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On the Elementary Obstruction to the Existence of Rational Points

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Abstract—The differentials of a certain spectral sequence converging to the Brauer–Grothendieck group of an algebraic variety X over an arbitrary field are interpreted as the $\cup$-product with the class of the so-called "elementary obstruction." This class is closely related to the cohomology class of the first-degree Albanese variety of X. If X is a homogeneous space of an algebraic group, then the elementary obstruction can be described explicitly in terms of natural cohomological invariants of X. This reduces the calculation of the Brauer–Grothendieck group to the computation of a certain pairing in the Galois cohomology.

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INTRODUCTION

Let X be a smooth algebraic variety over a field k of characteristic 0. The cohomological Brauer–Grothendieck group $\operatorname{Br} X = H^2_{\text{ét}}(X, \mathbb{G}_m)$ is important because of its birational invariance; moreover, if k is a number field, then the elements of $\operatorname{Br} X$ give conditions satisfied by the images of the k-points of X in the space of adelic points of X (the Manin–Brauer obstruction). The group $\operatorname{Br} X$ is calculated by using the Hochschild–Serre spectral sequence

$$H^p(k, H^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbb{G}_m), \tag{0.1}$$

where $\overline{X}$ denotes the variety over an algebraic closure $\bar{k}$, obtained from X by extending the base field, and the $H^i(k, \cdot\,)$ are the Galois cohomology groups of the field k. Suppose that there are no invertible everywhere regular nonconstant functions on $\overline{X}$. To calculate the quotient of the Brauer group $\operatorname{Br} X$ modulo the Brauer group $\operatorname{Br} k$ of the base field by using this spectral sequence, we need to know the Picard group $\operatorname{Pic} \overline{X}$, the Brauer group $\operatorname{Br} \overline{X}$, the action of the Galois group $\Gamma = \operatorname{Gal}(\bar{k}/k)$ on these groups, and the differentials

$$d_2^{1,1} \colon H^1(k, \operatorname{Pic} \overline{X}) \to H^3(k, \bar{k}^*), \qquad d_2^{0,2} \colon (\operatorname{Br} \overline{X})^\Gamma \to H^2(k, \operatorname{Pic} \overline{X}),$$
$$d_3^{0,2} \colon \operatorname{Ker}(d_2^{0,2}) \to H^3(k, \bar{k}^*).$$

The calculation is simplified in the following cases:

- $\operatorname{Br}(\overline{X}) = 0$ (an equivalent condition is that all cycles in $H^2(\overline{X}, \mathbb{Q}_\ell(1))$ are algebraic, and the groups $H^3(\overline{X}, \mathbb{Z}_\ell(1))$ are torsion-free for all ℓ; see [1, Sec. III.8]);
- there is a 0-cycle of degree 1 over k (e.g., a k-point) on X (in this case, $d_2^{1,1} = 0$; see Sec. 1 of this paper);

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- $H^3(k, \bar{k}^*) = 0$ (this is so if k is a number or local field).

Formulas for $\operatorname{Br} X$ encountered in the literature are often obtained under these simplifying assumptions. The objective of this paper is to study the Brauer group without the above additional assumptions.

We prove in the general case that, up to sign, the differentials $d_2^{i,1}$ coincide with the $\cup$-product with the class $e(X)$, which differs only in sign from the class of the natural 2-extension of Galois modules

$$1 \to \bar{k}^* \to \bar{k}(X)^* \to \operatorname{Div} \overline{X} \to \operatorname{Pic} \overline{X} \to 0$$

(see Proposition 1.1). This extension was considered by Colliot-Thélène and Sansuc [2] in relation to the necessity of the condition $e(X) = 0$ (which is equivalent to the existence of a Galois-equivariant section of the homomorphism $\bar{k}^* \to \bar{k}(X)^*$) for the existence of rational points (and 0-cycles of degree 1 defined over k) on X. The class $e(X)$ is called the *elementary obstruction* to the existence of rational points on X. If the group $\operatorname{Pic} \overline{X}$ is torsion-free, then the vanishing of $e(X)$ is necessary and sufficient for the existence of universal torsors on X; in the general case, it implies the existence of torsors of any given type (see [2] or [3, Ch. 2]). Proposition 1.1 gives an expression for the "algebraic" part of $\operatorname{Br} X$ in terms of the Galois cohomology (Corollary 1.2).

Suppose, in addition, that X is a projective variety. Let J be the Picard variety of X; the group of $\bar{k}$-points of J is identified with the component $\operatorname{Pic}^0 \overline{X}$ of the Picard group of $\overline{X}$, which parameterizes the classes of divisors algebraically equivalent to zero. Let A be the Albanese variety of X; it is dual to J as an Abelian variety. The choice of a $\bar{k}$-point on X determines the Albanese map $\overline{X} \to \overline{A}$ that takes this point to zero. This map descends to a map $X \to D$, where D is a principal homogeneous space with structure group A over k. We denote its class by $\delta(X) \in H^1(k, A)$. The classes $e(X)$ and $\delta(X)$ are closely related (see Proposition 2.1).

Now, suppose that X is a curve; then $J = A$ and $J(\bar{k}) = \operatorname{Pic}^0 \overline{X}$ parameterizes the classes of divisors of degree 0 on X, and $D(\bar{k}) = \operatorname{Pic}^1 \overline{X}$ parameterizes the classes of divisors of degree 1. Proposition 2.1 gives a formula for the Brauer group of the curve X. Indeed, because the Barsotti–Weil isomorphism determines a natural pairing

$$H^1(k, J) \times H^1(k, J) \to H^3(k, \bar{k}^*),$$

and $(\delta(X), \delta(X)) = 0$, we prove that the quotient group of $\operatorname{Br} X$ by the image of the Brauer group $\operatorname{Br} k$ of the field is canonically isomorphic to the quotient group of the orthogonal complement $\delta(X)^\perp \subset H^1(k, J)$ modulo the cyclic subgroup generated by $\delta(X)$ (Theorem 2.2).

Similar formulas are valid in the case where X is a principal homogeneous space of a semisimple algebraic group or a homogeneous space of a simply connected semisimple group with connected stabilizers (see Propositions 3.1 and 3.2).

The results obtained in this paper were applied by Borovoi, Colliot-Thélène, and the author in [4], where the equivalence of the vanishing of the elementary obstruction $e(X)$ and the injectivity of the natural map $\operatorname{Br} k \to \operatorname{Br} X$ for a local field k was proved. If X is a homogeneous space of a connected algebraic group with connected stabilizers over a local field, then the vanishing of $e(X)$ implies the existence of rational points on X. In [4], analogs of these results for number fields and some other fields were also obtained. For example, if k is a number field and X has points everywhere locally, then the vanishing of $e(X)$ implies the triviality of a part of Manin's obstruction to the Hasse principle, namely, the one related to the subgroup of locally constant classes. The converse is true if $\operatorname{Pic} \overline{X}$ is a finitely generated torsion-free Abelian group [2]; in the general case, the question of whether these two assertions are equivalent is, apparently, open. [1]

[1] The positive answer to this question has recently been given by O. Wittenberg.

1. A PROPOSITION FROM HOMOLOGICAL ALGEBRA

Suppose that $\mathscr{Y}$ and $\mathscr{X}$ are Abelian categories with sufficiently many injective objects and $f\colon \mathscr{Y} \to \mathscr{X}$ is a left exact additive functor having a left adjoint functor f^*. For $\mathscr{E} \in \mathrm{Ob}(\mathscr{Y})$ and $\mathscr{M} \in \mathrm{Ob}(\mathscr{X})$, the spectral sequence

$$\mathrm{E}_2 = \mathrm{Ext}^p_{\mathscr{X}}(\mathscr{M}, R^q f_*\mathscr{E}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{Y}}(f^*\mathscr{M}, \mathscr{E}) \tag{1.1}$$

is a special case of the spectral sequence of the composition of functors. Our objective is to describe the differentials

$$d_2^{i,j}\colon\ \mathrm{Ext}^i_{\mathscr{X}}(\mathscr{M}, R^j f_*\mathscr{E}) \to \mathrm{Ext}^{i+2}_{\mathscr{X}}(\mathscr{M}, R^{j-1} f_*\mathscr{E})$$

in terms of the $\cup$-product. Consider the auxiliary spectral sequence

$$\mathrm{Ext}^p_{\mathscr{X}}(R^j f_*\mathscr{E}, R^q f_*\mathscr{E}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{Y}}(f^*(R^j f_*\mathscr{E}), \mathscr{E}), \tag{1.2}$$

which is a special case of (1.1) (with $\mathscr{M} = R^j f_*\mathscr{E}$), and its differential

$$\partial = \partial_2^{0,j}\colon\ \mathrm{Hom}_{\mathscr{X}}(R^j f_*\mathscr{E}, R^j f_*\mathscr{E}) \to \mathrm{Ext}^2_{\mathscr{X}}(R^j f_*\mathscr{E}, R^{j-1} f_*\mathscr{E}).$$

The image $\partial(\mathrm{Id}_j)$ of the identity morphism $\mathrm{Id}_j \in \mathrm{Hom}_{\mathscr{X}}(R^j f_*\mathscr{E}, R^j f_*\mathscr{E})$ is a distinguished element in $\mathrm{Ext}^2_{\mathscr{X}}(R^j f_*\mathscr{E}, R^{j-1} f_*\mathscr{E})$. The following proposition is probably well known to experts, but we were unable to find it in the literature, so we give a complete proof here.

Proposition 1.1. *For any $i \geq 0$, $j \geq 1$, and $\alpha \in \mathrm{Ext}^i_{\mathscr{X}}(\mathscr{M}, R^j f_*\mathscr{E})$,*

$$d_2^{i,j}(\alpha) = (-1)^i \alpha \cup \partial(\mathrm{Id}_j),$$

where $\cup$ is the Yoneda pairing

$$\mathrm{Ext}^i_{\mathscr{X}}(\mathscr{M}, R^j f_*\mathscr{E}) \times \mathrm{Ext}^2_{\mathscr{X}}(R^j f_*\mathscr{E}, R^{j-1} f_*\mathscr{E}) \to \mathrm{Ext}^{i+2}_{\mathscr{X}}(\mathscr{M}, R^{j-1} f_*\mathscr{E}).$$

Remark A. For $i = 0$ and $j = 1$, this assertion coincides with Lemma 1.A.4 from [2].

Proof. First, suppose that $i = 1$. Let

$$0 \to R^j f_*\mathscr{E} \to \mathscr{N} \to \mathscr{M} \to 0 \tag{1.3}$$

be an extension whose class in $\mathrm{Ext}^1_{\mathscr{X}}(\mathscr{M}, R^j f_*\mathscr{E})$ is α, and let d denote the connecting homomorphism in the long exact sequence of Ext's of the first argument determined by the sequence (1.3). By the definition of the Yoneda pairing, for any $\xi \in \mathrm{Ext}^2_X(R^j f_*\mathscr{E}, R^{j-1} f_*\mathscr{E})$, we have $\alpha \cup \xi = d(\xi)$. To prove the assertion for $i = 1$, it suffices to show that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{X}}(R^j f_*\mathscr{E}, R^j f_*\mathscr{E}) & \xrightarrow{\partial} & \mathrm{Ext}^2_{\mathscr{X}}(R^j f_*\mathscr{E}, R^{j-1} f_*\mathscr{E}) \\
{\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
\mathrm{Ext}^1_{\mathscr{X}}(\mathscr{M}, R^j f_*\mathscr{E}) & \xrightarrow{d_2^{1,j}} & \mathrm{Ext}^3_{\mathscr{X}}(\mathscr{M}, R^{j-1} f_*\mathscr{E})
\end{array} \tag{1.4}$$

is anticommutative. Consider any exact sequence

$$0 \to A \to B \to C \to 0 \tag{1.5}$$

in $\mathscr{X}$. Let $\mathscr{D}^+(\mathscr{X})$ be the derived category of complexes bounded below, and let

$$\cdots \to C^\bullet[-1] \to A^\bullet \to B^\bullet \to C^\bullet \to \cdots$$

be the distinguished triangle in $\mathscr{D}^+(\mathscr{X})$ determined by (1.5). For any $F \in \mathscr{D}^+(\mathscr{X})$ and $i \in \mathbb{Z}$, the truncation functors determine the distinguished triangles

$$\cdots \to \tau_{\leq j-1}(F) \to F \to \tau_{\geq j}(F) \to \cdots.$$

We obtain the following natural diagram in the derived category of Abelian groups:

$$\begin{array}{ccc} \mathbb{R}\operatorname{Hom}_{\mathscr{X}}(A^\bullet, \tau_{\geq j}(F)) & \longrightarrow & \mathbb{R}\operatorname{Hom}_{\mathscr{X}}(A^\bullet, \tau_{\leq j-1}(F))[1] \\ {\scriptstyle d}\Big\downarrow & & {\scriptstyle d}\Big\downarrow \\ \mathbb{R}\operatorname{Hom}_{\mathscr{X}}(C^\bullet, \tau_{\geq j}(F))[1] & \longrightarrow & \mathbb{R}\operatorname{Hom}_{\mathscr{X}}(C^\bullet, \tau_{\leq j-1}(F))[2] \end{array}.$$

This diagram is anticommutative [5, Proposition 1.1.11]. In the particular case when $F = \tau_{[j-1,j]}(Rf_*\mathscr{E})$ and the exact sequence (1.5) is the sequence (1.3), we obtain the anticommutative diagram

$$\begin{array}{ccc} \mathbb{R}\operatorname{Hom}_{\mathscr{X}}((R^j f_*\mathscr{E})^\bullet, \tau_{[j]}(Rf_*\mathscr{E})) & \longrightarrow & \mathbb{R}\operatorname{Hom}_{\mathscr{X}}((R^j f_*\mathscr{E})^\bullet, \tau_{[j-1]}(Rf_*\mathscr{E}))[1] \\ {\scriptstyle d}\Big\downarrow & & {\scriptstyle d}\Big\downarrow \\ \mathbb{R}\operatorname{Hom}_{\mathscr{X}}(\mathscr{M}^\bullet, \tau_{[j]}(Rf_*\mathscr{E}))[1] & \longrightarrow & \mathbb{R}\operatorname{Hom}_{\mathscr{X}}(\mathscr{M}^\bullet, \tau_{[j-1]}(Rf_*\mathscr{E}))[2] \end{array}.$$

The required anticommutative diagram (1.4) is obtained by passing to the cohomology groups of degree j. Indeed, the identification of objects and vertical arrows is easy, and the fact that the differentials in the spectral sequence of the composition of functors are obtained from the connecting homomorphisms of truncated complexes follows readily from the explicit construction of this spectral sequence based on the Cartan–Eilenberg injective resolution (see, e.g., [6, Appendix B]).

To prove the proposition for $i > 1$, it suffices to decompose the i-fold extension α into a product of simple extensions and apply the above assertion i times.

It remains to consider the case $i = 0$. We have $\alpha \in \operatorname{Hom}_{\mathscr{X}}(\mathscr{M}, R^j f_*\mathscr{E})$. Since the spectral sequence (1.2) is functorial in the first argument, it follows that the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathscr{X}}(R^j f_*\mathscr{E}, R^j f_*\mathscr{E}) & \xrightarrow{\partial} & \operatorname{Ext}^2_{\mathscr{X}}(R^j f_*\mathscr{E}, R^{j-1} f_*\mathscr{E}) \\ {\scriptstyle \alpha^*}\Big\downarrow & & {\scriptstyle \alpha^*}\Big\downarrow \\ \operatorname{Hom}_{\mathscr{X}}(\mathscr{M}, R^j f_*\mathscr{E}) & \xrightarrow{d_2^{0,j}} & \operatorname{Ext}^2_{\mathscr{X}}(\mathscr{M}, R^{j-1} f_*\mathscr{E}) \end{array}$$

is commutative. This completes the proof of the proposition. □

Suppose that k is a field of characteristic 0 with algebraic closure $\bar{k}$, $\Gamma = \operatorname{Gal}(\bar{k}/k)$, X is a smooth geometrically irreducible reduced variety, $\overline{X} = X \times_k \bar{k}$, and there are no nonconstant invertible regular functions on $\overline{X}$, i.e., $H^0(\overline{X}, \mathbb{G}_m) = \bar{k}^*$. Consider the spectral sequence

$$\operatorname{Ext}^p_k(\mathscr{M}, H^q(\overline{X}, \mathbb{G}_m)) \Rightarrow \operatorname{Ext}^{p+q}_X(p^*\mathscr{M}, \mathbb{G}_m), \tag{1.6}$$

which is the sequence (1.1) in the special case where $\mathscr{Y}$ is the category of étale sheaves on X, $\mathscr{X}$ is the category of discrete Γ-modules (coinciding with the category of étale sheaves on $\operatorname{Spec}(k)$), $f\colon \mathscr{Y} \to \mathscr{X}$ is induced by the structure morphism $p\colon X \to \operatorname{Spec}(k)$, and $\mathscr{E} = \mathbb{G}_m$. Take $\mathscr{M} = \operatorname{Pic}\overline{X}$. We associate with X the canonical element

$$e(X) := \partial(\mathrm{Id}) \in \operatorname{Ext}^2_k(\operatorname{Pic}\overline{X}, \bar{k}^*)$$

considered above; here $\mathrm{Id} \in \operatorname{Hom}_k(\operatorname{Pic}\overline{X}, \operatorname{Pic}\overline{X})$ is the identity map. The class $e(X)$ is an important characteristic of the variety X; it has the following properties (see [3, Theorem 2.3.4]):

(1) the class $-e(X)$ coincides with that of the natural 2-extension of Γ-modules

$$1 \to \bar{k}^* \to \bar{k}(X)^* \to \operatorname{Div}\overline{X} \to \operatorname{Pic}\overline{X} \to 0; \tag{1.7}$$

(2) $e(X) = 0$ if and only if the natural homomorphism $\bar{k}^* \to \bar{k}(X)^*$ has a Γ-equivariant section;

(3) if there is a 0-cycle of degree 1 (e.g., a k-point) on X, then $\bar{k}^* \to \bar{k}(X)^*$ has a Galois equivariant section, whence $e(X) = 0$.

Remark B. The class $e(X)$ is functorial in the sense that if $f\colon Y \to X$ is a morphism of varieties admitting no nonconstant invertible regular functions over $\bar{k}$, then $e(X)$ is obtained from $e(Y)$ by using the homomorphism $f^*\colon \operatorname{Pic}\overline{X} \to \operatorname{Pic}\overline{Y}$. Indeed, since the spectral sequence (1.6) is functorial with respect to $\mathscr{M}$ and X, it follows that the diagram

$$\begin{array}{ccc}
\operatorname{Hom}_k(\operatorname{Pic}\overline{Y}, \operatorname{Pic}\overline{Y}) & \longrightarrow & \operatorname{Ext}^2_k(\operatorname{Pic}\overline{Y}, \bar{k}^*) \\
\downarrow & & \downarrow \\
\operatorname{Hom}_k(\operatorname{Pic}\overline{X}, \operatorname{Pic}\overline{Y}) & \longrightarrow & \operatorname{Ext}^2_k(\operatorname{Pic}\overline{X}, \bar{k}^*) \\
\uparrow & & \| \\
\operatorname{Hom}_k(\operatorname{Pic}\overline{X}, \operatorname{Pic}\overline{X}) & \longrightarrow & \operatorname{Ext}^2_k(\operatorname{Pic}\overline{X}, \bar{k}^*)
\end{array}$$

is commutative, which implies the required assertion.

Recall that $\operatorname{Br}_1 X$ denotes the so-called algebraic part of the Brauer group, i.e., the kernel of the natural homomorphism $\operatorname{Br} X \to \operatorname{Br}\overline{X}$, and $\operatorname{Br}_0 X$ is the image of $\operatorname{Br} k$ in $\operatorname{Br} X$. Since $\operatorname{Br}\bar{k} = 0$, it follows that $\operatorname{Br}_0 X$ is contained in $\operatorname{Br}_1 X$.

Corollary 1.2. *The quotient group* $\operatorname{Br}_1 X/\operatorname{Br}_0 X$ *is canonically isomorphic to the subgroup of* $H^1(k, \operatorname{Pic}\overline{X})$ *consisting of all elements* ξ *for which* $\xi \cup e(X) = 0$.

Proof. Setting $\mathscr{M} = \mathbb{Z}$ in (1.6), we obtain the spectral sequence (0.1). It remains to apply Proposition 1.1 with $i = j = 1$. □

If $e(X) = 0$, then the sequence

$$0 \to \operatorname{Br} k \to \operatorname{Br}_1 X \to H^1(k, \operatorname{Pic}\overline{X}) \to 0 \tag{1.8}$$

is exact; this follows from (0.1) and Proposition 1.1.

2. THE BRAUER GROUP AND THE BARSOTTI–WEIL ISOMORPHISM

Let k-gps be the category of commutative algebraic groups over a field k. This category is Abelian; for $A, B \in \operatorname{Ob}(k\text{-gps})$, $\operatorname{Ext}^i_{k\text{-gps}}(A, B)$ can be defined as the group of equivalence classes of i-fold extensions of A by B (a brief survey of the subject can be found in [7, Sec. I.0], [8]). If k is a perfect field, then there is a spectral sequence [9]

$$H^p(k, \operatorname{Ext}^q_{k\text{-gps}}(A, B)) \Rightarrow \operatorname{Ext}^{p+q}_{k\text{-gps}}(A, B). \tag{2.1}$$

Let Φ be the exact functor from k-gps to the category of discrete Γ-modules assigning the Galois module $A(\bar{k})$ to each group A, i.e., forgetting the algebraic group structure on $A(\bar{k})$.

According to the Barsotti–Weil formula, there exists a canonical isomorphism of Galois modules

$$\operatorname{Ext}^1_{k\text{-gps}}(A, \mathbb{G}_m) = A^t(\bar{k}),$$

where A is an Abelian variety and A^t is its dual Abelian variety [10, Sec. VII.3]. Moreover, $\operatorname{Ext}^i_{k\text{-gps}}(A, \mathbb{G}_m) = 0$ for $i \neq 1$ [8, Proposition 12.3]. These facts and the spectral sequence (2.1) give rise to the natural isomorphism

$$H^1(k, A^t) = \operatorname{Ext}^2_{k\text{-gps}}(A, \mathbb{G}_m). \tag{2.2}$$

Recall the construction of the Barsotti–Weil isomorphism. Let $\mathscr{P}$ be the linear Poincaré bundle on $A^t \times A$. To each $a \in A^t(\bar{k})$ we attach the principal homogeneous space W_a over A with structure group $\mathbb{G}_m$ obtained from the restriction of $\mathscr{P}$ to $a \times A$ by removing the zero section. The space W_a is endowed with the canonical structure of a commutative group scheme; thus, we obtain an extension in k-gps (see [7, Appendix C]), namely, $1 \to \mathbb{G}_m \to W_a \to A \to 1$.

We return to the situation considered at the end of the previous section. Let J be the Picard variety of a smooth complete geometrically irreducible reduced variety X, and let $A = J^t$ be the Albanese variety of X. The group of $\bar{k}$-points of J is identified with the subgroup $\operatorname{Pic}^0 \overline{X} \subset \operatorname{Pic} \overline{X}$, which consists of all divisors algebraically equivalent to zero. We denote the natural embedding of $J(\bar{k})$ into $\operatorname{Pic} \overline{X}$ by i.

If there is a rational point on X, then one can define the Albanese map $X \to A$ which takes this point to zero. In the general case, Weil [11] constructed a principal homogeneous space D over k with structure group A, and a morphism $\mathrm{Alb}\colon X \to D$ such that, when the base field is extended to $\bar{k}$, the morphism becomes the classical Albanese map. Let us denote the class $[D]$ by $\delta(X) \in H^1(k, A)$. The following proposition describes the relationship between the classes $\delta(X)$ and $e(X)$.

Proposition 2.1. *Suppose that X is a smooth complete geometrically irreducible reduced variety over a field k of characteristic 0, J is the Picard variety of X, and $A = J^t$ is the Albanese variety of X. Then $-i^*(e(X)) \in \operatorname{Ext}^2_k(J(\bar{k}), \bar{k}^*)$ is obtained from $\delta(X) \in H^1(k, A) = \operatorname{Ext}^2_{k\text{-gps}}(J, \mathbb{G}_m)$ by applying the forgetful functor Φ.*

Proof. Let $\operatorname{Div}^0 \overline{X}$ denote the group of divisors algebraically equivalent to zero. The element $-i^*(e(X))$ is the class of the extension

$$1 \to \bar{k}^* \to \bar{k}(X)^* \to \operatorname{Div}^0 \overline{X} \to J(\bar{k}) \to 0, \tag{2.3}$$

which is obtained from (1.7) by means of the homomorphism $i\colon J(\bar{k}) \to \operatorname{Pic} \overline{X}$. The corresponding 2-extension for D has the form

$$1 \to \bar{k}^* \to \bar{k}(D)^* \to \operatorname{Div}^0 \overline{D} \to J(\bar{k}) \to 0,$$

because the Γ-module $\operatorname{Pic}^0 \overline{D}$ is canonically isomorphic to $\operatorname{Pic}^0 \overline{X}$.

Fix a point $P \in D(\bar{k})$ in the image of the morphism Alb and let $\phi\colon \Gamma \to A(\bar{k})$ be a 1-cocycle such that, for any $g \in \Gamma$, the point $g(P)$ is obtained by translating P by $\phi(g)$. By definition, $\delta(X) \in H^1(k, A)$ is the class of ϕ. Let us identify $\overline{D}$ with $\overline{A}$, taking P for the zero of the group law. We denote the orbit of P under the action of the Galois group by $\mathfrak{P}$. Let $\mathscr{O}_{\mathfrak{P}}$ be the local ring of D in $\mathfrak{P}$, and let $\operatorname{Div}^0(\overline{D})_{\mathfrak{P}}$ denote the group of divisors algebraically equivalent to zero whose supports are disjoint from $\mathfrak{P}$. There is a natural 2-extension of Γ-modules

$$1 \to \bar{k}^* \to (\mathscr{O}_{\mathfrak{P}} \otimes_k \bar{k})^* \to \operatorname{Div}^0(\overline{D})_{\mathfrak{P}} \to J(\bar{k}) \to 0. \tag{2.4}$$

The morphism Alb defines a natural Γ-equivariant map from (2.4) to (2.3); therefore, these extensions represent the same class in $\operatorname{Ext}^1_k(J(\bar{k}), \bar{k}^*)$.

Let $k(P)$ be the residue field of the point P. Passing to the quotient group of $(\mathscr{O}_{\mathfrak{P}} \otimes_k \bar{k})^*$ modulo the subgroup of rational functions on D which are regular and take the value 1 at each point of $\mathfrak{P}$, we obtain an extension

$$1 \to \bar{k}^* \to (k(P) \otimes_k \bar{k})^* \to C_{\mathfrak{P}} \to J(\bar{k}) \to 0,$$

which is equivalent to (2.4) and can be taken as the definition of the group $C_{\mathfrak{P}}$. We claim that this extension comes from some 2-extension of commutative algebraic groups over k of the form

$$1 \to \mathbb{G}_m \to R_{k(P)/k}(\mathbb{G}_m) \to ? \to J \to 0,$$

where $R_{k(P)/k}$ denotes the Weil descent of the base field. To prove this, it suffices to show that the extension

$$1 \to (k(P) \otimes_k \bar{k})^*/\bar{k}^* \to C_{\mathfrak{P}} \to J(\bar{k}) \to 0 \tag{2.5}$$

is obtained from the extension of J by the k-torus $T = R_{k(P)/k}(\mathbb{G}_m)/\mathbb{G}_m$ in the category k-gps. Moreover, if we show that this is so over $\bar{k}$, then the general case will follow by applying the Galois descent. Finally, according to [10, Ch. VII, Sec. 1.4], it is sufficient to prove that (2.5) is determined by a *rational* symmetric set of factors, i.e., by a 2-cocycle of $J(\bar{k})$ with coefficients in $(\bar{k}^*)^{|\mathfrak{P}|}/\bar{k}^*$ determined by rational functions on J. Over $\bar{k}$, we can work with A instead of D.

Let $\mathscr{D}$ be the Poincaré divisor on $A \times J$. According to the theorem of the square and the "see-saw principle," there exists a rational function f on $A \times J \times J$ whose divisor is equal to

$$(f) = s_{23}^{-1}(\mathscr{D}) - p_{12}^{-1}(\mathscr{D}) - p_{13}^{-1}(\mathscr{D}),$$

where s_{23} denotes the sum of the second and third coordinates and p_{12} and p_{13} are the projections $A \times J \times J \to A \times J$. A rational section s of the homomorphism $C_{\mathfrak{P}} \to J(\bar{k})$ in (2.5) can be obtained by assigning the intersection of $\mathscr{D}$ with $A \times x$ to each point $x \in J(\bar{k})$. The corresponding set of factors is determined in [10]:

$$\delta(s)(x, y) := s(x + y) - s(x) - s(y).$$

In the case under consideration, this divisor coincides with the divisor of the restriction of f to $A \times x \times y$ up to divisors of functions taking equal nonzero values at all points conjugate to P. Thus, $\delta(s)(x, y)$ is equal to the value of f at these points up to a common factor from $\bar{k}^*$. The set of factors thus obtained is rational, because it is determined by a rational function. Thus, we have proved that (2.5) comes from an extension

$$1 \to \mathbb{G}_m \to R_{k(p)/k}(\mathbb{G}_m) \to J_{\mathfrak{P}} \to J \to 0 \tag{2.6}$$

of commutative algebraic groups over k for some commutative algebraic group $J_{\mathfrak{P}}$.

Remark 1. If X is a curve, then $J_{\mathfrak{P}}$ is the generalized Jacobian of X determined by the modulus $\mathfrak{P}$ (see [10, Sec. I.1]).

The extension (2.6) can be decomposed into a Yoneda product $\alpha \cup \beta$, where α is an extension of tori

$$1 \to \mathbb{G}_m \to R_{k(P)/k}(\mathbb{G}_m) \to T \to 1$$

and β is the extension of commutative algebraic groups

$$1 \to T \to J_{\mathfrak{P}} \to J \to 0$$

constructed above. Over $\bar{k}$, the torus $R_{k(P)/k}(\mathbb{G}_m)$ decomposes into a product of copies of $\mathbb{G}_m$; the number of copies is equal to that of conjugate points $g(P)$, where $g \in \Gamma$. Let $\pi_{g(P)}$ denote the projectors onto the corresponding components; these maps determine sections of α in the category of commutative $\bar{k}$-groups. The Milne spectral sequence (2.1) yields a canonical map

$$H^1(k, \mathrm{Hom}_{\bar{k}\text{-gps}}(T, \mathbb{G}_m)) \to \mathrm{Ext}^1_{k\text{-gps}}(T, \mathbb{G}_m)$$

(in fact, this is an isomorphism); the cocycle determining α has the form

$$g \longmapsto \psi(g) = \pi_{g(P)}/\pi_P \in \mathrm{Hom}_{\bar{k}\text{-gps}}(T, \mathbb{G}_m).$$

It follows that (2.6) is obtained by means of a similar map

$$H^1(k, \mathrm{Ext}^1_{\bar{k}\text{-gps}}(J, \mathbb{G}_m)) \to \mathrm{Ext}^2_{k\text{-gps}}(J, \mathbb{G}_m)$$

from the cocycle which takes $g \in \Gamma$ to the extension induced by β by means of the homomorphism $\pi_{g(P)}/\pi_P$. It is easy to find a set of factors for this extension of $\overline{J}$ by $\mathbb{G}_{m,\bar{k}}$. Taking s for a rational section and repeating the above calculations, we see that it has the form

$$f(g(P), x, y)/f(P, x, y) \in \bar{k}(\overline{J} \times \overline{J})^*.$$

According to the remark to Theorem 6 in [10, Sec. VII.3], the set of factors for the extension obtained by means of the Barsotti–Weil isomorphism from

$$\delta(X)(g) = g(P) - P \in A(\bar{k})$$

has the same form. This completes the proof of the proposition. $\square$

Remark 2. If X is a curve of genus 0, then it admits a Γ-invariant class of first-degree divisors; therefore, it is natural to assume that $\delta(X)=0$. However, $e(X)\in \operatorname{Ext}^2_k(\mathbb{Z},\bar{k}^*)=\operatorname{Br} X$ is the class of X as a Severi–Brauer variety, and $e(X)=0$ if and only if $X\simeq \mathbb{P}^1_k$. Similar examples exist for curves of larger genus. For example, let X be a curve of genus 2 which is a cyclic covering of degree 3 of a conic C ramified at four points such that C has no k-points. Then $e(X)\neq 0$. Indeed, otherwise, the homomorphism $\bar{k}^*\to \bar{k}(X)^*$ has a section, and therefore the corresponding homomorphism for C has a section as well, which contradicts the condition $e(C)\neq 0$. At the same time, X has a Γ-invariant class of divisors of degree 1 (the preimage of the $\bar{k}$-point of C minus the canonical class of X). Thus, in the general case, $\delta(X)=0$ does not imply $e(X)=0$. However, if X is a curve of genus 1, then the condition $\delta(X)=0$ is equivalent to the set $X(k)$ being nonempty and, therefore, implies $e(X)=0$.

Remark 3. Let X be any smooth projective variety, and let J be its Picard variety. If $e(X)=0$, then there exist torsors of any given type on X (see, e.g., [3, p. 31]). Let $Y\to X$ be a torsor whose type is the homomorphism $J[n](\bar{k})\to \operatorname{Pic}\overline{X}$. Then $Y=X\times_D B$, where $B\to D$ is a torsor of type $J[n](\bar{k})\to \operatorname{Pic}\overline{J}$. According to [3, Proposition 3.3.4 (b)], the existence of B is equivalent to the divisibility of the class $[D]\in H^1(k,A)$ by n. Hence $\delta(X)=[D]$ is a divisible element of the group $H^1(k,A)$. Thus, if $H^1(k,A)$ contains no divisible elements different from zero, then $e(X)=0$ implies $\delta(X)=0$. The field of real numbers does have this property, while the field of p-adic numbers does not. Nevertheless, even for this field (as well as for the more general field of fractions of an excellent Henselian discrete valuation ring with finite residue field), the vanishing of $e(X)$ implies that of $\delta(X)$ [12] (see also [4]). However, for some other fields, such as the Laurent series field $\mathbb{C}((t))$, this is not so; O. Wittenberg has recently shown that $e(X)=0$ for any variety over a field of cohomological dimension 1.

Remark 4. Finally, suppose that X is a curve over a number field having a 0-cycle of degree 1 over each completion of the field k. Then $\delta(X)=0$ implies $e(X)=0$. Indeed, each Γ-invariant element in $\operatorname{Pic}\overline{X}$ of degree 1 determines a decomposition of $\operatorname{Pic}\overline{X}$ into the sum $J(\bar{k})\oplus\mathbb{Z}$. Therefore,

$$\operatorname{Ext}^2_k(\operatorname{Pic}\overline{X},\bar{k}^*)\simeq \operatorname{Ext}^2_k(J(\bar{k}),\bar{k}^*)\oplus \operatorname{Br} k,$$

and $e(X)\in\operatorname{Br} k$. The presence of 0-cycles of degree 1 everywhere locally on X and Hasse's reciprocity law imply $e(X)=0$.

Under the assumption that the Tate–Shafarevich group of the Abelian variety J is finite, the converse implication is valid for any smooth projective variety X over a number field having points everywhere locally. In this case, according to Theorem 2.12 from [4], $e(X)=0$ implies that any adelic point on X (and, therefore, on D) is orthogonal in the sense of the Brauer–Manin pairing to the subgroup $Б(D)\subset \operatorname{Br} D$ consisting of locally constant classes. By a theorem of Yu. I. Manin (see [3, Theorem 6.2.3]), this implies that $\delta(X)$ is contained in the kernel of the Cassels–Tate pairing. If $Ш(J)$ is finite, then this pairing is nondegenerate, and hence $D\cong J$, i.e., $\delta(X)=0$.

Let A be an Abelian variety over a perfect field k, and let $J=A^t$ be the dual Abelian variety. The canonical pairing

$$(\ ,\ )\colon H^1(k,J)\times H^1(k,A)\to H^3(k,\bar{k}^*) \tag{2.7}$$

is induced by the composition of the isomorphism (2.2) and the homomorphism

$$H^1(k,A)\to \operatorname{Ext}^2_{k\text{-gps}}(J,\mathbb{G}_m)\to \operatorname{Ext}^2_k(J(\bar{k}),\bar{k}^*)$$

forgetting the algebraic group structure from the pairing determined by the Yoneda product

$$\cup\colon H^1(k,J)\times \operatorname{Ext}^2_k(J(\bar{k}),\bar{k}^*)\to H^3(k,\bar{k}^*)$$

in the Galois cohomology. Let J (respectively, A) be the Picard (respectively, Albanese) variety of a smooth complete geometrically irreducible reduced variety X over the field k. Proposition 2.1 and the functoriality of the last pairing imply that, for any $y\in H^1(k,J)$, we have

$$(y,\delta(X))=-y\cup i^*(e(X))=-i_*(y)\cup e(X). \tag{2.8}$$

Theorem 2.2. *Let X be a smooth complete geometrically irreducible reduced curve over a field k of characteristic 0. Then $\operatorname{Br} X/\operatorname{Br}_0 X$ is canonically isomorphic to the quotient group of the orthogonal complement to $\delta(X)$ in $H^1(k,J)$ with respect to the pairing (2.7) modulo the cyclic subgroup generated by $\delta(X)$.*

Proof. By a theorem of Tsen, we have $\operatorname{Br}\overline{X}=0$; therefore, $\operatorname{Br} X=\operatorname{Br}_1 X$. For a curve, we have $J=A$. It is well known and easy to prove that the exact sequence of Γ-modules

$$0\to J(\bar{k})\to \operatorname{Pic}\overline{X}\to \mathbb{Z}\to 0$$

represents the class $\delta(X)\in H^1(k,J)=\operatorname{Ext}^1_k(\mathbb{Z},J(\bar{k}))$. Hence $i_*(\delta(X))=0$, and (2.8) implies $(\delta(X),\delta(X))=0$. The same exact sequence also implies that $H^1(k,\operatorname{Pic}\overline{X})$ is the quotient group of $H^1(k,J)$ modulo the subgroup generated by $\delta(X)$. Now, the required assertion follows from Corollary 1.2 and Proposition 2.1. □

3. HOMOGENEOUS SPACES

Let G be a semisimple group over a field k of characteristic 0. The variety of G is geometrically irreducible and reduced, and there are no nonconstant regular invertible functions on $\overline{G}$ (by Rosenlicht's lemma, such functions are characters of $\overline{G}$ up to multiplication by a constant). Let $\widetilde{G}$ be the universal covering of G. It is well known that $\widetilde{G}$ is the central extension of G by some finite Abelian group μ, which is called the fundamental group of G. In [13], to such an extension was attached a natural map $d\colon H^1(k,G)\to H^2(k,\mu)$ of pointed sets.

Proposition 3.1. *Let X be a principal homogeneous space of a semisimple group G over k, and let $[X]$ be its class in $H^1(k,G)$. Then $\operatorname{Br} X/\operatorname{Br}_0 X$ coincides with the set of elements of $H^1(k,\hat{\mu})$ orthogonal to $d([X])$ with respect to the pairing*

$$H^1(k,\hat{\mu})\times H^2(k,\mu)\to H^3(k,\bar{k}^*),$$

where $\hat{\mu}=\operatorname{Hom}(\mu,\bar{k}^*)$ is the commutative k-group dual to μ.

Proof. According to [14], we have $\operatorname{Br}\overline{G}=0$; therefore, $\operatorname{Br} X=\operatorname{Br}_1 X$. The Galois module $\operatorname{Pic}\overline{X}$ is canonically isomorphic to $\hat{\mu}$ (see, e.g., [3, Sec. 3.2]). Therefore, the class $e(X)$ belongs to $\operatorname{Ext}^2_k(\hat{\mu},\bar{k}^*)=H^2(k,\mu)$. A result of Giraud [15, Sec. V.3.2.9] and Proposition 2.3.11 from [3] imply that $e(X)=d([X])$ (see [3, p. 54] for more details). Now, the required assertion follows from Corollary 1.2. □

Suppose that G is a simply connected group, i.e., $G=\widetilde{G}$, and X is a homogeneous space of G over k, not necessarily principal but such that the stabilizer $\overline{H}$ of some $\bar{k}$-point of X is connected. To X one canonically associates the Galois cohomology set $H^2(k,\overline{H})$ containing a distinguished subset of neutral elements (see [16] or [17] for details). Springer defined a class $\eta_X\in H^2(k,\overline{H})$, which is neutral if and only if X can be lifted to a principal homogeneous space of G [16] (see also [3, Sec. 9.2]). It is well known that X determines a canonical k-torus T such that $\overline{T}$ is the maximal toric quotient of $\overline{H}$ (see [17] or [18]). Let t denote the natural map of sets $H^2(k,\overline{H})\to H^2(k,T)$ (cf. [17, Sec. 1.7]), and let $\widehat{T}$ be the character module of the torus T.

Proposition 3.2. *Suppose that X is a homogeneous space of a simply connected semisimple group over k, and there is a $\bar{k}$-point with connected stabilizer on X. Then $\operatorname{Br}_1 X/\operatorname{Br}_0 X$ coincides with the set of elements of $H^1(k,\widehat{T})$ orthogonal to $t(\eta_X)\in H^2(k,T)$ with respect to the pairing*

$$H^1(k,\widehat{T})\times H^2(k,T)\to H^3(k,\bar{k}^*). \tag{3.1}$$

Proof. In the proof of Theorem 9.5.1 in [3], a canonical isomorphism of the Γ-modules $\operatorname{Pic}\overline{X}$ and $\widehat{T}$ was constructed (see the formula on p. 176; it uses the simply-connectedness of G). Thus, $e(X)$ belongs to $\operatorname{Ext}^2_k(\widehat{T},\bar{k}^*)=H^2(k,T)$. It was also proved in [3, p. 177] that $e(X)=t(\eta_X)$. It remains to apply Corollary 1.2. □

It is likely that the assumption that G is simply connected can be dispensed with at the expense of some complications; namely, instead of $\widehat{T}$, we must consider the hypercohomology of the natural complex of Γ-modules $\widehat{G} \to \widehat{T}$ (cf. [19]).

Recall that if M is a discrete Γ-module, then $Ш^i_\omega(k, M) \subset H^i(k, M)$ is defined as the intersection of the kernels of the restriction homomorphisms to all procyclic closed subgroups of Γ.

Corollary 3.3. *With the assumptions and the notation of Proposition 3.2, suppose that X_c is a smooth complete compactification of X. Then $\operatorname{Br} X_c/\operatorname{Br}_0 X$ coincides with the set of elements of the subgroup $Ш^1_\omega(k, \widehat{T}) \subset H^1(k, \widehat{T})$ which are orthogonal to $t(\eta_X)$ with respect to the pairing* (3.1).

Proof. Bogomolov proved that $\operatorname{Br} \overline{X_c} = 0$ [20]; so $\operatorname{Br} X_c = \operatorname{Br}_1 X_c$. The embedding $X \hookrightarrow X_c$ defines a natural map of the corresponding spectral sequences of the form (0.1). The restriction homomorphism $r\colon \operatorname{Pic} \overline{X_c} \to \operatorname{Pic} \overline{X} = \widehat{T}$ is surjective, because X_c is smooth. Its kernel is freely generated by the classes of divisors on $\overline{X_c}$ in the complement to $\overline{X}$, because there are no nonconstant invertible regular functions on $\overline{X}$. We obtain a short exact sequence of Γ-modules

$$0 \to \operatorname{Div}_{\overline{X_c} \setminus \overline{X}} \overline{X_c} \to \operatorname{Pic} \overline{X_c} \to \widehat{T} \to 0; \tag{3.2}$$

moreover, the Γ-module $\operatorname{Div}_{\overline{X_c} \setminus \overline{X}} \overline{X_c}$ is permutation, i.e., it has a Γ-invariant basis. According to a recent result of Colliot-Thélène and Kunyavskii [18], the Γ-module $\operatorname{Pic} \overline{X_c}$ is flasque, i.e.,

$$H^1(\Gamma', \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} \overline{X_c}, \mathbb{Z})) = 0$$

for any closed subgroup $\Gamma' \subset \Gamma$. It is well known and easy to prove that, in this situation, the homomorphism r induces an isomorphism between $H^1(k, \operatorname{Pic} \overline{X_c})$ and the subgroup $Ш^1_\omega(k, \widehat{T})$. The same homomorphism defines the commutative diagram of pairings

$$\begin{array}{ccccc} H^1(k, \operatorname{Pic} \overline{X}) & \times & \operatorname{Ext}^2_k(\operatorname{Pic} \overline{X}, \bar{k}^*) & \longrightarrow & H^3(k, \bar{k}^*) \\ \uparrow & & \downarrow & & \| \\ H^1(k, \operatorname{Pic} \overline{X_c}) & \times & \operatorname{Ext}^2_k(\operatorname{Pic} \overline{X_c}, \bar{k}^*) & \longrightarrow & H^3(k, \bar{k}^*) \end{array}.$$

Now, the required assertion follows from the functoriality of $e(X)$ (see the remark in Sec. 1). □

This corollary strengthens somewhat Theorem A (iii) of [18] in the case when G has no toric part.

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