number $\hat{L}$, if k is large enough, then the map Q_{ik} sends the disc $\hat{\mathbb{D}} := \{X^2 + Y^2 \le \hat{L}^2\}$ into a small neighborhood of M^+_i, whereas the map $\tilde{Q}_{ik}$ sends $\hat{\mathbb{D}}$ into a small neighborhood of M^-_i. Indeed, $\gamma_{i\,k} \to 0$ and $\beta_{i\,k} \to 0$ as $k \to +\infty$, by virtue of (51), and $\lambda^k \to 0$, $\mu_k \to 0$ by (59). Therefore, when $k \to \infty$, for any (X_i, Y_i) and $(\tilde{X}_i, \tilde{Y}_i)$ in $\hat{\mathbb{D}}$, when $k \to \infty$, the following limits holds true: $(x, y) \to (x^+_i, 0) = M^+_i$ in (60) and $(\tilde{x}, \tilde{y}) \to (0, y^-_i) = M^-_i$ in (61).

Note that the map Q_{1k} will be the map Q in the statement of the Rescaling Lemma.

Next, we are going to derive formulas (see (64),(77)) for the maps $Q^{-1}_{i+1,k} T_1 \tilde{Q}_{i+1,k}$ and $\tilde{Q}^{-1}_{i+1,k} T^k_0 Q_{ik}$ for the original map f and the perturbed map $\hat{f}$. As the composition:

$$(Q^{-1}_{N+1,k} T_1 \tilde{Q}_{N+1,k})(\tilde{Q}^{-1}_{N+1,k} T^k_0 Q_{Nk}) \circ \cdots (Q^{-1}_{i+1,k} T_1 \tilde{Q}_{i+1,k})(\tilde{Q}^{-1}_{i+1,k} T^k_0 Q_{ik}) \circ \cdots \circ$$
$$(Q^{-1}_{2,k} T_1 \tilde{Q}_{2,k})(\tilde{Q}^{-1}_{2,k} T^k_0 Q_{1k})$$

equals to $Q^{-1}_{1k} \circ \hat{f}^n \circ Q_{1k}$ for $n = N(ks + \bar{m})$, we will obtain formula (41) in this way and, thus, prove the Rescaling Lemma.

6.4. Renormalized iterations of the unperturbed map

We start with the map $\tilde{Q}^{-1}_{i+1,k} T^k_0 Q_i$, i.e., the map T^k_0 in the rescaled coordinates.

Lemma 6.6. *Take any $\hat{L} > 0$ and let $\hat{\mathbb{D}}$ be the disc $\{X^2 + Y^2 \le \hat{L}^2\}$ in $\mathbb{R}^2$. For every sufficiently large k there exist two real functions $\hat{\xi}_{ik}$, $\hat{\eta}_{ik}$ on $\hat{\mathbb{D}}$, which vanish at zero and satisfy for every $\rho \ge 1$:*

$$\|\tilde{\xi}_{ik}, \tilde{\eta}_{ik}\|_{C^\rho} = o(1)_{k \to +\infty}, \tag{63}$$

such that for every $(X_i, Y_i) \in \hat{\mathbb{D}}$ the points $T^j_0 Q_{ik}(X_i, Y_i)$ lie in U_0 for all $j = 1, \dots k$, and $(\tilde{X}_{i+1}, \tilde{Y}_{i+1}) = \tilde{Q}^{-1}_{i+1,k} T^k_0 Q_{ik}$ is given by

$$\begin{aligned} \tilde{X}_{i+1} &= X_i + \tilde{\xi}_{ik}(X_i, Y_i), \\ \tilde{Y}_{i+1} &= Y_i + \tilde{\eta}_{ik}(X_i, Y_i). \end{aligned} \tag{64}$$

Proof. We will use the notation $(x_i, y_i) = Q_{ik}(X_i, Y_i)$ and $(\tilde{x}_{i+1}, \tilde{y}_{i+1}) = \tilde{Q}_{i+1,k}(X_{i+1}, Y_{i+1}) = T^k_0(x_i, y_i)$. By (50), (60) and (61), the first k iterations of (x_i, y_i) by T_0 lie in U_0 if and only if:

$$\begin{cases} \tilde{x}_{i+1} = \lambda^k x_i + \xi_k(x_i, \tilde{y}_{i+1}), & y_i = \lambda^k \tilde{y}_{i+1} + \eta_k(x_i, \tilde{y}_{i+1}), \\ x_i = x_i^+ + b_i R_{i-1}\mu_k X_i, & y_i = \lambda^k (y_{i+1}^- + \gamma_{i\,k} + R_i \mu_k Y_i), \\ \tilde{x}_{i+1} = \lambda^k (x_i^+ + \beta_{i+1\,k} + b_i R_{i-1}\mu_k \tilde{X}_{i+1}), & \tilde{y}_{i+1} = y_{i+1}^- + R_i \mu_k \tilde{Y}_{i+1}. \end{cases} \tag{65}$$

Thus,

$$\begin{aligned} &\lambda^k(\beta_{i+1\,k} + b_i R_{i-1}\mu_k \tilde{X}_{i+1}) = \lambda^k b_i R_{i-1}\mu_k X_i + \xi_k(x_i^+ + b_i R_{i-1}\mu_k X_i, y_{i+1}^- + R_i\mu_k \tilde{Y}_{i+1}), \\ &\lambda^k(\gamma_{i\,k} + R_i\mu_k Y_i) = \lambda^k R_i \mu_k \tilde{Y}_{i+1} + \eta_k(x_i^+ + b_i R_{i-1}\mu_k X_i, y_{i+1}^- + R_i\mu_k \tilde{Y}_{i+1}). \end{aligned} \tag{66}$$

Using the definition of $\beta_{i+1\,k}$ and $\gamma_{i\,k}$, given in (62), we can see from formula (66) that the zero value of (X_i, Y_i) corresponds to the zero value of $(\tilde{X}_{i+1}, \tilde{Y}_{i+1})$. So, we can rewrite (66) as

$$\mu_k \tilde{X}_{i+1} = \mu_k X_i + \xi'_{ik}(\mu_k X_i, \mu_k \tilde{Y}_{i+1}), \qquad \mu_k Y_i = \mu_k \tilde{Y}_{i+1} + \eta'_{ik}(\mu_k X_i, \mu_k \tilde{Y}_{i+1}),$$

where, as follows from (51) and (59), the functions ξ'_{ik}, η'_{ik} uniformly tend to zero in the C^∞-topology as $k \to +\infty$; moreover, they vanish when $(X_i, \tilde{Y}_{i+1}) = 0$. So, we have that

$$\tilde{X}_{i+1} = X_i + \tilde{\xi}'_{ik}(X_i, \tilde{Y}_{i+1}), \qquad Y_i = \tilde{Y}_{i+1} + \tilde{\eta}'_{ik}(X_i, \tilde{Y}_{i+1}), \tag{67}$$

where the functions $\tilde{\xi}'_{ik}(\cdot) = \mu_k^{-1}\tilde{\xi}'_{ik}(\mu_k \cdot)$ and $\tilde{\eta}'_{ik}(\cdot) = \mu_k^{-1}\tilde{\eta}'_{ik}(\mu_k \cdot)$ are C^∞-small, and these functions vanish at zero. By the implicit function theorem, the second equation of (67) defines $\tilde{Y}_{i+1}$ as a vanishing at zero function of (X_i, Y_i) which is uniformly smooth. Therefore, formula (67) gives us the result of the lemma (i.e., relations (63) and (64)) with $\tilde{\xi}_{ik}(X_i, Y_i) := \tilde{\xi}'_{ik}(X_i, \tilde{Y}_{i+1})$ and $\tilde{\eta}_{ik}(X_i, Y_i) := -\tilde{\eta}'_{ik}(X_i, \tilde{Y}_{i+1})$. □

As the next step, we consider the map $Q^{-1}_{i+1,k} T_1 \tilde{Q}_{i+1,k}$ (the map T_1 in the rescaled coordinates) for the *unperturbed map* f.

Lemma 6.7. *There are two real functions $\tilde{\phi}_{1ik}$, $\tilde{\phi}_{2ik}$ on the disc $\hat{\mathbb{D}} := \{\tilde{X}^2 + \tilde{Y}^2 \le \hat{L}^2\}$ satisfying:*

$$\tilde{\phi}_{1ik} \xrightarrow{C^\infty} 0 \quad \text{and} \quad \tilde{\phi}_{2ik} \xrightarrow{C^\infty} 0 \quad \text{as } k \to \infty, \tag{68}$$

such that for every $(\tilde{X}_{i+1}, \tilde{Y}_{i+1}) \in \hat{\mathbb{D}}$ sent to (X_{i+1}, Y_{i+1}) by $Q^{-1}_{i+1,k} T_1 \tilde{Q}_{i+1,k}$ we have

$$X_{i+1} = \tilde{Y}_{i+1} + \tilde{\phi}_{1ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}), \qquad Y_{i+1} = C_{ik}\mu_k^{-1} + A_i \tilde{Y}_{i+1} - \tilde{X}_{i+1} + \tilde{\phi}_{2ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}), \tag{69}$$

where the constants[8] $C_{ik} = [c_{i+1}(x_i^+ + \beta_{i+1\,k}) - y_{i+2}^- - \gamma_{i+1\,k}]/R_{i+1}$ *and* $A_i = d_{i+1}(x_i^+ + \beta_{i+1,k})R_i/R_{i+1}$ *are uniformly bounded for all k.*

[8] See (54) and (60) for the definition of the coefficients in these formulas.

Proof. By definition, $Q_{i+1,k}(X_{i+1}, Y_{i+1}) = T_1 \tilde{Q}_{i+1,k}(\tilde{X}_{i+1}, \tilde{Y}_{i+1})$. Let $(x_{i+1}, y_{i+1}) := Q_{i+1,k}(X_{i+1}, Y_{i+1})$ and $(\tilde{x}_{i+1}, \tilde{y}_{i+1}) = \tilde{Q}_{i+1,k}(X_{i+1}, Y_{i+1})$. By (54), (60), and (61)

$$\begin{cases} x_{i+1} = x^+_{i+1} + b_{i+1}(\tilde{y}_{i+1} - y^-_{i+1}) + \varphi_{1\,i+1}(\tilde{x}_{i+1}, \tilde{y}_{i+1} - y^-_{i+1}), \\ y_{i+1} = \tilde{x}_{i+1}(c_{i+1} + d_{i+1}(\tilde{y}_{i+1} - y^-_{i+1}) + \varphi_{2\,i+1}(\tilde{x}_{i+1}, \tilde{y}_{i+1} - y^-_{i+1})), \\ x_{i+1} = x^+_{i+1} + b_{i+1} R_i \mu_k X_{i+1}, \quad y_{i+1} = \lambda^k (y^-_{i+2} + \gamma_{i+1\,k} + R_{i+1}\mu_k Y_{i+1}) , \\ \tilde{x}_{i+1} = \lambda^k (x^+_i + \beta_{i+1\,k} + b_i R_{i-1}\mu_k \tilde{X}_{i+1}), \quad \tilde{y}_{i+1} = y^-_{i+1} + R_i \mu_k \tilde{Y}_{i+1} . \end{cases}$$

We replace x_{i+1} and y_{i+1} in the first and second lines by their expressions from the third line, and then eliminate the term x^+_{i+1} which now appears on both sides of the first line. This gives us

$$\begin{cases} b_{i+1} R_i \mu_k X_{i+1} = b_{i+1}(\tilde{y}_{i+1} - y^-_{i+1}) + \varphi_{1\,i+1}(\tilde{x}_{i+1}, \tilde{y}_{i+1} - y^-_{i+1}), \\ \lambda^k (y^-_{i+2} + \gamma_{i+1\,k} + R_{i+1}\mu_k Y_{i+1}) = \tilde{x}_{i+1}(c_{i+1} + d_{i+1}(\tilde{y}_{i+1} - y^-_{i+1}) + \varphi_{2\,i+1}(\tilde{x}_{i+1}, \tilde{y}_{i+1} - y^-_{i+1})), \\ \tilde{x}_{i+1} = \lambda^k (x^+_i + \beta_{i+1\,k} + b_i R_{i-1}\mu_k \tilde{X}_{i+1}), \quad \tilde{y}_{i+1} - y^-_{i+1} = R_i \mu_k \tilde{Y}_{i+1} . \end{cases}$$

We can now isolate the terms X_{i+1} and Y_{i+1}, and replace $\tilde{y}_{i+1} - y^-_{i+1}$ by its expression $R_i \mu_k \tilde{Y}_{i+1}$ from the last line:

$$\begin{cases} X_{i+1} = \tilde{Y}_{i+1} + \frac{1}{b_{i+1} R_i \mu_k} \varphi_{1\,i+1}(\tilde{x}_{i+1}, R_i \mu_k \tilde{Y}_{i+1}), \\ Y_{i+1} = -\frac{y^-_{i+2} + \gamma_{i+1\,k}}{R_{i+1}\mu_k} + \frac{\tilde{x}_{i+1}}{\lambda^k R_{i+1}\mu_k}(c_{i+1} + d_{i+1} R_i \mu_k \tilde{Y}_{i+1} + \varphi_{2\,i+1}(\tilde{x}_{i+1}, R_i \mu_k \tilde{Y}_{i+1})), \\ \tilde{x}_{i+1} = \lambda^k (x^+_i + \beta_{i+1\,k} + b_i R_{i-1}\mu_k \tilde{X}_{i+1}) . \end{cases}$$

Replacing $\tilde{x}_{i+1}$ in the first and second line by the right-hand side of the third line, we obtain

$$\begin{aligned} X_{i+1} &= \tilde{Y}_{i+1} + \tilde{\phi}_{1ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}), \\ Y_{i+1} &= C_{ik}\mu_k^{-1} + A_i \tilde{Y}_{i+1} + \frac{c_{i+1} b_i R_{i-1}}{R_{i+1}} \tilde{X}_{i+1} + \tilde{\phi}_{2ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}) , \end{aligned} \tag{70}$$

where the coefficients C_{ik} and A_i are as defined in the statement of the lemma, and

$$\tilde{\phi}_{1ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}) = \frac{1}{b_{i+1} R_i} \tilde{\varphi}_{1ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}), \tag{71}$$

$$\begin{aligned} l\tilde{\phi}_{2ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}) = &\frac{x^+_i + \beta_{i+1\,k} + b_i R_{i-1}\mu_k \tilde{X}_{i+1}}{R_{i+1}} \tilde{\varphi}_{2ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}) \\ &+ \frac{b_i R_{i-1} \tilde{X}_{i+1}}{R_{i+1}} d_{i+1} R_i \mu_k \tilde{Y}_{i+1}, \end{aligned} \tag{72}$$

where

$$\tilde{\varphi}_{jik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}) = \mu_k^{-1} \varphi_{j\,i+1}(\lambda^k (x^+_i + \beta_{i+1\,k} + b_i R_{i-1}\mu_k \tilde{X}_{i+1}), R_i \mu_k \tilde{Y}_{i+1}), \quad j = 1, 2. \tag{73}$$

Since the coefficient $\frac{c_{i+1}b_iR_{i-1}}{R_{i+1}}$ in the second equation of (70) equals to -1 by (58), formula (70) gives us the lemma (cf. formula (69)), once we show that the functions $\tilde{\phi}_{jik}$ defined by (71),(72) satisfy (68). To do this, we just need to notice that for every given ρ, the C^ρ-norms of the functions $\tilde{\varphi}_{ijk}$ defined by (73) tend to zero as $k \to +\infty$. Indeed, since the C^∞-functions $\varphi_{j\,i+1}$ are independent of k, all their derivatives up to the order ρ are uniformly bounded. Therefore, the derivatives $\frac{\partial^\rho \tilde{\varphi}_{jik}}{(\partial \tilde{X}_{i+1})^{\rho-\sigma}(\partial \tilde{Y}_{i+1})^\sigma}$ are estimated as $O(\lambda^{(\rho-\sigma)k}\mu_k^{\sigma-1})$. Hence, by (59), all the derivatives of order 2 and higher tend to zero as $k \to +\infty$, as well as the first derivative with respect to $\tilde{X}_{i+1}$. From (56)-(59), we also have that $\tilde{\varphi}_{jik} = o(1)_{k\to+\infty}$ and $\partial_{\tilde{Y}_{i+1}}\tilde{\varphi}_{jik} = o(1)_{k\to+\infty}$, and the required C^∞-smallness of the functions $\tilde{\varphi}_{ijk}$ follows. $\square$

6.5. *Construction of the perturbation g*

The perturbation we will now add to the map f does not change the map T_0 (because the perturbed map $\hat{f}$ equals to $g \circ f$ where g will be supported in U_+ which does not intersect $f(U_0)$, so the difference $\hat{f} - f$ is supported outside of U_0). Therefore, Lemma 6.6 and its formula (64) for $\tilde{Q}^{-1}_{i+1,k}T_0^kQ_{ik}$ remains the same when f is replaced by the perturbed map $\hat{f}$. The map T_1 will be affected by the perturbation, hence Lemma 6.7 and its formula (69) for the map $Q^{-1}_{i+1,k}T_1\tilde{Q}_{i+1,k}$ will be modified. In order to construct the perturbation map g, we take the functions $\psi_1, \dots, \psi_N$ from the statement of the lemma and define

$$\begin{aligned}\hat{\psi}_{i+1} : x \mapsto &- \lambda^k C_{ik}R_{i+1} - \lambda^k \frac{A_iR_{i+1}}{b_{i+1}R_i}(x - x^+_{i+1}) \\ &+ \lambda^k \mu_k R_{i+1}\psi_i(\mu_k^{-1}(x - x^+_{i+1})/(b_{i+1}R_i)),\end{aligned} \tag{74}$$

where the constants C_{ik}, A_i are the same as in (69). As condition (59) implies that $|\lambda|^k = o(\mu_k^r)$, we have

$$\|\hat{\psi}_{i+1}\|_{C^r} = o(\mu_k) \to 0 \text{ as } k \to +\infty.$$

Recall that we use the notation (x, y) for the non-rescaled coordinates near the set $\{M_i^+; 1 \le i \le N\} = \{(x_i^+, 0); 1 \le i \le N\} \subset W^u_{loc}(O)$. Take $\varepsilon > 0$ and define neighborhoods V_i and $V_i' \subset V_i$ of the homoclinic points M_i^+:

$$V_i := [-\varepsilon, \varepsilon]^2 + M_i^+ = [-\varepsilon + x_i^+, \varepsilon + x_i^+] \times [-\varepsilon, \varepsilon] \qquad V_i' := [-\varepsilon/2, \varepsilon/2]^2 + M_i^+.$$

We assume that ε is small enough, so the discs V_i all lie in U_+ and are mutually disjoint. Denote $V' = \sqcup_i V_i'$ and $V = \sqcup_i V_i$, so $V' \subset V$. We choose k large enough, so $Q_{ik}(\hat{\mathbb{D}}) \subset V_i'$, where $\hat{\mathbb{D}}$, the domain of the rescaling coordinates, is the disc from the formulation of Lemmas 6.6 and 6.7. Below we define the perturbation map g such that the support of $g - id$ lies in V.

Let $\Psi \in C^\infty(\sqcup_i[-\varepsilon + x_i^+, \varepsilon + x_i^+], \mathbb{R})$ be such that its derivative $D\Psi$, in restriction to $[-\varepsilon + x_i^+, \varepsilon + x_i^+]$, satisfies $D\Psi = \hat{\psi}_i$. Let $\rho \in C^\infty(\mathbb{R}, [0,1])$ be the bump function equal to 1 identically in V' and equal to 0 identically outside of V. Let the perturbation map g be equal to the time-1 map for the differential equation

$$\dot{x} = -\Psi(x)\partial_y\rho(x,y), \qquad \dot{y} = D\Psi(x)\rho(x,y) + \Psi(x)\partial_x\rho(x,y).$$

This is a Hamiltonian system in $\mathbb{R}^2$ (with the Hamiltonian $-\Psi(x)\rho(x,y)$), so g is a symplectic diffeomorphism. Since $\|\hat{\psi}_i\|_{C^r} \to 0$ as $k \to +\infty$, it follows that $\|g - id\|_{C^r}$ can be made smaller than any given constant δ if k is taken sufficiently large. Note also that the support of $g - id$ lies in $V \subset U_+$. All this is in agreement with the statement of Rescaling Lemma.

By construction, the restriction of g to V_i' is given by

$$g|_{V_i'} : (x, y) \mapsto (x, y + \hat{\psi}_i(x)). \tag{75}$$

Plugging this into (54), we find that for the perturbed map $\hat{f} = g \circ f$ the transition map $T_1 = \hat{f}^m = g \circ f^m$ from a neighborhood of M_{i+1}^- to a neighborhood of M_{i+1}^+ is given by

$$\begin{aligned} x &= x_{i+1}^+ + b_{i+1}(\tilde{y} - y_{i+1}^-) + \varphi_{1\,i+1}(\tilde{x}, \tilde{y} - y_{i+1}^-), \\ y &= \hat{\psi}_{i+1}(x) + \tilde{x}(c_{i+1} + d_{i+1}(\tilde{y} - y_{i+1}^-) + \varphi_{2\,i+1}(\tilde{x}, \tilde{y} - y_{i+1}^-)), \end{aligned} \tag{76}$$

with the same functions $(\varphi_{1\,i+1}, \varphi_{2\,i+1})$ and coefficients x_{i+1}^+, y_{i+1}^-, b_{i+1}, c_{i+1}, d_{i+1} as for the unperturbed map f, and with the single additional term $\hat{\psi}_{i+1}(x)$ in the second equation. Therefore, because the rescaling map $Q_{i+1,k}$ is affine, the only correction to the map $Q_{i+1,k}^{-1} T_1 \tilde{Q}_{i+1,k}$ due to this perturbation will be the additional term

$$DQ_{i+1,k}^{-1} \circ (0, \hat{\psi}_{i+1}) \circ Q_{i+1,k}(X_{i+1}, Y_{i+1}) = (0,\ \lambda^{-k}\mu_k^{-1} R_{i+1}^{-1} \hat{\psi}_{i+1}(x_{i+1}^+ + b_{i+1} R_i \mu_k X_{i+1}))$$

in the right-hand side of the respective equations in (69) (we use here the rescaling formula (60) with i replaced by $i+1$; DQ denotes the linear part of the affine map Q). By (74), this formula can be rewritten as

$$DQ_{i+1,k}^{-1} \circ (0, \hat{\psi}_{i+1}) \circ Q_{i+1,k}(X_{i+1}, Y_{i+1}) = (0,\ -\mu_k^{-1} C_{ik} - A_i X_{i+1} + \psi_i(X_{i+1})),$$

so the equation (69) for the map $Q_{i+1,k}^{-1} T_1 \tilde{Q}_{i+1,k}$ changes to

$$\begin{aligned} X_{i+1} &= \tilde{Y}_{i+1} + \tilde{\phi}_{1ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}), \\ Y_{i+1} &= -\tilde{X}_{i+1} + \psi_i(X_{i+1}) + \tilde{\phi}_{2ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}) - A_i \tilde{\phi}_{1ik}(\tilde{X}_{i+1}, \tilde{Y}_{i+1}). \end{aligned} \tag{77}$$

By this formula and formula (64) from Lemma 6.6, the map $T_1 T_0^k = \hat{f}^{\bar{m}+sk} = g \circ f^{\bar{m}+sk}$ from a small neighborhood of M_i^+ to a small neighborhood of M_{i+1}^+ is written, in the rescaled coordinates, as

$$X_{i+1} = Y_i + \phi_{1ik}(X_i, Y_i), \qquad Y_{i+1} = -X_i + \psi_i(X_{i+1}) + \phi_{2ik}(X_i, Y_i), \tag{78}$$

where

$$\phi_{1ik} = \hat{\eta}_{ik} + \tilde{\phi}_{1ik} \circ (id + (\xi_{ik}, \eta_{ik})), \qquad \phi_{2ik} = -\hat{\xi}_{ik} + (\tilde{\phi}_{2ik} - A_i\tilde{\phi}_{1ik}) \circ (id + (\xi_{ik}, \eta_{ik})). \tag{79}$$

By (63),(68), the functions ϕ_{1ik}, ϕ_{2ik} tend to zero uniformly in C^∞ on any compact as $k \to +\infty$. Importantly, the functions $(\phi_{1ik}, \phi_{2ik})_i$ do not depend on the choice of the perturbation functions $(\psi_i)_i$.

Formula (78), in fact, completes the proof of the Rescaling Lemma. Indeed, it can be rewritten as

$$(X_{i+1}, Y_{i+1}) = H_{\psi_i} \circ \Phi_i(X_i, Y_i),$$

where

$$\Phi_i(X_i, Y_i) = (X_i - \phi_{2ik}(X_i, Y_i), Y_i + \phi_{ik}(X_i, Y_i)).$$

Thus, the map $(T_1 T_0^k)^N$ from a small neighborhood of M_1^+ takes indeed the required form (41) in the coordinates $(X, Y) = Q_{1k}^{-1}(x, y)$. In order to prove relation (44) (with $Q = Q_{1k}$ and $n = N(ks + \bar{m})$), note that the functions ψ_i are uniformly bounded by the constant L, and the functions ϕ_{jik} are small, so for any given constant L_i the image of the disc $\{X_i^2 + Y_i^2 \leq L_i^2\}$ by the map (78) lies inside the disc $\{X_{i+1}^2 + Y_{i+1}^2 \leq L_{i+1}^2\}$ where $L_{i+1} = L_i + L + 1$. Therefore, the images of the unit disc $\mathbb{D}$ by the first N iterations of (78) lie in the disc $\hat{\mathbb{D}}$ from Lemmas 6.6 and 6.7 if its radius $\hat{L}$ satisfies $\hat{L} > N(L+1)$. In other words, $\hat{f}^{i(ks+\bar{m})} \circ Q_{1k}(\mathbb{D}) \subset Q_{i+1,k}(\hat{\mathbb{D}}) \subset V_{i+1} \subset U_0$ for all $i = 0, \dots, N-1$. Since the discs V_i are mutually disjoint, we have that $\hat{f}^{i(ks+\bar{m})} \circ Q_{1k}(\mathbb{D}) \cap Q_{1k}(\mathbb{D})$ for all $i = 0, \dots, N-1$. The images $\hat{f}^m \circ Q_{1k}(\mathbb{D})$ with $m \neq i(ks + \bar{m})$ cannot intersect $Q_{1k}(\mathbb{D})$ too, because they lie outside of U_+ by construction. Indeed, by Lemma 6.6, $\hat{f}^{ls} Q_{ik}(\hat{\mathbb{D}}) = T_0^l Q_{ik}(\hat{\mathbb{D}}) \subset U_0 \cap T_0 U_0$ for $l = 1, \dots, k$ – this implies that $\hat{f}^j Q_{ik}(\hat{\mathbb{D}}) \cap U_+ = \emptyset$ for $1 \leq j \leq ks$ (see (53)), and $\hat{f}^{ks} Q_{ik}(\hat{\mathbb{D}}) \subset U_-$ – this implies that $\hat{f}^{ks+j} Q_{ik}(\hat{\mathbb{D}}) \cap U_+ = \emptyset$ for $j = 1, \dots, m-1$ (see (52)). Thus, relation (44) is fulfilled for all $m = 1, \dots, N(ks + \bar{m}) - 1$, as required.

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